

BRANCHING GRAPHS FOR FINITE UNITARY GROUPS IN NON-DEFINING CHARACTERISTIC

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ABSTRACT. We show that the modular branching rule (in the sense of Harish-Chandra) on unipotent modules for finite unitary groups is piecewise described by particular connected components of the crystal graph of well-chosen Fock spaces, under favourable conditions. Besides, we give the combinatorial formula to pass from one to the other in the case of modules arising from cuspidal modules of defect 0. This partly proves a recent conjecture of Jacon and the authors.

1. INTRODUCTION

In [25], Nicolas Jacon and the authors have presented several conjectures about the distribution of the unipotent modules for finite unitary groups based on the concept of weak Harish-Chandra series. The present paper is a sequel to that work, complementing it in several ways. In particular we take some steps towards proving the main conjecture stated there, namely [25, Conjecture 5.7].

Our first objective is the description of the branching graph of a series of Ariki-Koike algebras $\mathcal{H}_{k,d,n}$ over fields k of positive characteristic ℓ , where n varies and where k , d and the parameters are fixed. Ariki has shown [4, Theorem 6.1] that this branching graph is equal to the crystal graph of a Fock space representation of a quantum algebra of affine type A . Ariki's result uses a version of the Fock space defined in [31] leading to a labelling of the vertices of the crystal graph by Kleshchev multipartitions. To apply this result to the conjectures of [25], we need Uglov's realization of the crystal graph. In Section 2 of our paper we review Ariki's result and discuss the relation between the crystal graphs arising from either Kleshchev's or Uglov's realization. We also comment on the connection to canonical basic sets, thus giving a further motivation for the preference of Uglov's version.

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In Section 3 we first recall the definition of weakly cuspidal pairs from [25] and the corresponding endomorphism algebras. The main statement here is Proposition 3.1, which extends a result by Howlett and Lehrer; it essentially shows that the covariant Hom-functors commute with Harish-Chandra restriction and the restriction in the endomorphism algebras, respectively.

These results are applied in Section 4 to the Harish-Chandra branching graphs of the unitary and symplectic groups and the orthogonal groups of odd degree. Such a graph is defined with respect to a series of groups $G_0 \hookrightarrow G_1 \hookrightarrow \cdots \hookrightarrow G_n \hookrightarrow \cdots$, where G_n is one of the classical groups above and G_{n-1} is the Levi subgroup of the stabilizer in G_n of an isotropic vector. A further ingredient is an algebraically closed field k of characteristic ℓ different from the defining characteristic of the groups G_n . The connected components of a Harish-Chandra branching graph correspond to the weak Harish-Chandra series of the groups G_{m+n} , $n \geq 0$, arising from a fixed weakly cuspidal pair (G_m, X) (Proposition 4.2). Define \mathcal{H}_n as the endomorphism algebra of the module obtained by Harish-Chandra inducing X from G_m to G_{m+n} . In Proposition 4.3 we prove that the corresponding family of Hom-functors yields an isomorphism between the connected component of the Harish-Chandra branching graph arising from (G_m, X) and the branching graph of the family \mathcal{H}_n , $n \geq 0$ of endomorphism algebras.

We expose in Section 5 consequences of these results for the Harish-Chandra branching graphs of the unitary groups. Provided the algebras \mathcal{H}_n are Iwahori-Hecke algebras of type B_n with a particular pair of parameters, the Harish-Chandra branching graphs are isomorphic to crystal graphs as in [25, Conjecture 5.7]; this is Proposition 5.1. This result does not yet, however, yield the conjectured matching of the vertices of the two graphs involved. Under the same hypothesis we obtain a strong condition on the structures of Harish-Chandra restricted unipotent modules in a minimal situation. Each direct summand of such a module has a simple socle and a simple head isomorphic to each other (Proposition 5.2). Ultimately, this derives from a result of Grojnowski and Vazirani (see [27, Theorem B]). Finally, we prove [25, Conjecture 5.7] for the principal series and the other series arising from cuspidal unipotent defect 0 modules (Theorem 5.5). It is remarkable that these results do not require any restriction on ℓ .

2. THE BRANCHING GRAPH OF ARIKI-KOIKE ALGEBRAS

In this section we are interested in the combinatorial description of the branching graph for modular Ariki-Koike algebras over a field

of positive characteristic ℓ in terms of crystals of Fock spaces. This is achieved via Ariki's classic categorification theorem, more precisely with the results of his paper [4]. The only slight adjustment here is that instead of using Kleshchev's realization of the Fock space crystal, as is done by Ariki, we favor Uglov's version. This is motivated by the fact that we expect [25, Conjecture 5.7] to hold for Uglov's realization. Throughout this section modules are left modules, and we write $A\text{-mod}$ for the category of finitely generated left modules of the algebra A .

2.1. Ariki-Koike algebras. Let $d \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}_{\geq 0}$, and let k be a field. Let $u, v_1, \dots, v_d \in k$ with u non-zero. Following [34], we define the Ariki-Koike algebra with parameters u, v_1, \dots, v_d to be the k -algebra $\mathcal{H}_{k,d,n}$ defined by generators T_0, T_1, \dots, T_{n-1} and relations:

- the braid relations of type B ,
- the relations $(T_0 - v_1) \dots (T_0 - v_d) = 0$ and $(T_i - u)(T_i + 1) = 0$ for all $i = 1, \dots, n - 1$.

In particular, if $d = 2$, then $\mathcal{H}_{k,d,n}$ is an Iwahori-Hecke algebra of type B_n with parameters u and $-v_1 v_2^{-1}$ in the sense of [21, Definition 4.4.1 and Remark 8.1.3], via the change of generators $T'_i = T_i$ for $i = 1, \dots, n - 1$ and $T'_0 = -v_2^{-1} T_0$. This is of importance since we will use this identification from Proposition 5.1 on.

If $\mathcal{H}_{k,d,n}$ is semisimple, there is a labelling of the simple $\mathcal{H}_{k,d,n}$ -modules by d -partitions of n . The problem of labelling the simple modules in the non-semisimple case is much more complicated. We recall some facts in Sections 2.3 and 2.4 below. We also know in particular that $\mathcal{H}_{k,d,n}$ is non-semisimple as soon as there exist integers s_1, \dots, s_d such that $v_i = u^{s_i}$ for all $i = 1, \dots, d$, see e.g. [34, Corollary 3.3]. In case $\mathcal{H}_{k,d,n}$ is non-semisimple, there is a decomposition map and a decomposition matrix relating its representation theory to that of the generic (hence semisimple) Ariki-Koike algebra; see [3, Sections 13.3 and 13.4] for details.

2.2. Crystal of the Fock space. For v an indeterminate, $e \in \mathbb{Z}_{>1}$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$, we consider the level d Fock space representation of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ with charge \mathbf{s} , denoted by $\mathcal{F}_{\mathbf{s},e}$, see for instance [20, Chapter 6]. It is the $\mathbb{Q}(v)$ -vector space with basis all d -partitions. We know in particular that $\mathcal{F}_{\mathbf{s},e}$ is an integrable representation, and is therefore endowed with a crystal structure.

Note that the action of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ on $\mathcal{F}_{\mathbf{s},e}$ requires an order on the nodes of a d -partition. There are essentially two different orders that yield two isomorphic $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ -module structures on $\mathcal{F}_{\mathbf{s},e}$, which we recall here. The first one is defined as follows.

