An extremal even unimodular lattice of dimension 72

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- The Hamming weight of a codeword $c \in C$ is $\mathrm{wt}(c) := |\{i \mid c_i \neq 0\}|$.
- C is called doubly-even if $\mathrm{wt}(c) \in 4\mathbb{Z}$ for all $c \in C$.
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- The minimum distance $d(C) := \min\{\mathrm{wt}(c) \mid 0 \neq c \in C\}$.
- The weight enumerator of C is $p_C := \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)} \in \mathbb{C}[x, y]_n$.

The minimum distance measures the error correcting quality of a self-dual code.

Self-dual codes

Remark

- The all-one vector $\mathbf{1}$ lies in the dual of every even code since $\mathrm{wt}(c) \equiv_2 (c, c) \equiv_2 (c, \mathbf{1})$.
- $C = C^{\perp} \leq \mathbb{F}_2^n$ then $n = 2\dim(C)$.
- Self-dual doubly-even codes correspond to totally isotropic subspaces in the quadratic space $\mathbf{1}^{\perp}/\langle \mathbf{1} \rangle$.

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The extended Hamming code

$$h_8 : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

has $p_{h_8}(x, y) = x^8 + 14x^4y^4 + y^8$ and is the unique doubly-even self-dual code of length 8.

Extremal codes

The binary Golay code $\mathcal{G}_{24}$ is the unique doubly-even self-dual code of length 24 with minimum distance ≥ 8.

$$p_{\mathcal{G}_{24}} = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24}$$

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Theorem (Gleason, ICM 1970)

Let $C = C^{\perp} \leq \mathbb{F}_2^n$ be doubly even. Then

- $n \in 8\mathbb{Z}$
- $p_C \in \mathbb{C}[p_{h_8}, p_{\mathcal{G}_{24}}]$
- $d(C) \leq 4 + 4\lfloor \frac{n}{24} \rfloor$

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length	8	16	24	32	48	72	80
$d(C)$	4	4	8	8	12	16	16
extremal codes	h_8	$h_8 \perp h_8, d_{16}^+$	$\mathcal{G}_{24}$	5	QR_{48}	?	≥ 5

Turyn's construction of the Golay code

Construction of Golay code

Choose two copies C and D of h_8 such that

$$C \cap D = \langle \mathbf{1} \rangle, \ C + D = \mathbf{1}^{\perp} \leq \mathbb{F}_2^8$$

$$\mathcal{G}_{24} := \{(c + d_1, c + d_2, c + d_3) \mid c \in C, d_i \in D, d_1 + d_2 + d_3 \in \langle \mathbf{1} \rangle\}$$

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Proof: (a) unique expression if c represents classes in $h_8/\langle \mathbf{1} \rangle$, so

$$|\mathcal{G}_{24}| = 2^3 \cdot 2^4 \cdot 2^4 \cdot 2 = 2^{12}$$

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(b) Follows since C and D are doubly-even, so generators have weight divisible by 4.

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Turyn applied to Golay

will not yield an extremal code of length 72. Such an extremal code has no automorphism of order 2 which has fixed points.

A generalization of Turyn's construction.

Theorem

Let $C = C^\perp, D = D^\perp \leq \mathbb{F}_q^n$ and $X \leq \mathbb{F}_q^m$ such that $X \cap X^\perp = \{0\}$. Then

$$\mathcal{T}(C, D, X) := C \otimes X + D \otimes X^\perp \leq \mathbb{F}_q^{nm}$$

is a self-dual code, which is doubly-even, if C and D are doubly-even.

Proof: Let $c, c' \in C$, $d, d' \in D$, $x, x' \in X$ and $y, y' \in X^\perp$. Then

$$\begin{array}{ll} (c \otimes x, c' \otimes x') = 0 & \text{since } C \subseteq C^\perp \\ (d \otimes y, d' \otimes y') = 0 & \text{since } D \subseteq D^\perp \\ (c \otimes x, d \otimes y) = 0 & \text{since } x \in X, y \in X^\perp \end{array}$$

so $\mathcal{T} \subset \mathcal{T}^\perp$. Moreover

$$\dim(\mathcal{T}) = \dim(C \otimes X) + \dim(D \otimes X^\perp) - \dim(C \otimes X \cap D \otimes X^\perp) = nm/2 - 0$$

since $X \cap X^\perp = \{0\}$.

