Automorphisms of extremal codes

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Plan

The use of symmetry

- Beautiful objects have symmetries.
- Symmetries help to reduce the search space for nice objects
- and hence make huge problems acessible to computations.

The use of challenge problems

- Applications for classical theories and theorems such as
- Burnside orbit counting
- Invariant theory of finite groups
- Theory of quadratic forms
- Representation theory of finite groups
- Provide a practical introduction to abstract theory.

Self-dual codes

Definition

- A linear binary code C of length n is a subspace $C \leq \mathbb{F}_2^n$.
- The dual code of C is
$$C^\perp := \{x \in \mathbb{F}_2^n \mid (x, c) := \sum_{i=1}^n x_i c_i = 0 \text{ for all } c \in C\}$$
- C is called self-dual if $C = C^\perp$.
- $\mathrm{Aut}(C) = \{\sigma \in S_n \mid \sigma(C) = C\}$.

Facts

- $\dim(C) + \dim(C^\perp) = n$ so $C = C^\perp \Rightarrow \dim(C) = \frac{n}{2}$.
- Let $\mathbf{1} = (1, \ldots, 1)$. Then $(c, c) = (c, \mathbf{1})$.
- So if $C = C^\perp$ then $\mathbf{1} \in C$.

Doubly-even self-dual codes

The Hamming weight.

- The Hamming weight of a codeword $c \in C$ is
$$\mathrm{wt}(c) := |\{i \mid c_i \neq 0\}|.$$
- $\mathrm{wt}(c) \equiv_2 (c, c)$, so $C \subseteq C^\perp$ implies $\mathrm{wt}(C) \subset 2\mathbb{Z}$.
- C is called doubly-even if $\mathrm{wt}(C) \subset 4\mathbb{Z}$.
- Fact: $C = C^\perp \leq \mathbb{F}_2^n$ doubly-even $\Rightarrow n \in 8\mathbb{Z}$.
- The minimum distance $d(C) := \min\{\mathrm{wt}(c) \mid 0 \neq c \in C\}$.
- A self-dual code $C \leq \mathbb{F}_2^n$ is called extremal if $d(C) = 4 + 4\lfloor \frac{n}{24} \rfloor$.
- The weight enumerator of C is
$$p_C := \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)} \in \mathbb{C}[x, y]_n.$$

Examples for self-dual doubly-even codes

Hamming Code

$$h_8 : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

the extended Hamming code, the unique doubly-even self-dual code of length 8,

$$p_{h_8}(x, y) = x^8 + 14x^4y^4 + y^8$$

and $\mathrm{Aut}(h_8) = 2^3 : L_3(2)$.

Golay Code

The binary Golay code $\mathcal{G}_{24}$ is the unique doubly-even self-dual code of length 24 with minimum distance ≥ 8. $\mathrm{Aut}(\mathcal{G}_{24}) = M_{24}$

$$p_{\mathcal{G}_{24}} = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$$

Application of invariant theory

The weight enumerator of C is $p_C := \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)} \in \mathbb{C}[x,y]_n$.

Theorem (Gleason, ICM 1970)

Let $C = C^\perp \leq \mathbb{F}_2^n$ be doubly even. Then $d(C) \leq 4 + 4\lfloor \frac{n}{24} \rfloor$
Doubly-even self-dual codes achieving equality are called extremal.

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Proof:

- $p_C(x,y) = p_C(x,iy)$, $p_C(x,y) = p_{C^\perp}(x,y) = p_C(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}})$
- $G_{192} := \langle \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \rangle$.
- $p_C \in \mathrm{Inv}(G_{192}) = \mathbb{C}[p_{h_8}, p_{\mathcal{G}_{24}}]$
- $\exists ! f \in \mathbb{C}[p_{h_8}, p_{\mathcal{G}_{24}}]_{8m}$ such that

$$f(1,y) = 1 + 0y^4 + \ldots + 0y^{4\lfloor \frac{m}{3} \rfloor} + a_m y^{4\lfloor \frac{m}{3} \rfloor + 4} + b_m y^{4\lfloor \frac{m}{3} \rfloor + 8} + \ldots$$

- $a_m > 0$ for all m

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- $a_m > 0$ for all m

Proposition

$b_m < 0$ for all $m \geq 494$ so there is no extremal code of length ≥ 3952.