Let (a, b, c) and (a', b', c') be nodes of (the Young diagram of) a d -partition λ ,¹ such that $b - a + s_c = b' - a' + s_{c'} \pmod{e}$. We write

$$(a, b, c) \prec_{\mathcal{U}} (a', b', c') \quad \text{if} \quad \begin{cases} b - a + s_c < b' - a' + s_{c'} & \text{or} \\ b - a + s_c = b' - a' + s_{c'} & \text{and} \quad c > c'. \end{cases}$$

We refer to the module structure afforded by this order as *Uglov's realization* of the Fock space. This is the order originally used in [31], and then in [14], [37], [20].

The second order is defined by

$$(a, b, c) \prec_{\mathcal{K}} (a', b', c') \quad \text{if} \quad \begin{cases} c > c' & \text{or} \\ c = c' & \text{and} \quad a > a', \end{cases}$$

and we call *Kleshchev's realization* the module structure afforded by this order. This is used in particular in [3], [34], [6].

Remark 2.1. Note that the definition of the second order does not require the charge \mathbf{s} . However, we always want to compare nodes such that $b - a + s_c = b' - a' + s_{c'} \pmod{e}$. This means that the order $\prec_{\mathcal{K}}$ is in particular invariant under translation of any component of \mathbf{s} by a multiple of e . On the other hand, the order $\prec_{\mathcal{U}}$ strictly depends on \mathbf{s} .

The two orders yield isomorphic Kashiwara crystals (in the sense that the two crystal graphs are the same as colored oriented graphs up to a relabelling of the vertices), where the action of the crystal operators corresponds to adding, respectively removing a so-called good node. Denote by $\mathcal{G}_{\mathbf{s},e}$ the crystal graph of the Fock space, and $\mathcal{B}_{\mathbf{s},e}$ the connected component of this graph containing the empty d -partition \emptyset .

Then $\mathcal{B}_{\mathbf{s},e}$ is the crystal graph of the irreducible highest weight subrepresentation $V_{\mathbf{s},e} = \mathcal{U}_v(\widehat{\mathfrak{sl}}_e) \cdot \emptyset$ of $\mathcal{F}_{\mathbf{s},e}$. The vertices appearing in Uglov's (respectively Kleshchev's) realization of the crystal graph $\mathcal{B}_{\mathbf{s},e}$ are called the *Uglov* (respectively *Kleshchev*) d -partitions, and denoted by $\mathcal{U}_{\mathbf{s},e}$ (respectively $\mathcal{K}_{\mathbf{s},e}$). We call *rank* of a d -partition the total number of nodes it contains. For n a fixed integer, we denote by $\mathcal{U}_{\mathbf{s},e}(n)$ (respectively $\mathcal{K}_{\mathbf{s},e}(n)$) the Uglov (respectively Kleshchev) d -partitions of rank n .

Remark 2.2. According to [23, Proposition 5.1], the orders $\prec_{\mathcal{U}}$ and $\prec_{\mathcal{K}}$ coincide on the nodes of a d -partition of rank n , if and only if $s_i - s_j \geq n - e + 1$ for all $1 \leq i < j \leq d$. In particular, if we denote $M = \min\{s_i - s_j \mid i < j\}$, this implies that the two crystal graphs are the same up to rank $M + e - 1$ (provided of course that $M + e - 1 \geq 0$).

¹Here, $c \in \{1, \dots, d\}$ stands for the component of the node, a for the row of the node in its component, and b for the column of the node in its component.

Algebraic interpretations aside (which are exposed in Section 2.4), one can first notice that there is an advantage to work with Uglov rather than Kleshchev d -partitions. In the particular case where $0 \leq s_1 \leq \dots \leq s_d \leq e - 1$, the Uglov d -partitions are known as FLOTW d -partitions and have a non-recursive combinatorial characterisation, by [14, Theorem 2.10]. In general, there also exists a pathfinding-free combinatorial characterisation of Uglov d -partitions, see [22, Theorem 6.3]. However, finding a non-recursive characterisation of the Kleshchev d -partitions in the general case is still an open problem (though when $d = 2$, this can be achieved via the results of [5, Section 9]).

2.3. The branching rule for modular Ariki-Koike algebras. Let $\mathcal{H}_n = \mathcal{H}_{k,d,n}$ be a non-semisimple Ariki-Koike algebra such that u has order e in k , and $v_i = u^{s_i}$ for all $i = 1, \dots, d$ for some $(s_1, \dots, s_d) = \mathbf{s} \in \mathbb{Z}^d$. Following Ariki's book [3, Section 13.6], there exist i -restriction and i -induction functors (for $i = 0, \dots, e - 1$) which refine the restriction (respectively induction) functors between \mathcal{H}_{n+1} -mod and \mathcal{H}_n -mod (respectively between \mathcal{H}_n -mod and \mathcal{H}_{n+1} -mod), denoted by $i\text{-Res}_n^{n+1}$ and $i\text{-Ind}_n^{n+1}$. Then Grojnowski and Vazirani [27, Theorem B] have proved that the functors \tilde{e}_i and \tilde{f}_i defined by

$$\begin{aligned} \tilde{e}_i M &= \text{Soc}(i\text{-Res}_n^{n+1}(M)) && \text{for } M \in \mathcal{H}_{n+1}\text{-mod}, \\ \tilde{f}_i M &= \text{Hd}(i\text{-Ind}_n^{n+1}(M)) && \text{for } M \in \mathcal{H}_n\text{-mod} \end{aligned}$$

send simple modules to simple modules. This yields a coloring of the arrows of the branching graph of \mathcal{H}_n , $n \geq 0$. In fact, as was shown by Ariki (see [4, Theorem 4.1]), it even defines the structure of an abstract crystal in the sense of [32, Section 7.2] on the set $\text{Irr}(\mathcal{H}) = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \text{Irr}(\mathcal{H}_n)$.

Using the cellular approach, there is a natural parametrisation of $\text{Irr}(\mathcal{H}_n)$ by the set $\mathcal{K}_{\mathbf{s},e}(n)$ of Kleshchev d -partitions of rank n ; this is the main result of [2]. Hence, write $\text{Irr}(\mathcal{H}_n) = \{D^\lambda \mid \lambda \in \mathcal{K}_{\mathbf{s},e}(n)\}$. Actually, we have more. The following result is due to Ariki. Importantly, it holds regardless of the characteristic of the field k .²

Theorem 2.3. [4, Theorem 6.1] *Under the identification $D^\lambda \leftrightarrow \lambda$, the branching graph on $\text{Irr}(\mathcal{H})$ is exactly the crystal graph $\mathcal{B}_{\mathbf{s},e}$ in Kleshchev's realization.*

Because of the discussion of Section 2.2, there is a crystal isomorphism φ between the crystal graphs $\mathcal{B}_{\mathbf{s},e}$ in Kleshchev's and Uglov's

²This is important to notice, since Ariki's theorem in its general version only holds for Ariki-Koike algebras defined over a field of characteristic zero.

realization:

$$\begin{array}{ccc} \varphi : \mathcal{K}_{\mathbf{s},e} & \xrightarrow{\sim} & \mathcal{U}_{\mathbf{s},e} \\ & \lambda \mapsto & \varphi(\lambda). \end{array}$$

By definition, φ preserves the rank, and φ is the identity up to rank $M + e - 1$ by Remark 2.2. However, it appears to be difficult to determine φ in general.

The following statement is then straightforward from Theorem 2.3.