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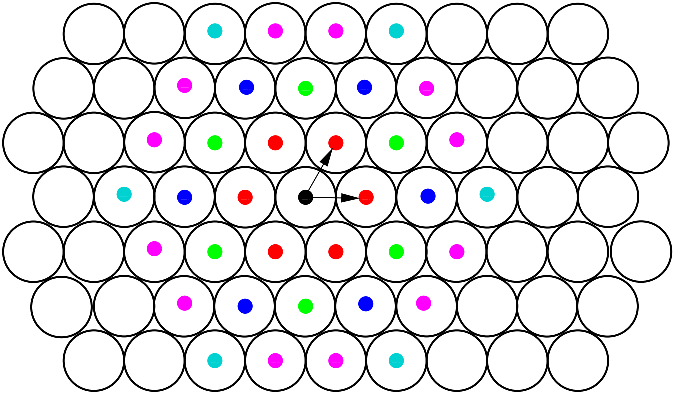
Turyn's example.

$X = \langle (1,1,1) \rangle$, $C \cong D \cong h_8$ such that $C \cap D = \langle \mathbf{1} \rangle$ then
$\mathcal{T}(C, D, X) \cong \mathcal{G}_{24}$.

Example: Bachoc/Nebe.

$C \cong D \cong h_8$, $C \cap D = \langle \mathbf{1} \rangle$.
$X \cong X^{\perp}$ a $[10, 5, 4]$-code, such that $X \cap X^{\perp} = \langle \mathbf{1} \rangle$. Then $\mathcal{T}(C, D, X)$ is a self-orthogonal $[80, 39, 16]$-code contained in a unique extremal doubly-even self-dual code.

Lattices and sphere packings



Hexagonal Circle Packing

$$\theta = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \ldots.$$

Dense lattice sphere packings

- Classical problem to find densest sphere packings:
- Dimension 2: Lagrange (lattices), Fejes Tóth (general)
- Dimension 3: Kepler conjecture, proven by T.C. Hales (1998)
- Dimension ≥ 4: open
- Densest lattice sphere packings:
- Voronoi algorithm ($\sim$1900) all locally densest lattices.
- Densest lattices known in dimension 1,2,3,4,5, Korkine-Zolotareff (1872) 6,7,8 Blichfeldt (1935) and 24 Cohn, Kumar (2003).
- Density of lattice measures error correcting quality.

Even unimodular lattices

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The sphere packing density of an even unimodular lattice is proportional to its minimum.

Lattices and codes

Construction A

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$$L_C := \{\sum_{i=1}^{n} \frac{a_i}{2} e_i \mid (\overline{a}_1, \ldots, \overline{a}_n) \in C\} \subset \mathbb{R}^n$$

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Duality

- $L_C^{\#} = L_{C^\perp}$
- L_C is even if C is doubly-even
- L_C is even unimodular, if C is self-dual and doubly-even.

$L_{h_8} = E_8$ the unique even unimodular lattice of dimension 8.

The Leech lattice and the Golay code

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- Then L is an even unimodular lattice and $\{\pm e_1, \ldots, \pm e_{24}\} = \{v \in L \mid Q(v) = 1\}$.

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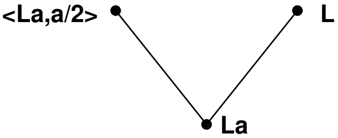
- Then L is an even unimodular lattice and $\{\pm e_1, \ldots, \pm e_{24}\} = \{v \in L \mid Q(v) = 1\}$.
- Let $a := \frac{3}{2}e_1 + \frac{1}{2}e_2 + \ldots + \frac{1}{2}e_{24}$.
- Then $Q(\frac{1}{2}a) = 2$.
- Let $L_a := \{v \in L \mid (v, a) \in 2\mathbb{Z}\}$ and
- $\Lambda_{24} := L^{(a)} := \langle \frac{1}{2}a, L_a \rangle$. Then $\min(\Lambda_{24}) = 2$.



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  where $a_k = |\{\ell \in L \mid Q(\ell) = k\}|$.
- θ_L defines a holomorphic function on the upper half plane by substituting $q := \exp(2\pi i z)$.
- Then θ_L is a modular form of weight $\frac{n}{2}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.
- n is a multiple of 8.

Theta-series of lattices

Let (L, Q) be an even unimodular lattice of dimension n so a regular positive definite integral quadratic form $Q : L \to \mathbb{Z}$.