Automorphism groups of extremal codes

length	8	24	32	40	48	72	80	96	104	≥ 3952
$d(C)$	4	8	8	8	12	16	16	20	20	
extremal	h_8	$\mathcal{G}_{24}$	5	16, 470	QR_{48}	?	≥ 15	?	≥ 1	0

$\mathrm{Aut}(C) = \{\sigma \in S_n \mid \sigma(C) = C\}$ is the automorphism group of $C \leq \mathbb{F}_2^n$.

- $\mathrm{Aut}(h_8) = 2^3.L_3(2)$
- $\mathrm{Aut}(\mathcal{G}_{24}) = M_{24}$
- Length 32: $L_2(31)$, $2^5.L_5(2)$, $2^8.S_8$, $2^8.L_2(7).2$, $2^5.S_6$.
- Length 40: 10,400 extremal codes with $\mathrm{Aut} = 1$.
- $\mathrm{Aut}(QR_{48}) = L_2(47)$.
- Sloane (1973): Is there a $(72, 36, 16)$ self-dual code?
- If C is such a $(72, 36, 16)$ code then $\mathrm{Aut}(C)$ has order ≤ 5.

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- Length 40: 10,400 extremal codes with $\mathrm{Aut} = 1$.
- $\mathrm{Aut}(QR_{48}) = L_2(47)$.
- Sloane (1973): Is there a $(72, 36, 16)$ self-dual code?
- If C is such a $(72, 36, 16)$ code then $\mathrm{Aut}(C)$ has order ≤ 5.
- There is no beautiful $(72, 36, 16)$ self-dual code.

The Type of an automorphism

Definition

Let $\sigma \in S_n$ of prime order p. Then σ is of Type (z, f), if σ has z p-cycles and f fixed points. $zp + f = n$.

- Let p be odd, $\sigma = (1, 2, .., p)(p + 1, .., 2p)...((z - 1)p + 1, .., zp)$.
- $\mathbb{F}_2^n = \mathrm{Fix}(\sigma) \perp E(\sigma) \cong \mathbb{F}_2^{z+f} \perp \mathbb{F}_2^{z(p-1)}$ with

$$\mathrm{Fix}(\sigma) = \left\langle \begin{array}{cccccccc} 1\dots1 & 0\dots0 & \dots & 0\dots0 & 0 & 0 & \dots & 0 \\ 0\dots0 & 1\dots1 & \dots & 0\dots0 & 0 & 0 & \dots & 0 \\ 0\dots0 & 0\dots0 & \dots & 1\dots1 & 0 & 0 & \dots & 0 \\ 0\dots0 & 0\dots0 & \dots & 0\dots0 & 1 & 0 & \dots & 0 \\ 0\dots0 & 0\dots0 & \dots & 0\dots0 & 0 & 1 & \dots & 0 \\ \underbrace{0\dots0}_{p} & \underbrace{0\dots0}_{p} & \dots & \underbrace{0\dots0}_{p} & 0 & 0 & \dots & 1 \end{array} \right\rangle$$

$$E(\sigma) = \mathrm{Fix}(\sigma)^{\perp} = \{(x_1, \dots, x_p, x_{p+1}, \dots, x_{2p}, \dots, x_{(z-1)p+1}, \dots, x_{zp}, 0, \dots, 0) \mid x_1 + \dots + x_p = x_{p+1} + \dots + x_{2p} = \dots = x_{(z-1)p+1} + \dots + x_{zp} = 0\}$$

Two self-dual codes of smaller length

- Let $C \leq \mathbb{F}_2^n$ and p an odd prime,
- $\sigma = (1, 2, .., p)(p+1, .., 2p)...((z-1)p+1, .., zp) \in \mathrm{Aut}(C)$.
- Then $C = C \cap \mathrm{Fix}(\sigma) \oplus C \cap E(\sigma) =: \mathrm{Fix}_C(\sigma) \oplus E_C(\sigma)$.