Corollary 2.4. *Under the identification $D^\lambda \leftrightarrow \varphi(\lambda)$, the branching graph on $\text{Irr}(\mathcal{H})$ is exactly the crystal graph $\mathcal{B}_{\mathbf{s},e}$ in Uglov's realization.*

In particular, we get a labelling

$$\begin{array}{ccc} \mathcal{U}_{\mathbf{s},e}(n) & \longleftrightarrow & \text{Irr}(\mathcal{H}_n) \\ \boldsymbol{\mu} & \longleftrightarrow & C^\mu \end{array}$$

with $C^\mu = D^\lambda$ if and only if $\boldsymbol{\mu} = \varphi(\lambda)$.

2.4. Compatibility with the theory of canonical basic sets in characteristic zero. Up to now, this labelling of $\text{Irr}(\mathcal{H}_n)$ by $\mathcal{U}_{\mathbf{s},e}(n)$ may seem a bit superficial. Actually, it is not, since this class of d -partitions naturally appear in the context of canonical basic sets for \mathcal{H}_n . For this section, we mainly refer to [20, Chapter 6]. The theory of canonical basic sets provides a way to label the simple modules of a Hecke algebra. In fact, this is the suitable labelling for our purpose, see Theorem 5.5. In the case of \mathcal{H}_n , this labelling is achieved as follows.

Recall that to \mathcal{H}_n is associated a charge $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$, and the integer e . According to [29], there is a generalization to \mathcal{H}_n of Lusztig's \mathbf{a} -function for Iwahori-Hecke algebras, depending on a parameter $\mathbf{m} \in \mathbb{Q}^d$ and denoted by $\mathbf{a}^{\mathbf{m}}$. This induces an order $<_{\mathbf{m}}$ on the ordinary Specht modules (i.e. on d -partitions of n), by setting $\lambda <_{\mathbf{m}} \mu$ if and only if $\mathbf{a}^{\mathbf{m}}(\lambda) < \mathbf{a}^{\mathbf{m}}(\mu)$. If the decomposition matrix of \mathcal{H}_n is unitriangular with respect to this order (for the exact definition see [20, Definition 5.5.19]), then this gives a labelling of the simple modules of \mathcal{H}_n by a subset of d -partitions of n , which we call the *canonical basic set* for \mathcal{H}_n with respect to $<_{\mathbf{m}}$. The following result is due to Geck and Jacon.

Theorem 2.5. [20, Theorem 6.7.2] *Suppose that $\text{char}(k) = 0$. Let $\mathbf{m} = (m_1, \dots, m_d)$ such that $0 < (s_j - m_j) - (s_i - m_i) < e$ for $i < j$. Then $\mathcal{U}_{\mathbf{s},e}(n)$ is the canonical basic set for \mathcal{H}_n with respect to $<_{\mathbf{m}}$.*

The proof requires Ariki's theorem to identify the decomposition matrix with the specialisation at $v = 1$ of the matrix of the canonical basis of $V_{\mathbf{s},e} \leq \mathcal{F}_{\mathbf{s},e}$, whence the restriction to the zero characteristic.

Write

$$\begin{array}{ccc} \mathcal{U}_{\mathbf{s},e} & \longleftrightarrow & \text{Irr}(\mathcal{H}_n) \\ \boldsymbol{\mu} & \longleftrightarrow & M^\mu \end{array}$$

for the labelling given by this theorem. This way of labelling $\text{Irr}(\mathcal{H}_n)$ does not a priori give any information about the branching. However, it is indeed compatible with the crystal structure of Corollary 2.4.

Proposition 2.6. *Suppose that $\text{char}(k) = 0$. Then for all $\boldsymbol{\mu} \in \mathcal{U}_{\mathbf{s},e}$, we have $M^\mu = C^\mu$.*

Proof. Recall that the labelling by C^μ is given through the labelling by Kleshchev d -partitions (arising from Ariki's use of the cellular theory), i.e. $C^\mu = D^\lambda$ with $\boldsymbol{\mu} = \varphi(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \mathcal{K}_{\mathbf{s},e}$. Now, for fixed n , choose a charge $\mathbf{s}' = (s'_1, \dots, s'_d) \in \mathbb{Z}^d$ such that $s'_i = s_i + t_i e$ for some $(t_1, \dots, t_d) \in \mathbb{Z}^d$ and $s'_i - s'_j \geq n - e + 1$ for all $1 \leq i < j \leq d$. By combining Remarks 2.1 and 2.2, we have $\mathcal{K}_{\mathbf{s},e}(n) = \mathcal{K}_{\mathbf{s}',e}(n) = \mathcal{U}_{\mathbf{s}',e}(n)$. Moreover, it is clear that if we write \mathcal{H}'_n for the Ariki-Koike algebra with parameters u a primitive e -th root of 1 and $v_i = u^{s'_i}$, then $\mathcal{H}'_n = \mathcal{H}_n$.

Let $\mathbf{m} = (m_1, \dots, m_d)$ such that $0 < (s_j - m_j) - (s_i - m_i) < e$ for $i < j$ and $\mathbf{m}' = (m'_1, \dots, m'_d)$ such that $0 < (s'_j - m'_j) - (s'_i - m'_i) < e$ for $i < j$. By Theorem 2.5, we know that $\mathcal{U}_{\mathbf{s},e}(n)$ (respectively $\mathcal{K}_{\mathbf{s},e}(n)$) is the canonical basic set for \mathcal{H}_n with respect to $<_{\mathbf{m}}$ (respectively $<_{\mathbf{m}'}$).

Denote $\{G_v(\boldsymbol{\mu}, \mathbf{s}) \mid \boldsymbol{\mu} \in \mathcal{U}_{\mathbf{s},e}\}$ the canonical basis of $V_{\mathbf{s},e}$. Similarly, denote $\{G'_v(\boldsymbol{\lambda}, \mathbf{s}') \mid \boldsymbol{\lambda} \in \mathcal{K}_{\mathbf{s},e}\}$ the canonical basis of $V_{\mathbf{s}',e}$. Decompose the elements

$$G_v(\boldsymbol{\mu}, \mathbf{s}) = \sum_{\boldsymbol{\nu} \vdash_d |\boldsymbol{\mu}|} g_{\boldsymbol{\nu}, \boldsymbol{\mu}}(v) \boldsymbol{\nu} \quad \text{and} \quad G'_v(\boldsymbol{\lambda}, \mathbf{s}') = \sum_{\boldsymbol{\nu} \vdash_d |\boldsymbol{\lambda}|} g'_{\boldsymbol{\nu}, \boldsymbol{\lambda}}(v) \boldsymbol{\nu}$$

on the basis of d -partitions. Ariki's theorem ensures that the decomposition numbers of \mathcal{H}_n are given by the evaluations $g_{\boldsymbol{\nu}, \boldsymbol{\mu}}(1)$ (or equivalently $g'_{\boldsymbol{\nu}, \boldsymbol{\lambda}}(1)$).