- The theta series of L is

$$\theta_L = \sum_{\ell \in L} q^{Q(\ell)} = 1 + \sum_{k=\min(L)}^{\infty} a_k q^k$$

  where $a_k = |\{\ell \in L \mid Q(\ell) = k\}|$.
- θ_L defines a holomorphic function on the upper half plane by substituting $q := \exp(2\pi i z)$.
- Then θ_L is a modular form of weight $\frac{n}{2}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.
- n is a multiple of 8.
- $\theta_L \in \mathcal{M}_{\frac{n}{2}}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, \Delta]_{\frac{n}{2}}$ where $E_4 := \theta_{E_8} = 1 + 240q + \ldots$ is the normalized Eisenstein series of weight 4 and

$$\Delta = q - 24q^2 + 252q^3 - 1472q^4 + \ldots \text{ of weight 12}$$

Extremal modular forms

Basis of $\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$:

$$\begin{array}{lcccccc}
E_4^k = & & 1+ & 240kq+ & *q^2+ & \dots & \\
E_4^{k-3}\Delta = & & & q+ & *q^2+ & \dots & \\
E_4^{k-6}\Delta^2 = & & & & q^2+ & \dots & \\
\vdots & & & & & & \\
E_4^{k-3m_k}\Delta^{m_k} = & & & \dots & & q^{m_k}+ & \dots
\end{array}$$

where $m_k = \lfloor \frac{n}{24} \rfloor = \lfloor \frac{k}{3} \rfloor$.

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Definition

This space contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \ldots + 0q^{m_k} + a(f^{(k)})q^{m_k+1} + b(f^{(k)})q^{m_k+2} + \ldots$$

$f^{(k)}$ is called the extremal modular form of weight $4k$.

$f^{(1)} = 1 + 240q + \ldots = \theta_{E_8}$, $f^{(2)} = 1 + 480q + \ldots = \theta_{E_8}^2$,
$f^{(3)} = 1 + 196,560q^2 + \ldots = \theta_{\Lambda_{24}}$,
$f^{(6)} = 1 + 52,416,000q^3 + \ldots = \theta_{P_{48p}} = \theta_{P_{48q}} = \theta_{P_{48n}}$,
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Extremal even unimodular lattices

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Let L be an n-dimensional even unimodular lattice. Then

$$\min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor = 1 + m_{n/8}.$$

Lattices achieving this bound are called extremal.

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Extremal even unimodular lattices L$\leq \mathbb{R}^n$

n	8	16	24	32	40	48	72	80
min(L)	2	2	4	4	4	6	8	8
number of extremal lattices	1	2	1	$\geq 10^7$	$\geq 10^{51}$	≥ 3	≥ 1	≥ 4

Extremal even unimodular lattices

Theorem (Siegel)

$a(f^{(k)}) > 0$ for all k and $b(f^{(k)}) < 0$ for large k ($k \geq 5200$).

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- The density of the associated sphere packing realises a local maximum of the density function on the space of all $24m$-dimensional lattices.

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- The 196560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realise the maximal kissing number in dimension 24.
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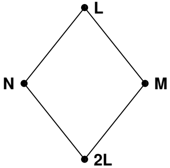
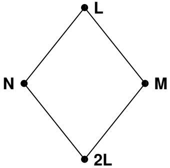
- For $m = 2, 3$ these lattices are the densest known lattices and realise the maximal known kissing number.



Turyn's construction

- Let (L, Q) be an even unimodular lattice of dimension n.
- Choose sublattices $M, N \leq L$ such that $M + N = L$, $M \cap N = 2L$, and $(M, \frac{1}{2}Q)$, $(N, \frac{1}{2}Q)$ even unimodular.
- Such a pair (M, N) is called a polarisation of L.



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- Such a pair (M, N) is called a polarisation of L.
- For $k \in \mathbb{N}$ let $\mathcal{L}(M, N, k) :=$

  $$\{(m+x_1, \ldots, m+x_k) \in \perp^k L \mid m \in M, x_i \in N, x_1 + \ldots + x_k \in 2L\}.$$

- Define $\tilde{Q} : \mathcal{L}(M, N, k) \to \mathbb{Z}$,

  $$\tilde{Q}(y_1, \ldots, y_k) := \frac{1}{2}(Q(y_1) + \ldots + Q(y_k)).$$

- $(\mathcal{L}(M, N, k), \tilde{Q})$ is an even unimodular lattice of dimension nk.

Obtaining Leech from E_8

Theorem (Lepowsky, Meurman; Tits)

Let $(L, Q) \cong E_8$ be the unique even unimodular lattice of dimension 8. Then for any polarisation (M, N) of E_8 the lattice $\mathcal{L}(M, N, 3)$ has minimum ≥ 2.

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Only one $y_i \neq 0$ then $y_i \in 2L$ and $\tilde{Q}(y) \geq 2$.