$$\begin{aligned}
\mathrm{Fix}_C(\sigma) &= \{(\underbrace{c_p \dots c_p}_{p}\, \underbrace{c_{2p} \dots c_{2p}}_{p} \dots \underbrace{c_{zp} \dots c_{zp}}_{p}\, c_{zp+1} \dots c_n) \in C\} \cong \\
\pi(\mathrm{Fix}_C(\sigma)) &= \{(c_p c_{2p} \dots c_{zp} c_{zp+1} \dots c_n) \in \mathbb{F}_2^{z+f} \mid c \in \mathrm{Fix}_C(\sigma)\}
\end{aligned}$$

- and $C^\perp = C^\perp \cap \mathrm{Fix}(\sigma) \oplus C^\perp \cap E(\sigma)$.
- $C = C^\perp$ then $\mathrm{Fix}_C(\sigma)$ is self-dual in $\mathrm{Fix}(\sigma)$ and $E_C(\sigma)$ is (Hermitian) self-dual in $E(\sigma)$.

Fact

$\pi(\mathrm{Fix}_C(\sigma))$ is a self-dual code of length $z + f$, in particular

$$\dim(\mathrm{Fix}_C(\sigma)) = \frac{z+f}{2} \text{ and } |\mathrm{Fix}_C(\sigma)| = 2^{(z+f)/2}.$$

Application of Burnside's orbit counting theorem

Theorem (Conway, Pless, 1982)

Let $C = C^\perp \leq \mathbb{F}_2^n$, $\sigma \in \mathrm{Aut}(C)$ of odd prime order p and Type (z, f).

Then $$2^{(z+f)/2} \equiv 2^{n/2} \pmod{p}.$$

Proof: Apply orbit counting:
The number of G-orbits on a finite set M is $\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}_M(g)|$.
Here $G = \langle\sigma\rangle$, $M = C$, $\operatorname{Fix}_C(g) = \operatorname{Fix}_C(\sigma)$ for all $1 \neq g \in G$, and the number of $\langle\sigma\rangle$-orbits on C is $\frac{1}{p}(2^{n/2} + (p-1)2^{(z+f)/2}) \in \mathbb{N}$.

Corollary

$C = C^\perp \leq \mathbb{F}_2^n$, $p > n/2$ an odd prime divisor of $|\mathrm{Aut}(C)|$, then $p \equiv \pm 1 \pmod{8}$.

Here $z = 1$, $f = n - p$, $(z + f)/2 = (n - (p - 1))/2$, so $2^{(p-1)/2}$ is 1 mod p and hence 2 must be a square modulo p.

Application of quadratic forms

Remark

- $C = C^\perp \Rightarrow \mathbf{1} = (1, \ldots, 1) \in C$, since $(c, c) = (c, \mathbf{1})$.
- If C is self-dual then $n = 2\dim(C)$ is even and

$$\mathbf{1} \in C^\perp = C \subset \mathbf{1}^\perp = \{c \in \mathbb{F}_2^n \mid \mathrm{wt}(c) \text{ even } \}.$$

- Self-dual doubly-even codes correspond to totally isotropic subspaces in the quadratic space

$$\mathbf{E_{n-2}} := (\mathbf{1}^\perp/\langle \mathbf{1} \rangle, q), q(c + \langle \mathbf{1} \rangle) = \frac{1}{2}\mathrm{wt}(c) \pmod 2 \in \mathbb{F}_2.$$

- $C = C^\perp \leq \mathbb{F}_2^n$ doubly-even $\Rightarrow n \in 8\mathbb{Z}$.

Theorem (A. Meyer, N. 2009)

Let $C = C^\perp \leq \mathbb{F}_2^n$ doubly-even. Then $\mathrm{Aut}(C) \leq \mathrm{Alt}_n$.

Application of quadratic forms: Some background

- Assume $n \in 8\mathbb{Z}$.
- $\mathbf{E_{n-2}} := (\mathbf{1}^\perp/\langle \mathbf{1}\rangle, q), q(c + \langle \mathbf{1}\rangle) = \frac{1}{2}\,\mathrm{wt}(c) \pmod 2 \in \mathbb{F}_2$. is an $(n-2)$-dimensional quadratic space over $\mathbb{F}_2$.
- There is $X \le \mathbf{E_{n-2}}$ with $X = X^\perp$ and $q(X) = \{0\}$ call such X self-dual isotropic.
- $C = C^\perp \le \mathbb{F}_2^n$, doubly-even, then $X = C/\langle \mathbf{1}\rangle \le \mathbf{E_{n-2}}$ is self-dual isotropic.
- $O(\mathbf{E_{n-2}}) = \{g \in \mathrm{GL}(\mathbf{E_{n-2}}) \mid q(g(x)) = q(x)$ for all $x \in \mathbf{E_{n-2}}\}$ the orthogonal group of $\mathbf{E_{n-2}}$.