Let us focus on $\mathcal{K}_{\mathbf{s},e}(n)$. The fact that it is the canonical basic set with respect to $<_{\mathbf{m}'}$ means that for all $\boldsymbol{\lambda} \in \mathcal{K}_{\mathbf{s},e}(n)$, $\boldsymbol{\lambda}$ is the unique element which is smaller (with respect to $<_{\mathbf{m}'}$) than all d -partitions $\boldsymbol{\nu}$ with $g_{\boldsymbol{\nu}, \boldsymbol{\lambda}}(v) \neq 0$. Moreover, $g_{\boldsymbol{\lambda}, \boldsymbol{\lambda}}(1) = 1$. Now, because of the particular value of \mathbf{m}' we have chosen, the order $<_{\mathbf{m}'}$ coincides with the classic dominance order, see e.g. [24, Proof of Proposition 1.2.11]. Besides, the same properties holds in the cellular theory for this dominance ordering, namely $\boldsymbol{\lambda}$ is the unique element which is smaller (with respect to the dominance order) than all d -partitions $\boldsymbol{\nu}$ such that the decomposition number $d_{\boldsymbol{\nu}, \boldsymbol{\lambda}}$ is non-zero; and moreover $d_{\boldsymbol{\lambda}, \boldsymbol{\lambda}} = 1$. For this, see for instance to [2, Theorem 2.2]. This means that the labelling of $\text{Irr}(\mathcal{H}_n)$ by $\mathcal{K}_{\mathbf{s},e}(n)$ as in Theorem 2.3 on the one hand and by the theory of

canonical basic sets on the other hand is exactly the same. Precisely, if $\lambda \in \mathcal{K}_{\mathbf{s},e}(n)$ labels the simple module M by the theory of canonical basic sets, then $M = D^\lambda$. In particular, the theory of canonical basic sets provides the same branching information.

Finally, let us go back to the labelling of $\text{Irr}(\mathcal{H}_n)$ by $\mathcal{U}_{\mathbf{s},e}(n)$ via the theory of canonical basic sets. We already know the existence of the crystal isomorphism $\mathcal{K}_{\mathbf{s},e} \xrightarrow{\varphi} \mathcal{U}_{\mathbf{s},e}$. The characteristic property of the canonical basis elements $G_v(\boldsymbol{\mu}, \mathbf{s})$ and $G'_v(\boldsymbol{\lambda}, \mathbf{s}')$ [37, Section 4] ensures that this crystal isomorphism maps the Kleshchev d -partition $\boldsymbol{\lambda}$ labelling an irreducible module M to the Uglov d -partition $\varphi(\boldsymbol{\lambda})$ labelling the same module M (which we had denoted $M^{\varphi(\boldsymbol{\lambda})}$). Since $M = D^\lambda$, we deduce $M^\boldsymbol{\mu} = M = D^\lambda = C^\boldsymbol{\mu}$ where $\boldsymbol{\mu} = \varphi(\boldsymbol{\lambda})$. \square

We end this section by mentioning that it is also relevant to work with Uglov's realization of the crystal when we want to link the representation theory of Ariki-Koike algebras with that of rational Cherednik algebras. In [35], Shan has defined i -induction and i -restriction functors on the category \mathcal{O}_c for the family of rational Cherednik algebras of type $G(d, 1, n)$ with parameter c , for $n \geq 0$. Similarly to Ariki's i -induction and i -restriction (for the Ariki-Koike algebra), she has proved that these operators induce the structure of an abstract crystal, which is isomorphic to that of a Fock space $\mathcal{F}_{\mathbf{s},e}$, where the parameters \mathbf{s} and e are determined by c . Moreover, Losev [33] has used Uglov's realization of $\mathcal{F}_{\mathbf{s},e}$ to give an explicit combinatorial rule for the computation of this crystal.

Now, the representation theory of Ariki-Koike algebras on the one hand, and rational Cherednik algebras on the other hand, are known to be related by an exact functor $KZ : \mathcal{O}_c \rightarrow \bigoplus_{n \geq 0} \mathcal{H}_n\text{-mod}$. This functor has the nice property of commuting with Ariki's (respectively Shan's) i -induction and i -restriction functors. Therefore, it maps the Uglov crystal of $\mathcal{F}_{\mathbf{s},e}$ encoding Shan's branching rule to the Uglov crystal $\mathcal{B}_{\mathbf{s},e}$ encoding Ariki's branching rule. Note that relying on previous knowledge ([10, Corollary 5.8]) about canonical basic sets, this was already mentioned in [26, Paragraph 4.15].

3. BRANCHING IN ENDOMORPHISM ALGEBRAS

Let G be a finite group with a split BN -pair of characteristic p satisfying the commutator relations. The set of N -conjugates of the standard Levi subgroups is denoted by \mathcal{L} . We let $\mathcal{L}' := \mathcal{L}'_G$ denote an N -stable subset of \mathcal{L} containing G and $B \cap N$ and satisfying $L \cap^x M \in \mathcal{L}'$ for all $L, M \in \mathcal{L}'$ and all $x \in N$. For $M \in \mathcal{L}'$ we put $\mathcal{L}'_M := \{L \in \mathcal{L}' \mid L \leq M\}$.

Next, let k be a field of characteristic different from p , such that k is a splitting field for all subgroups of G . If A is a k -algebra, we write $A\text{-mod}$ and $\text{mod-}A$ for the category of finite dimensional left, respectively right, A -modules.

For $L \in \mathcal{L}$ we denote R_L^G and $*R_L^G$ the Harish-Chandra induction and restriction functors, respectively. These are defined using a choice of a parabolic subgroup of G with Levi complement L , but are known to be independent of this choice up to a natural isomorphism of functors (see e.g. [8, Theorem 3.10]).

Following [25, Section 2.3], we call a simple kG -module X weakly cuspidal (with respect to \mathcal{L}'), if $*R_L^G(X) = 0$ for all $L \in \mathcal{L}'$ with $L \neq G$. A weakly cuspidal pair (L, X) consists of an $L \in \mathcal{L}'$ and a simple kL -module X which is weakly cuspidal with respect to \mathcal{L}'_L .

Now fix a weakly cuspidal pair (L, X) and put

$$Y := R_L^G(X).$$

Write $H_G := \text{End}_{kG}(Y)$ and $kG\text{-mod}_Y$ for the full subcategory of $kG\text{-mod}$ consisting of those modules which are quotients and submodules of a finite direct sum of copies of Y . The covariant Hom-functor with respect to Y is denoted by F_Y :

$$F_Y : kG\text{-mod} \rightarrow \text{mod-}H_G, \quad V \mapsto \text{Hom}_{kG}(Y, V).$$

Here, H_G acts on the right of $\text{Hom}_{kG}(Y, V)$ by composition of maps. A result of Cabanes states that F_Y induces an equivalence between $kG\text{-mod}_Y$ and $\text{mod-}H_G$, provided that H_G is self-injective (see [7, Theorem 2]).

Suppose now that $M \in \mathcal{L}'$ with $L \leq M$ (then $L \in \mathcal{L}'_M$ by definition of \mathcal{L}'_M) and put

$$Z := R_L^M(X).$$

Write $H_M := \text{End}_{kM}(Z)$ and F_Z for the covariant Hom-functor between $kM\text{-mod}$ and $\text{mod-}H_M$. There is a natural embedding

$$i : H_M \rightarrow \text{End}_{kG}(R_M^G(Z)),$$

and as $R_M^G(Z) \cong Y$, the map i induces a restriction functor

$$\text{Res}_{H_M}^{H_G} : \text{mod-}H_G \rightarrow \text{mod-}H_M.$$

The following result is essentially due to Howlett and Lehrer (see [28, Theorem 1.13]) although they formulate it for the contravariant Hom-functor and in the semisimple case. The analogous statement in the case of defining characteristic is contained in [7, 2.3].