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(m+a,m+b,m+c) in L(M,N,3)

2L⊥2L⊥2L

m in M

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a+b+c in 2L

$$d := \min(L, Q) = \min(M, \tfrac{1}{2}Q) = \min(N, \tfrac{1}{2}Q)$$

Then $\lceil \frac{3d}{2} \rceil \leq \min(\mathcal{L}(M, N, 3)) \leq 2d.$

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$(a, 0, 0)$ $a = 2\ell \in 2L$ with $\frac{1}{2}Q(2\ell) = 2\mathbb{Q}(\ell) \ge 2d$.

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72-dimensional lattices from Leech (Griess)

If $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$ then $3 \leq \min(\mathcal{L}(M, N, 3)) \leq 4$.

The vectors v with $Q(v) = 3$

Assume that $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$

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- All 4095 non-zero classes of $M/2L$ are represented by vectors m with $Q(m) = 4$.
- For $m \in M$ let $N_m := \{a \in N \mid (a, m) \in 2\mathbb{Z}\}$ and $N^{(m)} := \langle N_m, m \rangle$.
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Enumerate short vectors in $\mathfrak{L}(M, N, 3)$

For all 4095 nonzero classes $m + 2L \in M/2L$ and all 24^2 pairs (y_1, y_2) of roots in $N^{(m)}$ check if $\langle 2L, m + y_1 + y_2 \rangle$ has minimum ≥ 3.

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At most $4095 \cdot 8 \cdot 16 = 524,160$ lattices of dimension 24.

Stehlé, Watkins proof of extremality

Theorem (Stehlé, Watkins (2010))

Let L be an even unimodular lattice of dimension 72 with $\min(L) \geq 3$. Then L is extremal, if and only if it contains at least $6, 218, 175, 600$ vectors v with $Q(v) = 4$.

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$$\theta_L = 1 + a_3 q^3 + a_4 q^4 + \ldots = f^{(9)} + a_3 \Delta^3.$$

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So $a_4 = 6,218,175,600 - 72a_3 \geq 6,218,175,600$ if and only if $a_3 = 0$.

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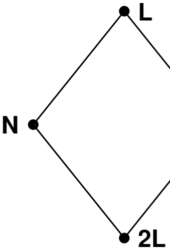
Remark

A similar proof works in all jump dimensions $24k$ (extremal minimum = $k+1$) for lattices of minimum $\geq k$.
For dimensions $24k + 8$ and lattices of minimum $\geq k$ one needs to count vectors v with $Q(v) = k + 2$.

How to find polarisations



- Rough estimate shows that there are about 10^{10} orbits of polarisations (M, N) of the Leech lattice such that $(M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q) \cong \Lambda_{24}$.

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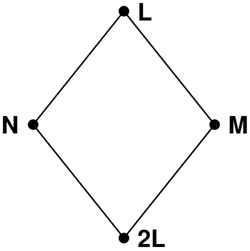

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- Griess proposes $M = (f-1)L$, $N = (g-1)L$, $g^2 = f^2 = -1$, (fg) fixed point free odd order: No extremal lattice.
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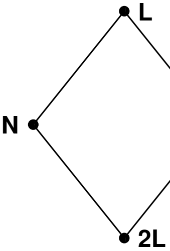

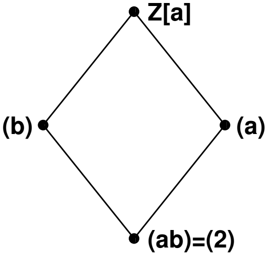

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- Bachoc and Nebe (1995) used Hermitian polarisations to construct extremal 80-dimensional lattices.
- $\alpha, \beta \in \mathrm{End}(L)$ such that $(\alpha x, y) = (x, \beta y)$ and $\alpha\beta = 2$.
- $M := \alpha L$, $N := \beta L$.

Hermitian polarisations

Let $\alpha \in \mathrm{End}(L)$ such that

- $\alpha^2 - \alpha + 2 = 0$ ($\mathbb{Z}[\alpha] =$ integers in $\mathbb{Q}[\sqrt{-7}]$).
- $(\alpha x, y) = (x, \beta y)$ where $\beta = 1 - \alpha = \overline{\alpha}$.

Then $M := \alpha L$, $N := \beta L$ defines a polarisation of L such that $(L, Q) \cong (M, \frac{1}{2}Q) \cong (N, \frac{1}{2}Q)$.