Definition

Fix $X_0 \le \mathbf{E_{n-2}}$ self-dual isotropic. $D : O(\mathbf{E_{n-2}}) \to \{1, -1\}$,
$D(g) := (-1)^{\dim(X_0/(X_0 \cap g(X_0)))}$ the Dickson invariant.

Fact $g \in \mathrm{Stab}_{O(\mathbf{E_{n-2}})}(X) \Rightarrow D(g) = 1$.

Application of quadratic forms

$\mathrm{Aut}(C) = \{\sigma \in S_n \mid \sigma(C) = C\}$ is the automorphism group of $C \leq \mathbb{F}_2^n$.

Theorem (A. Meyer, N. 2009)

Let $C = C^\perp \leq \mathbb{F}_2^n$ doubly-even. Then $\mathrm{Aut}(C) \leq \mathrm{Alt}_n$.

- **Proof.** (sketch)
- $\mathbf{E_{n-2}} = (\mathbf{1}^\perp/\langle \mathbf{1} \rangle, q), q(c + \langle \mathbf{1} \rangle) = \frac{1}{2}\,\mathrm{wt}(c) \pmod 2 \in \mathbb{F}_2$.
- $C/\langle \mathbf{1} \rangle$ is a self-dual isotropic subspace $\mathbf{E_{n-2}}$.
- The stabilizer in the orthogonal group of $\mathbf{E_{n-2}}$ of such a space has trivial Dickson invariant.
- $S_n \leq O(\mathbf{E_{n-2}})$, $\mathrm{Aut}(C) = \mathrm{Stab}_{S_n}(C)$.
- The restriction of the Dickson invariant to S_n is the sign.

Application of Representation Theory

G finite group, $\mathbb{F}_2 G = \{\sum_{g \in G} a_g g \mid a_g \in \mathbb{F}_2\}$ group ring.
Then G acts on $\mathbb{F}_2 G \cong \mathbb{F}_2^{|G|}$ by permuting the basis elements.

Theorem (Sloane, Thompson, 1988)

There is a G-invariant self-dual doubly-even code $C \leq \mathbb{F}_2 G$, if and only if $|G| \in 8\mathbb{N}$ and the Sylow 2-subgroups of G are not cyclic.

Theorem (A. Meyer, N., 2009)

Given $G \leq S_n$. Then there is $C = C^\perp \leq \mathbb{F}_2^n$ doubly-even such that $G \leq \mathrm{Aut}(C)$, if and only if

(1) $n \in 8\mathbb{N}$,

(2) all self-dual composition factors of the $\mathbb{F}_2 G$-module $\mathbb{F}_2^n$ occur with even multiplicity, and

(3) $G \leq \mathrm{Alt}_n$.

General theoretical results (Summary)

- Invariant Theory:
  $C = C^{\perp} \leq \mathbb{F}_2^n$ extremal if $d(C) = 4 + 4\lfloor \frac{n}{24} \rfloor$
- Orbit Counting:
  $C = C^{\perp}$, $\sigma \in \mathrm{Aut}(C)$ of odd prime order p and Type (z, f), then $2^{(z+f)/2} \equiv 2^{n/2} \pmod{p}$
- Quadratic Forms:
  $C = C^{\perp}$ doubly even, then $n \in 8\mathbb{Z}$ and $\mathrm{Aut}(C) \leq \mathrm{Alt}_n$.
- Equivariant Witt groups and Representation Theory:
  Characterisation of the permutation groups admitting a self-dual doubly-even invariant code.

$C = C^{\perp} \leq \mathbb{F}_2^{72}$ extremal, $G = \mathrm{Aut}(C)$.

Theorem (Conway, Huffmann, Pless, Bouyuklieva, O'Brien, Willems, Feulner, Borello, Yorgov, N., ..)

Let $C \leq \mathbb{F}_2^{72}$ be an extremal doubly even code,
$G := \mathrm{Aut}(C) := \{\sigma \in S_{72} \mid \sigma(C) = C\}$, $\sigma \in G$ of prime order p.