Proposition 3.1. *The following diagram of functors is commutative up to a natural isomorphism of functors.*

$$\begin{array}{ccc}
kG\text{-mod} & \xrightarrow{F_Y} & \text{mod-}H_G \\
\downarrow *R_M^G & & \downarrow \text{Res}_{H_M}^{H_G} \\
kM\text{-mod} & \xrightarrow{F_Z} & \text{mod-}H_M
\end{array}$$

Proof. We may assume $Y = R_M^G(Z)$. To proceed, we choose a parabolic subgroup Q of G with Levi complement M to define the functors R_M^G and $*R_M^G$; in particular,

$$Y = R_M^G(Z) = kG \otimes_{kQ} \text{Infl}_M^Q(Z).$$

Then for every $V \in kG\text{-mod}$ we have

$$F_Y(V) = \text{Hom}_{kG}(Y, V) = \text{Hom}_{kG}(kG \otimes_{kQ} \text{Infl}_M^Q(Z), V).$$

Applying Frobenius reciprocity, we find

$$\text{Hom}_{kG}(kG \otimes_{kQ} \text{Infl}_M^Q(Z), V) \cong \text{Hom}_{kQ}(\text{Infl}_M^Q(Z), \text{Res}_Q^G(V))$$

as right $H_M = \text{End}_{kQ}(\text{Infl}_M^Q(Z))$ -modules. (Indeed, the above isomorphism equals η^* , with $\eta : \text{Infl}_M^Q(Z) \rightarrow kG \otimes_{kQ} \text{Infl}_M^Q(Z)$, $z \mapsto 1 \otimes z$.) Clearly,

$$\begin{aligned}
\text{Hom}_{kQ}(\text{Infl}_M^Q(Z), \text{Res}_Q^G(V)) &\cong \text{Hom}_{kM}(Z, \text{Fix}_{O_p(Q)}(\text{Res}_Q^G(V))) \\
&= F_Z(*R_M^G(V))
\end{aligned}$$

as right H_M -modules. As all of the above isomorphisms are natural, the result follows. \square

Lemma 3.2. *Let $S \in kM\text{-mod}_Z$ and $T \in kG\text{-mod}_Y$. Then $R_M^G(S) \in kG\text{-mod}_Y$ and $*R_M^G(T) \in kM\text{-mod}_Z$.*

Proof. As R_M^G is exact, and as $R_M^G(Z)$ is isomorphic to Y , the first assertion is clear. To prove the second assertion, assume that T is a submodule and a quotient of $Y' = R_L^G(X')$, where Y' and X' are direct sums of the same number of copies of Y , respectively X . Hence $*R_M^G(T)$ is a submodule and a quotient of $*R_M^G(R_L^G(X'))$. By Mackey's theorem, the latter is isomorphic to a direct sum of modules of the form

$$(1) \quad R_{M \cap {}^x L}^M(*R_{M \cap {}^x L}^{xL}({}^x X'))$$

for suitable $x \in N$. As (L, X) is weakly cuspidal, so is $({}^x L, {}^x X)$ for all such x . It follows that a summand (1) is zero unless ${}^x L \leq M$, in which case (1) is isomorphic to a direct sum of copies of Z . The claim follows. \square

Proposition 3.3. *Let $S \in kM\text{-mod}_Z$ and $T \in kG\text{-mod}_Y$. Suppose that H_M is self-injective. Then*

$$\text{Hom}_{H_M}(F_Z(S), \text{Res}_{H_M}^{H_G}(F_Y(T))) \cong \text{Hom}_{kM}(S, {}^*R_M^G(T)).$$

Proof. We have

$$\begin{aligned} \text{Hom}_{H_M}(F_Z(S), \text{Res}_{H_M}^{H_G}(F_Y(T))) &\cong \text{Hom}_{H_M}(F_Z(S), F_Z({}^*R_M^G(T))) \\ &\cong \text{Hom}_{kM}(S, {}^*R_M^G(T)), \end{aligned}$$

where the first isomorphism follows from Proposition 3.1, and the second from [7, Theorem 2] together with Lemma 3.2. \square

Remark 3.4. By [25, Proposition 2.3], the simple submodules of Z and of Y belong to $kM\text{-mod}_Z$ and $kG\text{-mod}_Y$, respectively. Thus if H_M is self-injective, Proposition 3.3 applies to these simple modules.

4. THE HARISH CHANDRA BRANCHING GRAPH

In [25, Section 4], we introduced the Harish Chandra branching graph for unipotent modules of certain classical groups. Let us recall this definition. Fix primes $p \neq \ell$ and a power q of p . Let k denote an algebraically closed field of characteristic ℓ . For every $n \in \mathbb{Z}_{\geq 0}$ we consider the groups $\text{GU}_{2n}(q)$, $\text{GU}_{2n+1}(q)$, $\text{SO}_{2n+1}(q)$ and $\text{Sp}_{2n}(q)$ (with $\text{GU}_0(q)$ and $\text{Sp}_0(q)$ the trivial group), and call n the *rank* of such a group. The *Dynkin type* of these groups is 2A_0 , 2A_1 , B and C , respectively.

We now fix one of these Dynkin types \mathcal{D} , say, and write G_n for the group of Dynkin type \mathcal{D} and rank n . Then G_n is a group with a split BN -pair of characteristic p satisfying the commutator relations. If r, m are non-negative integers with $r + m = n$, there is a standard Levi subgroup $L_{r,m}$ of G_n isomorphic to $G_r \times \text{GL}_1(q^\delta) \times \cdots \times \text{GL}_1(q^\delta)$ with m factors $\text{GL}_1(q^\delta)$ and $\delta = 1$ if \mathcal{D} is B or C , and $\delta = 2$, otherwise. These Levi subgroups and their N -conjugates are called *pure Levi subgroups* of G_n . The set \mathcal{L}'_n of pure Levi subgroups of G_n is N -stable and satisfies $L \cap {}^xM \in \mathcal{L}'_n$ for all $L, M \in \mathcal{L}'_n$.

The Harish-Chandra branching graph $\mathcal{G}_{\mathcal{D},q,\ell}$ for the Dynkin type \mathcal{D} (and fixed q and ℓ) is defined as follows. Its vertices are the isomorphism classes of the unipotent kG_n -modules, where n runs over the integers. There is an arrow $[X] \rightarrow [Y]$ between the vertices $[X]$ and $[Y]$ of $\mathcal{G}_{\mathcal{D},q,\ell}$, if and only if there is $n \in \mathbb{Z}_{\geq 0}$ such that X and Y are kG_n - and kG_{n+1} -modules, respectively, and the inflation of X to $L_{n,1}$ occurs in the socle of ${}^*R_{L_{n,1}}^{G_{n+1}}(Y)$, i.e. if and only if $\text{Hom}_{kL_{n,1}}(\text{Infl}_{G_n}^{L_{n,1}}(X), {}^*R_{L_{n,1}}^{G_{n+1}}(Y)) \neq 0$. (Recall that $L_{n,1} \cong G_n \times \text{GL}_1(q^\delta)$.)