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Remark

$\mathcal{L}(\alpha L, \beta L, 3) = L \otimes_{\mathbb{Z}[\alpha]} P_b$ where

$$P_b = \langle (\beta, \beta, 0), (0, \beta, \beta), (\alpha, \alpha, \alpha) \rangle \mathbb{Z}[\alpha]^3$$

with the half the standard Hermitian form

$$h : P_b \times P_b \to \mathbb{Z}[\alpha], h((a_1, a_2, a_3), (b_1, b_2, b_3)) = \frac{1}{2} \sum_{i=1}^{3} a_i \overline{b_i}.$$

P_b is Hermitian unimodular and $\mathrm{Aut}_{\mathbb{Z}[\alpha]}(P_b) \cong \pm \mathrm{PSL}_2(7)$. So $\mathrm{Aut}(\mathcal{L}(\alpha L, \beta L, 3)) \geq \mathrm{Aut}_{\mathbb{Z}[\alpha]}(L) \times \mathrm{PSL}_2(7)$.

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures of the Leech lattice.

	group	order	
1	$\mathrm{SL}_2(25)$	$2^4 3 \cdot 5^2 13$	
2	$2.A_6 \times D_8$	$2^7 3^2 5$	
3	$\mathrm{SL}_2(13).2$	$2^4 3 \cdot 7 \cdot 13$	
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	
6	soluble	$2^9 3^3$	
7	$\pm \mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2^4 3^2 7^2$	
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2^7 3^3 5 \cdot 7^2$	
9	$2.J_2.2$	$2^9 3^3 5^2 7$	

Hermitian structures of the Leech lattice

Theorem (M. Hentschel, 2009)

There are exactly nine $\mathbb{Z}[\alpha]$-structures of the Leech lattice.

	group	order	# Q(v) = 3
1	$\mathrm{SL}_2(25)$	$2^4 3 \cdot 5^2 13$	0
2	$2.A_6 \times D_8$	$2^7 3^2 5$	$2 \cdot 20,160$
3	$\mathrm{SL}_2(13).2$	$2^4 3 \cdot 7 \cdot 13$	$2 \cdot 52,416$
4	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	$2 \cdot 100,800$
5	$(\mathrm{SL}_2(5) \times A_5).2$	$2^6 3^2 5^2$	$2 \cdot 100,800$
6	soluble	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm \mathrm{PSL}_2(7) \times (C_7 : C_3)$	$2^4 3^2 7^2$	$2 \cdot 306,432$
8	$\mathrm{PSL}_2(7) \times 2.A_7$	$2^7 3^3 5 \cdot 7^2$	$2 \cdot 504,000$
9	$2.J_2.2$	$2^9 3^3 5^2 7$	$2 \cdot 1,209,600$

The extremal 72-dimensional lattice Γ

Main result

- Γ is an extremal even unimodular lattice of dimension 72.
- $\mathrm{Aut}(\Gamma)$ contains $\mathcal{U} := (\mathrm{PSL}_2(7) \times \mathrm{SL}_2(25)) : 2$.
- $\mathcal{U}$ is an absolutely irreducible subgroup of $\mathrm{GL}_{72}(\mathbb{Q})$.
- All $\mathcal{U}$-invariant lattices are similar to Γ.
- Γ realises the densest known sphere packing
- and maximal known kissing number in dimension 72.
- Structure of Γ can be used for decoding: Annika Meyer

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Remark (Masaaki Harada, 2010)

The lattice Γ gives extremal doubly-even codes over $\mathbb{Z}/4k\mathbb{Z}$ of length 72 for $k \geq 2$.
Certain odd neighbors of Γ yield optimal odd unimodular lattices.

A generalisation of Turyn's construction for lattices.

Theorem (Quebbemann)

Let $(L,Q) \leq \mathbb{R}^n$ be an even lattice, p a prime not dividing $\det(L)$.
Then L has a polarisation mod p:
$L = M + N,\ M \cap N = pL$ and $(M, \frac{1}{p}Q), (N, \frac{1}{p}Q)$ even.
Let $X \leq \mathbb{F}_p^m$. Then

$$\mathcal{L}(M,N,X) := \langle (x_1 a, \ldots, x_m a), (y_1 b, \ldots, y_m b) \mid \\ a \in M, b \in N, (\overline{x}_1, \ldots, \overline{x}_m) \in X, (\overline{y}_1, \ldots, \overline{y}_m) \in X^\perp \rangle$$

is an even lattice of dimension nm of determinant $\det(L)^m$.

Examples.

- $X = \langle (1,1) \rangle$, $p = 3$, $L = \Lambda_{24}$, $M = \alpha L$, $N = (1-\alpha)L$ with $\alpha = (1+\sqrt{-11})/2$ such that $\mathrm{Aut}_\alpha(\Lambda_{24}) \cong \mathrm{SL}_2(13)$ yields extremal 48-dimensional lattice P_{48n}.
- $L \cong E_8$, $X = [10,5,4]$-code, $p = 2$ (+neighbor): Two 80-dimensional extremal lattices (Bachoc, Nebe 1995).