- If $p = 2$ or $p = 3$ then σ has no fixed points. (B)
- If $p = 5$ or $p = 7$ then σ has 2 fixed points. (CHPB)
- G contains no element of prime order ≥ 7. (BYFN)
- G has no subgroup S_3, D_{10}, $C_3 \times C_3$. (BFN)
- If $p = 2$ then C is a free $\mathbb{F}_2\langle\sigma\rangle$-module. (N)
- G has no subgroup C_{10}, $C_4 \times C_2$, Q_8.
- $G \not\cong \mathrm{Alt}_4$, $G \not\cong D_8$, $G \not\cong C_2 \times C_2 \times C_2$ (BN)
- G contains no element of order 6. (Borello)
- and hence $|G| \leq 5$.
- G contains no element of order 4. (Y)

Existence of an extremal code of length 72 is still open.

The Type of a permutation of prime order

Theoretical results, p odd.

Definition (recall)

Let $\sigma \in S_n$ of prime order p. Then σ is of Type (z, f), if σ has z p-cycles and f fixed points. $zp + f = n$.

Theorem (Conway, Pless) (recall)

Let $C = C^{\perp} \leq \mathbb{F}_2^n$, $\sigma \in \mathrm{Aut}(C)$ of odd prime order p and Type (z, f).

Then $$2^{(z+f)/2} \equiv 2^{n/2} \pmod{p}.$$

Corollary. $n = 72 \Rightarrow p \neq 37, 43, 53, 59, 61, 67.$

Corollary. If $n = 8$ then $p \neq 5$ and $p = 3 \Rightarrow$ Type $(2, 2)$.

$2^4 \not\equiv 2^{(1+3)/2} \pmod{5}$, $2^4 \not\equiv 2^{(1+5)/2} \pmod{3}$.

Computational results, p odd.

BabyTheorem: $n = 8, p = 3$

All doubly even self-dual codes of length 8 that have an automorphism of order 3 are equivalent to h_8.

- $\sigma = (1,2,3)(4,5,6)(7)(8) \in \mathrm{Aut}(C)$
- $e_0 = 1 + \sigma + \sigma^2$, $e_1 = \sigma + \sigma^2$ idempotents in $\mathbb{F}_2\langle\sigma\rangle$
- $C = Ce_0 \perp Ce_1 \leq \mathbb{F}_2^8 e_0 \perp \mathbb{F}_2^8 e_1 \cong \mathbb{F}_2^4 \perp \mathbb{F}_4^2$
- $Ce_0 = \mathrm{Fix}_C(\sigma)$ isomorphic to a self-dual code in $\mathbb{F}_2^4$, so

$$Ce_0 : \left[\begin{array}{cccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

- $Ce_1 = E_C(\sigma) \leq \mathbb{F}_4^2$ Hermitian self-dual, $Ce_1 \cong [1,1]$, so

$$Ce_1 : \left[\begin{array}{cccccccc} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

and hence

$$C \quad : \left[\begin{array}{cccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

Computational results, p odd.

Theorem. (Borello, Feulner, N. 2012, 2013)

Let $C = C^{\perp} \leq \mathbb{F}_2^{72}$, extremal, so $d(C) = 16$.
Then $\mathrm{Aut}(C)$ has no subgroup C_7, $C_3 \times C_3$, D_{10}, S_3.