Let (L, X) be a weakly cuspidal pair, where $L = G_m$ for some $m \in \mathbb{Z}_{\geq 0}$, and X is unipotent. For a non-negative integer n put

$$Y_n := R_{L_{m,n}}^{G_{m+n}}(X),$$

where X is viewed as a $kL_{m,n}$ -module via inflation. Moreover, we put

$$\mathcal{H}_n := \mathcal{H}_n(X) := \text{End}_{kG_{m+n}}(Y_n).$$

Then there are natural inclusions

$$\nu_n : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}.$$

We also write F_n for the Hom-functor

$$F_n : kG_n\text{-mod}_{Y_n} \rightarrow \text{mod-}\mathcal{H}_n.$$

The branching graph for \mathcal{H}_n , $n \geq 0$, is defined as follows. Its vertices are the isomorphism classes of the simple \mathcal{H}_n -modules, where n runs through the non-negative integers. Two such vertices $[S]$ and $[T]$ are connected by an arrow $[S] \rightarrow [T]$, if and only if there is $n \in \mathbb{Z}_{\geq 0}$, such that S is an \mathcal{H}_n -module and T is an \mathcal{H}_{n+1} -module such that S occurs in the socle of $\text{Res}_{\mathcal{H}_n}^{\mathcal{H}_{n+1}}(T)$, i.e. if and only if $\text{Hom}_{\mathcal{H}_n}(S, \text{Res}_{\mathcal{H}_n}^{\mathcal{H}_{n+1}}(T)) \neq 0$.

Definition 4.1. Define $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ to be the induced subgraph of $\mathcal{G}_{\mathcal{D},q,\ell}$, consisting of the vertices $[Y]$ such that there is a directed path from $[X]$ to $[Y]$.

Proposition 4.2. *The subgraph $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ is a connected component of $\mathcal{G}_{\mathcal{D},q,\ell}$ (with respect to the underlying undirected graph), and every connected component of $\mathcal{G}_{\mathcal{D},q,\ell}$ is of this form.*

Proof. By definition, every vertex of $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ is connected to $[X]$, and thus $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ is connected.

Consider a path $[Y] \rightarrow [Z]$ of length 1 in $\mathcal{G}_{\mathcal{D},q,\ell}$. We claim that $[Y]$ and $[Z]$ belong to $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ if and only if one of $[Y]$ or $[Z]$ belongs to this subgraph. To prove this, it suffices to show that $[Y]$ is a vertex of $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ if $[Z]$ is one. Consider a source vertex $[X']$ of $\mathcal{G}_{\mathcal{D},q,\ell}$ such that there is a directed path from $[X']$ to $[Y]$. Suppose that X' and Z are unipotent modules of $G_{m'}$ and G_{m+n} , respectively. Then Z lies in the weak Harish-Chandra series defined by $(L_{m,n}, X)$ and $(L_{m',n'}, X')$ with $m+n = m'+n'$. It follows that $L_{m,n}$ and $L_{m',n'}$ are conjugate in G_{m+n} and thus are equal. Moreover, X and X' are conjugate by an element in the relative Weyl group of $L_{m,n}$. As the latter group fixes X , it follows that $[X] = [X']$, thus proving our claim.

The claim implies that the connected component of $\mathcal{G}_{\mathcal{D},q,\ell}$ containing $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ is equal to $\mathcal{G}_{\mathcal{D},q,\ell}(X)$.

As every vertex of the Harish-Chandra branching graph belongs to some weak Harish-Chandra series, it is clear that every connected component of $\mathcal{G}_{\mathcal{D},q,\ell}$ is of the asserted form. \square

Proposition 4.3. *The collection of functors F_n , $n \in \mathbb{Z}_{\geq 0}$ yields an isomorphism between $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ and the branching graph of \mathcal{H}_n , $n \geq 0$.*

Proof. As already noted in the proof of [25, Proposition 2.3], the general results of Cabanes and Enguehard [8, Theorems 1.20(iv), 2.27] imply that \mathcal{H}_n is symmetric, hence self-injective, for all non-negative integers n . Our claim now follows from Proposition 3.3 and Remark 3.4. \square

By [25, Theorem 3.2], we have that \mathcal{H}_n is an Iwahori-Hecke algebra of type B_n , with parameters q^δ and Q , where Q occurs on the doubly laced end node of the Dynkin diagram. Although the value of Q can only be determined explicitly in some cases, it is clear from the proof of [25, Theorem 3.2] that Q only depends on L and not on n .

5. THE UNITARY GROUPS

As in Section 4, we fix primes $p \neq \ell$ and a power q of p . Again, k denotes an algebraically closed field of characteristic ℓ . We write e for the order of $-q$ in the finite field \mathbb{F}_ℓ , assuming henceforth that e is odd and larger than 1.

Assume that \mathcal{D} is one the Dynkin types 2A_0 or 2A_1 . Thus, if n is a non-negative integer, G_n now denotes one of the groups $\mathrm{GU}_{2n}(q)$ or $\mathrm{GU}_{2n+1}(q)$.

We also fix a weakly cuspidal unipotent kG_m -module X for some non-negative integer m . For every non-negative integer n , we view X as a module for $L_{m,n}$ via inflation, so that $(L_{m,n}, X)$ is a weakly cuspidal pair in G_{m+n} .

Notice that we have worked with the categories of finitely generated right \mathcal{H}_n -modules for the endomorphism algebras arising in the previous section. We are now going to apply the results of Section 2 to these algebras, where we have worked with left modules. This is no loss since duality of vector spaces induces an equivalence between A -mod and $\mathrm{mod}\text{-}A$ for a finite-dimensional k -algebra A .

Proposition 5.1. *Suppose that for some non-negative integer s and any non-negative integer n ,*

$$\mathcal{H}_n := \mathrm{End}_{kG_{m+n}}(R_{L_{m,n}}^{G_{n+m}}(X))$$

is an Iwahori-Hecke algebra of type B_n with parameters q^2 and $Q = q^{2s+1}$.

Then the Harish-Chandra branching graph $\mathcal{G}_{\mathcal{D},q,\ell}(X)$ is isomorphic to the crystal graph $\mathcal{B}_{\mathbf{s},e}$, in which the colors of the arrows are neglected, where $\mathbf{s} = (s + (1 - e)/2, 0)$. (For the notation see Section 2.2.)

Proof. Notice that an Iwahori-Hecke algebra over k of type B_n with parameters q^2 and q^{2s+1} is equal to the Ariki-Koike algebra $\mathcal{H}_n := \mathcal{H}_{k,2,n}$ with parameters $u = q^2$, $v_1 = -q^{2s+1}$ and $v_2 = 1$, as explained in Section 2.1. As q is a primitive 2ℓ th root of unity in k , we have $v_1 = u^{s+(1-e)/2}$.

By Proposition 4.3, the Harish-Chandra branching graph is isomorphic to the branching graph of \mathcal{H}_n , $n \geq 0$. The latter is isomorphic to the crystal graph $\mathcal{B}_{\mathbf{s},e}$ by the results of Section 2.3. \square

It seems reasonable to expect that the hypothesis of the above proposition is always satisfied. Unfortunately, a general proof of this result appears to be out of reach at the moment. It would, however, follow from [25, Conjecture 5.5] in conjunction with Lemma 5.5 and Theorem 3.2 of [25], at least if $\ell > 2m + 1$. Thus, in view of Proposition 5.1, Conjecture 5.7 of [25] follows from [25, Conjecture 5.5], up to the labelling of the vertices.

