What is a block?

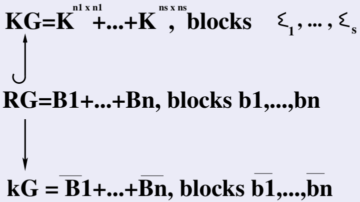
Definition

Let A be a ring. Then a block b in A is a central primitive idempotent.
($b^2 = b \in Z(A)$ not a sum)
Also the indecomposable ring $B := bA$ is called a block.
Then (under certain finiteness assumption)

$$A = b_1 A \oplus \ldots \oplus b_n A = B_1 \oplus \ldots \oplus B_n$$

Blocks of finite group rings.

- G finite group
- R local integral domain
- $K = frac(R)$ of char. 0
- $R/\pi R = k$ field of characteristic p
- K, k big enough.



b block in RG is sum $b = \sum_{t \in T(b)} \epsilon_t$.

Central idempotents

- $\Delta_t : G \to \mathrm{GL}_{n_t}(K)$ $(t = 1, \ldots, s)$ the irreducible representations
- $\chi_t(g) := trace(\Delta_t(g))$, so $n_t = \chi_t(1)$, irreducible character.

Blocks of RG.

$$\epsilon_t = \frac{\chi_t(1)}{|G|} \sum_{g \in G} \chi_t(g) g^{-1} \in KG.$$

The blocks of RG are unique sums of the ϵ_t:

$$b = \sum_{t \in T(b)} \epsilon_t = \sum_{g \in G} (\sum_{t \in T(b)} \frac{\chi_t(1)\chi_t(g)}{|G|}) g^{-1} \in RG$$

if and only if $\sum_{t \in T(b)} \chi_t(1)\chi_t(g) \in |G|R$ for all $g \in G$.

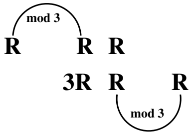
The character χ_t belongs to the block b if $t \in T(b)$, so $b\epsilon_t = \epsilon_t$.

$G = S_4,\ p = 3,\ R = \mathbb{Z}_3,\ K = \mathbb{Q}_3,\ k = \mathbb{F}_3,\ |G| = 2^3 3.$

$C_{S_4}(g)$	S_4	V_4	D_8	C_4	C_3
conj.classes	$(.)$	$(..)$	$(..)(..)$	$(....)$	$(...)$
1	1	1	1	1	1
2	1	-1	1	-1	1
3	2	0	2	0	-1
4	3	1	-1	-1	0
5	3	-1	-1	1	0

- $b_1 = \epsilon_4 = \frac{3}{24}(3(.)^+ + (..)^+ - (..)(..)^+ - (....)^+) \in RG$
- $b_2 = \epsilon_5 = \frac{3}{24}(3(.)^+ - (..)^+ - (..)(..)^+ + (....)^+) \in RG$
- $b = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1}{4}((.)^+ + (..)(..)^+) = \frac{1}{4}\sum_{g \in V_4} g \in RG$

$$\mathbb{Z}_3 S_4 = B \oplus B_1 \oplus B_2 = \mathbb{Z}_3 S_3 \oplus \mathbb{Z}_3^{3\times 3} \oplus \mathbb{Z}_3^{3\times 3}$$



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The defect group of a block

Definition.

Let $b = \sum_{g \in G} a_g g \in RG$ be a block. Then

$$D(b) = \max\{\mathrm{Syl}_p(C_G(g)) \mid a_g \text{ mod } \pi \neq 0\}$$

is called the defect group of b (unique up to conjugacy).

$G = S_4$ above, then $D(b_4) = D(b_5) = 1$, $D(b) = C_3$

Numerical properties

$|G| = p^a m$, $|D(b)| = p^{d(b)}$, $d := d(b)$ is the defect of b.

- $d = 0 \Leftrightarrow |T(b)| = 1$. Then $B = R^{n \times n}$ is a matrix ring.
- $\gcd\{\chi_t(1) \mid t \in T(b)\} = p^{a-d} y$ with $p \nmid y$.
- If $\chi_t(1) = p^{a-d+h(t)} y'$ with $p \nmid y'$ then $h(t)$ is the height of χ_t.
- The principal block b is the one containing the trivial character. Its defect groups are the Sylow-p-subgroups of G.

Brauer's first main theorem

Let $B = bRG$ be a block and $D := D(b)$ its defect group. Let $N := N_G(D)$ be the normalizer in G of D.

The Brauer correspondence

The blocks of RG with defect group D are in bijection with the blocks of RN with defect group D.

The Donovan conjecture

If k is algebraically closed, then there are only finitely many Morita equivalence classes of blocks with a given defect group.

Donovan's conjecture for special cases

Donovan's conjecture holds for blocks with cyclic defect groups.
It also holds for tame blocks.
Scopes proved the Donovan conjecture for symmetric groups.
This allows to deduce it for the alternating groups.

The group structure of defect groups

Cyclic defect group

Let b be a block of RG such that $D(b)$ is cyclic. Then b is a Brauer tree algebra and its structure is well understood.
Brauer, Dade, Green, Thompson, Plesken
There is a similar (slightly weaker) result for tame blocks those with dihedral, semidihedral or quaternion defect group.
Erdmann

Abelian defect group

- Brauer's height zero conjecture:
  All characters in b have height 0, if and only if $D(b)$ is abelian.
- Broué's perfect isometry conjecture:
  If $D(b)$ is abelian then b and its Brauer correspondent are perfectly isometric.
  See http://www.maths.bris.ac.uk/∼majcr/adgc/adgc.html for results on this conjecture.

Exercises.

Blocks from character tables.

- Compute the blocks of A_4 for $p = 3$.
- Compute the blocks of S_5 for $p = 2, 3, 5$.

$C_{S_5}(g)$	S_5	$S_3 \times C_2$	D_8	C_6	C_4	C_5	C_6
conj.classes	$(.)$	$(..)$	$(..)(..)$	$(...)$	$(....)$	$(.....)$	$(..)(...)$
1	1	1	1	1	1	1	1
2	1	-1	1	1	-1	1	-1
3	4	2	0	1	0	-1	-1
4	4	-2	0	1	0	-1	1
5	5	1	1	-1	-1	0	1
6	5	-1	1	-1	1	0	-1
7	6	0	-2	0	0	1	0