- **Proof.** for $S_3 = \langle \sigma, \tau \mid \sigma^3, \tau^2, (\sigma\tau)^2 \rangle$
- $\sigma = (1,2,3)(4,5,6)\cdots(67,68,69)(70,71,72)$
- $\tau = (1,4)(2,6)(3,5)\cdots(67,70)(68,72)(69,71)$
- $C \cong \mathrm{Fix}_C(\sigma) \oplus E_C(\sigma)$ with $\mathrm{Fix}_C(\sigma) \cong (1,1,1) \otimes \mathcal{G}_{24}$ and
- $E_C(\sigma) \leq \mathbb{F}_4^{24}$ Hermitian self-dual, minimum distance ≥ 8.
- τ acts on $E_C(\sigma)$ by $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{23}, \epsilon_{24})^{\tau} = (\overline{\epsilon_2}, \overline{\epsilon_1}, \ldots, \overline{\epsilon_{24}}, \overline{\epsilon_{23}})$
- $\mathrm{Fix}_{E_C(\sigma)}(\tau) = \{\epsilon := (\overline{\epsilon_2}, \epsilon_2 \ldots, \overline{\epsilon_{24}}, \epsilon_{24}) \in E_C(\sigma)\}$
- $\cong \pi(\mathrm{Fix}_{E_C(\sigma)}(\tau)) = \{(\epsilon_2, \ldots, \epsilon_{24}) \mid \epsilon \in \mathrm{Fix}_{E_C(\sigma)}(\tau)\} \leq \mathbb{F}_4^{12}$
- is trace Hermitian self-dual additive code, minimum distance ≥ 4.
- There are 195,520 such codes.
- $\langle \mathrm{Fix}_{E_C(\sigma)}(\tau) \rangle_{\mathbb{F}_4} = E_C(\sigma)$.
- No $E_C(\sigma)$ has minimum distance ≥ 8.

$C = C^{\perp} \leq \mathbb{F}_2^{72}$, doubly-even.

Theorem. (A. Meyer, N.) (recall)

Let $C = C^{\perp} \leq \mathbb{F}_2^n$ doubly-even. Then $\mathrm{Aut}(C) \leq \mathrm{Alt}_n$.

Corollary. $\mathrm{Aut}(C)$ **has no element of order** 8**.**

$\sigma \in \mathrm{Aut}(C)$ of order 8. Then

$$\sigma = (1, 2, \ldots, 8)(9, \ldots, 16) \ldots (65, \ldots, 72)$$

since σ^4 has no fixed points. So sign$(\sigma) = -1$, a contradiction.

(This corollary was known before and is already implied by the Sloane-Thompson Theorem.)

$C = C^{\perp} \leq \mathbb{F}_2^{72}$, doubly even, extremal, so $d(C) = 16$

Theorem. (N. 2012)

Let $\tau \in \mathrm{Aut}(C)$ of order 2. Then C is a free $\mathbb{F}_2\langle\tau\rangle$-module.

- Let $R = \mathbb{F}_2\langle\tau\rangle$ the free $\mathbb{F}_2\langle\tau\rangle$-module, $S = \mathbb{F}_2$ the simple one.
- Then $C = R^a \oplus S^b$ with $2a + b = 36$.
- $F := \mathrm{Fix}_C(\tau) = \{c \in C \mid c\tau = c\} \cong S^{a+b}$, $C(1-\tau) \cong S^a$.
- $\tau = (1,2)(3,4)\ldots(71,72)$.
- $F \cong \pi(F), \pi(c) = (c_2, c_4, c_6, \ldots, c_{72}) \in \mathbb{F}_2^{36}$.
- **Fact**: $\pi(F) = \pi(C(1-\tau))^{\perp} \supseteq D = D^{\perp} \supseteq \pi(C(1-\tau))$.
- $d(F) \geq d(C) = 16$, so $d(D) \geq d(\pi(F)) \geq 8$.
- There are 41 such extremal self-dual codes D (Gaborit etal).
- No code D has a proper overcode with minimum distance ≥ 8.
- This can also be seen a priori considering weight enumerators.
- So $\pi(F) = D$ and hence $a + b = 18$, so $a = 18$, $b = 0$.

Theorem: C is a free $\mathbb{F}_2\langle\tau\rangle$-module.

Corollary. $\mathrm{Aut}(C)$ has no element of order 8.

$g \in \mathrm{Aut}(C)$ of order 8. Then C is a free $\mathbb{F}_2\langle g^4\rangle$-module, hence also a free $\mathbb{F}_2\langle g\rangle$-module of rank $\dim(C)/8 = 36/8 = 9/2$ a contradiction.

Corollary. $\mathrm{Aut}(C)$ has no subgroup Q_8.