A more conceptual approach to proving the latter conjecture, again up to the labelling of the vertices, would be to reveal a categorification phenomenon, in the very same spirit as [1] for Ariki-Koike algebras, [35] for cyclotomic rational Cherednik algebras, and [6] for cyclotomic quiver Hecke algebras. In our case, it is particularly crucial to understand how a coloring of the arrows in the Harish-Chandra branching graph would arise, and, in turn, give an interpretation of the rest of the abstract crystal data (namely the weight and the functions φ_i and ε_i in Kashiwara's notation [32, Section 7]). Actually, according to Uglov's work [37], the Fock space $\mathcal{F}_{\mathbf{s},e}$ can be seen as a submodule of the integrable $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ -module $\Lambda^{s+(1-e)/2}$ consisting of semi-infinite wedge products. Besides, there is an integrable action of level e of $\mathcal{U}'_{v^{-1}}(\widehat{\mathfrak{sl}}_2)$ on $\Lambda^{s+(1-e)/2}$, as well as an action of a Heisenberg algebra. These three actions pairwise commute, and each element of $\Lambda^{s+(1-e)/2}$ is obtained by acting on some very particular elements ([37, Theorem 4.8]). It would be interesting to use this approach to the Fock space and to look for categorical actions (in the sense of [11]) of these three algebras on the Grothendieck group of category of kG_n -modules which have a filtration by unipotent modules, for n varying. Note that a categorification of the action of the Heisenberg algebra is achieved in [36].

Proposition 5.2. *Let the notation and assumptions be as in Proposition 5.1. Put $G := G_{m+n}$ and $L := L_{m,n}$. Let S be a simple kG -module in the head (or socle) of $R_L^G(X)$. Let $M := L_{m+n-1,1}$ be the maximal pure Levi subgroup of G . Suppose that*

$${}^*R_M^G(S) = S_1 \oplus \cdots \oplus S_r$$

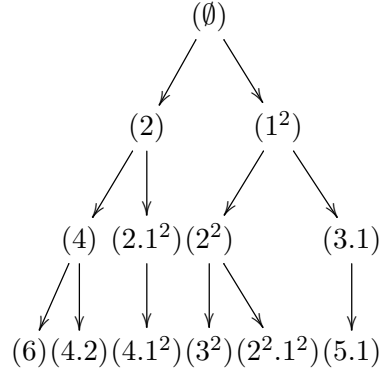
with indecomposable kM -modules S_i , $1 \leq i \leq r$. Then the socle of each S_i is simple and isomorphic to its head.

Proof. Lemma 3.2 implies that each direct summand S_i is contained in $kM\text{-mod}_Z$ with $Z = R_L^M(X)$. The assertion now follows from the corresponding properties of the simple \mathcal{H}_n -modules discussed in Section 2.3 (more precisely [27, Theorem B]), together with Proposition 3.1 and [7, Theorem 2)]. \square

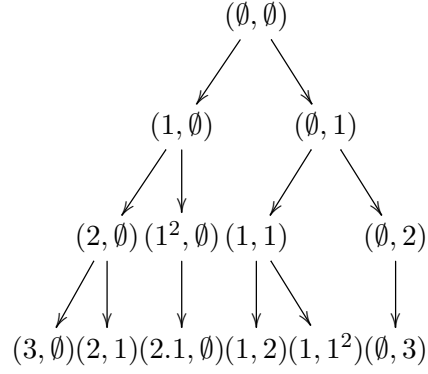
It follows from Proposition 5.1 that the modules in a weak Harish-Chandra series can be labelled by Uglov bipartitions, provided the hypothesis of the proposition is satisfied. The unipotent kG -modules of $G := \text{GU}_r(q)$ are also labelled by partitions of r , as explained in [25, Section 5.3]. Let ν be a partition of r . We then write X_ν for the unipotent kG -module labelled by ν . We will now discuss the question of matching the two labellings in special cases.

We first give an example illustrating why we want to consider Uglov's crystal structure rather than Kleshchev's. In fact, Kleshchev's realization gives rise to bipartitions that do not naturally appear as twisted 2-quotients of the labels of the vertices in the Harish-Chandra branching graph. Recall the combinatorial notions introduced in [25, Section 5.2]. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$, let $\Delta_t := (t, t-1, \dots, 1) = \lambda_{(2)}$ be the 2-core of λ , and $(\lambda^1, \lambda^2) = \lambda^{(2)}$ the 2-quotient of λ . The *twisted 2-quotient* of λ is the bipartition $\bar{\lambda}^{(2)}$ defined by

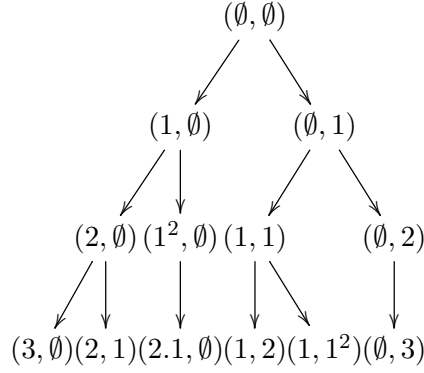
$$\bar{\lambda}^{(2)} = \begin{cases} (\lambda^1, \lambda^2) & \text{if } t \text{ is even} \\ (\lambda^2, \lambda^1) & \text{if } t \text{ is odd.} \end{cases}$$



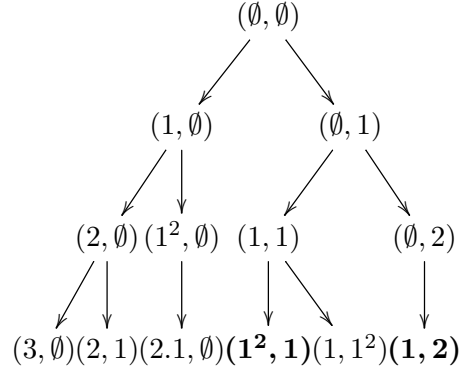
The connected component of the Harish-Chandra branching graph giving the principal series for $\mathrm{GU}_r(q)$, $0 \leq r \leq 6$ even, $e = 3$, under the identification $X_\lambda \leftrightarrow \lambda$.



The connected component of the Harish-Chandra branching graph giving the principal series for $\mathrm{GU}_r(q)$, $0 \leq r \leq 6$ even, $e = 3$, under the identification $X_\lambda \leftrightarrow \overline{\lambda}^{(2)}$.



The crystal graph $\mathcal{B}_{\mathbf{s},e}$ in Uglov's realization, for $\mathbf{s} = (-1, 0)$, $e = 3$, up to rank 3.



The crystal graph $\mathcal{B}_{\mathbf{s},e}$ in Kleshchev's realization, for $\mathbf{s} = (-1, 0)$, $e = 3$, up to rank 3.

The first graph can be derived from the decomposition matrices computed by Dudas and Malle in [13]. Notice that the results of Dudas and Malle require the condition $\ell > 6$, but $e = 3$ implies $\ell \geq 7$ as $2e \mid \ell - 1$. The third graph agrees with the second graph (in particular the labels of the vertices match). However, Uglov's and Kleshchev's realizations of $\mathcal{B}_{\mathbf{s},e}$ already differ in rank 3, as illustrated in the fourth graph. The differences with the Uglov crystal are indicated in boldface

type. Such an example has first been found by Gunter Malle and the second author.

Let us proceed to the main result of this section. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and 2-core $\Delta_t := (t, t-1, \dots, 1)$. Now, let $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$. Further, write $\boldsymbol{\lambda} = \overline{\lambda}^{(2)}$, $\mathbf{c} = (t, 0)$ and set $\mathbf{n}(\boldsymbol{\lambda}) = \sum_{j \geq 1} (j-1)(\kappa_j - \kappa_j^0)$, where κ_j (respectively κ_j^0) denote the elements of the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{c})$ (respectively $\mathfrak{B}(\emptyset, \mathbf{c})$) in decreasing order, see [30, Section 2.2] for the notation.