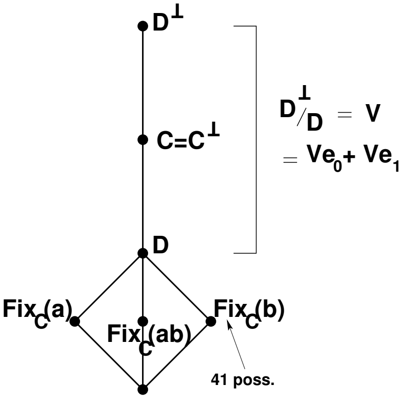
Use a theorem by J. Carlson: If M is an $\mathbb{F}_2 Q_8$-module such that the restriction of M to the center of Q_8 is free, then M is free.

Corollary. $\mathrm{Aut}(C)$ has no subgroup $U \cong C_2 \times C_4$, C_8 or C_{10}.

- Let $\tau \in U$ of order 2, $F = \mathrm{Fix}_C(\tau) \cong \pi(F) = D = D^\perp \leq \mathbb{F}_2^{36}$.
- Then D is one of the 41 extremal codes classified by Gaborit etal.
- $U/\langle\tau\rangle \cong C_4$ or C_5 acts on D.
- None of the 41 extremal codes D has a fixed point free automorphism of order 4 or an automorphism of order 5 with exactly one fixed point.

$\mathrm{Alt}_4 = \langle a, b, \sigma\rangle \trianglerighteq \langle a, b\rangle = V_4$, (Borello, N. 2013)

Computational results: No $\mathrm{Alt}_4 \leq \mathrm{Aut}(C)$.



3 possibilities for D
$\dim(D^\perp/D) = 20, 20, 22$.
$C/D \leq D^\perp/D$
maximal isotropic subspace.
V_4 acts trivially on $D^\perp/D =: V$.
$V = Ve_0 \oplus Ve_1$
is an $\mathbb{F}_2\langle\sigma\rangle$-module.
Unique possibility for Ce_0.
$Ce_1 \leq Ve_1$ Hermitian
maximal singular $\mathbb{F}_4$-subspace.
Compute
all these subspaces as orbit
under the unitary group of Ve_1.
No extremal code is found.

$\tau \in \mathrm{Aut}(C)$ order 2

Situation

$C = C^\perp \leq \mathbb{F}_2^{24m}$, extremal, i.e. $d(C) = 4m+4$, $\tau \in \mathrm{Aut}(C)$ of order 2.

- Bouyuklieva: $\tau \sim (1,2)\cdots(24m-1,24m)$ (Type $(12m,0)$) unless $m=5$ where Type $(48,24)$ might be possible.
- Assume $\tau \sim (1,2)\cdots(24m-1,24m)$.
- $D' := \pi(\mathrm{Fix}_C(\tau)) \leq \mathbb{F}_2^{12m}$ is the dual of some self-orthogonal code
$$(D')^\perp \subseteq D'$$
and $d(D') \geq 2m+2$.
- C is a free $\mathbb{F}_2\langle\tau\rangle$-module, if and only if D' is self-dual.

Theorem (Borello, N. 2015)

If $D' \neq (D')^\perp$, then $d(D') \leq 4\lfloor \frac{m}{2} \rfloor + 2$.

Theoretical results, $p = 2$.

Theorem (Borello, N. 2015)

Let $m \geq 3$ be odd and $C = C^\perp$ an extremal doubly-even binary code of length $24m$.

- If $\tau \in \mathrm{Aut}(C)$ is of order 2 and fixed point free then C is a free $\mathbb{F}_2\langle\tau\rangle$-module.
- If 8 divides $|\mathrm{Aut}(C)|$, then the Sylow 2-subgroups of $\mathrm{Aut}(C)$ are isomorphic to $C_2 \times C_2 \times C_2$, $C_2 \times C_4$, or D_8.

Conclusion

Search for extremal codes with automorphisms provides a nice application for

- Classical theories in particular
- Quadratic Forms:
  $C = C^{\perp}$ doubly even, then $n \in 8\mathbb{Z}$ and $\mathrm{Aut}(C) \leq \mathrm{Alt}_n$.
- which provides a characterisation of the permutation groups admitting a self-dual doubly-even invariant code.
- Modular Representation Theory and Invariant Theory
  $n = 24m$, $d(C) = 4m + 4$, $\tau \in \mathrm{Aut}(C)$ of Type $(12m, 0)$.
  If m is odd then C is a free $\mathbb{F}_2\langle\tau\rangle$-module.

They are also the motivation for explicit computations with a practical and detailed use of the structure of the automorphism group.