Remark 5.3. *Lusztig's \mathbf{a} -function for the Iwahori-Hecke algebra of type B_n with parameters q^2 and q^{2t+1} is the function $\mathbf{a} := \mathbf{a}^{\mathbf{m}}$ of Section 2.4 with $\mathbf{m} = (t, 0)$, and is given by the formula $\mathbf{a}(\boldsymbol{\mu}) = 2\mathbf{n}(\boldsymbol{\mu})$, for all $\boldsymbol{\mu} \vdash_2 n$. This follows from [20, Proposition 5.5.11, Example 5.5.14 and Example 1.3.9].*

Lemma 5.4. *For all partition λ , we have $n(\lambda) = \mathbf{a}(\boldsymbol{\lambda})$.*

Proof. The reverse combinatorial procedure to taking the twisted 2-quotient is explained in [25, Section 7.2]. Accordingly, for fixed $t \in \mathbb{N}$, we denote $\Phi_t(\boldsymbol{\lambda})$ the (unique) partition such that $\Phi_t(\boldsymbol{\lambda})_{(2)} = \Delta_t$ and $\overline{\Phi_t(\boldsymbol{\lambda})}^{(2)} = \boldsymbol{\lambda}$. Therefore, we write $\lambda = \Phi_t(\boldsymbol{\lambda})$. We remark that the construction of $\Phi_t(\boldsymbol{\lambda})$ is done via the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{c})$, so that the parts of $\Phi_t(\boldsymbol{\lambda}) = \lambda$ can be read off $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{c})$: they are precisely the integers $2(\kappa_j - \kappa_j^0)$. This, together with Remark 5.3, proves the claim. \square

We are now ready to prove those parts of [25, Conjecture 5.7] which concern the Harish-Chandra series arising from cuspidal modules lifting to cuspidal unipotent defect 0 modules. By formula [15, (8.5)] of Fong and Srinivasan, an ordinary unipotent module labelled by the partition λ has defect 0 if and only if λ is an e -core. Our result generalizes [19, Theorem 5.4] in level 2 as well as [25, Theorem 6.2]. The proof is inspired by the considerations in [18, 2.5]. Remarkably, there is no restriction on ℓ .

Theorem 5.5. *Let $0 \leq s < (e-1)/2$ be an integer, put $m := \lfloor s(s+1)/2 \rfloor$, and let X denote the unipotent cuspidal kG_m -module labelled by Δ_s . Then under the identification $X_\lambda \leftrightarrow \overline{\lambda}^{(2)}$, the Harish-Chandra branching graph $\mathcal{G}_{\mathcal{D}, q, \ell}(X)$ is exactly the crystal graph $\mathcal{B}_{\mathbf{s}, e}$ in Uglov's realization, with $\mathbf{s} = (s + (1-e)/2, 0)$.*

Proof. Fix a non-negative integer n , put $L := L_{m, n}$ and $G := G_{m+n}$. Let $r := 2(m+n) + \iota$ with $\iota \in \{0, 1\}$ such that $2m + \iota = s(s+1)/2$ and $G = G_{m+n} = \mathrm{GU}_r(q)$. Choose an ℓ -modular system (K, \mathcal{O}, k) such that K is large enough for G . By our assumption on s , the triangular

partition Δ_s is an e -core. Hence the cuspidal unipotent KL -module Y labelled by Δ_s is of ℓ -defect 0, and thus reduces irreducibly to the kL -module X . In particular, X is cuspidal.

Let \hat{X} denote an $\mathcal{O}L$ -lattice in Y . Then X and \hat{X} are projective, as Y is of defect 0. It follows that $R_L^G(\hat{X})$ is projective. Write

$$R_L^G(X) = X_1 \oplus X_2 \oplus \cdots \oplus X_h$$

with projective indecomposable kG -modules X_i , $i = 1, \dots, h$. Then

$$R_L^G(\hat{X}) = \hat{X}_1 \oplus \hat{X}_2 \oplus \cdots \oplus \hat{X}_h$$

with indecomposable projective $\mathcal{O}G$ -lattices \hat{X}_i lifting X_i , $i = 1, \dots, h$. For each $i = 1, \dots, h$, the irreducible constituents of $K \otimes_{\mathcal{O}} \hat{X}_i$ are unipotent and thus of the form Y_ν for partitions ν of r with 2-core Δ_s , as they lie in the ordinary Harish-Chandra series determined by (L, Y) .

Now let \leq denote the lexicographic order on the set of partitions of r . For $i = 1, \dots, h$ let

$$\mu(i) := \max\{\nu \mid [K \otimes_{\mathcal{O}} \hat{X}_i : Y_\nu] \neq 0\},$$

and put

$$\Lambda^0 := \{\mu(i) \mid 1 \leq i \leq h\}.$$

Let $1 \leq i, j \leq h$. Geck's result [17] on the triangular shape of the ℓ -modular decomposition matrix of $\mathrm{GU}_r(q)$ implies that $\mu(i) = \mu(j)$ if and only if $X_i \cong X_j$. For each $\mu \in \Lambda^0$ choose i with $1 \leq i \leq h$ and $\mu = \mu(i)$ and write X_μ for the simple head composition factor of X_i . Then the set $\{X_\mu \mid \mu \in \Lambda^0\}$ equals the Harish-Chandra series determined by (L, X) . As the elements of Λ^0 are partitions of r with 2-core Δ_s , the set

$$\bar{\Lambda}^0 := \{\bar{\mu}^{(2)} \mid \mu \in \Lambda^0\}$$

consists of bipartitions of n .

For $R \in \{K, \mathcal{O}, k\}$ put

$$H_R(L, R \otimes_{\mathcal{O}} \hat{X}) := \mathrm{End}_{RG}(R_L^G(R \otimes_{\mathcal{O}} \hat{X})).$$

Then $H_R(L, R \otimes_{\mathcal{O}} \hat{X})$ is an Iwahori-Hecke algebra over R of type B_n with parameters q^2 and q^{2s+1} , viewed as elements of R (see [9, p. 464]).

We denote the Hom-functor with respect to $R_L^G(R \otimes_{\mathcal{O}} \hat{X})$ by F_R .

For a bipartition ν of n and $R \in \{K, k\}$, let S_R^ν denote the Specht module of $H_R(L, R \otimes_{\mathcal{O}} \hat{X})$ associated to ν by Dipper James and Murphy [12, Theorem 4.22] (where we follow [20] in our notational convention). By the results of Fong and Srinivasan in the appendix of [16], we have $F_K(Y_\nu) = S_K^\nu$ with $\nu = \bar{\nu}^{(2)}$ for all $\nu \vdash r$ with 2-core Δ_s . For $\mu \in \Lambda^0$ and $\boldsymbol{\mu} := \bar{\mu}^{(2)}$ put $M^\boldsymbol{\mu} := F_k(X_\mu)$. Then $\{M^\boldsymbol{\mu} \mid \boldsymbol{\mu} \in \bar{\Lambda}^0\}$ is a set of representatives for the simple $H_k(L, X)$ -modules.

We claim that $\bar{\Lambda}^0$ is a canonical basic set for $H_k(L, X)$ as defined in [20, Definition 3.2.1] or [19, Definition 2.4]. For a partition ν of r with 2-core Δ_s and $\mu = \mu(i) \in \Lambda^0$, where $1 \leq i \leq h$, put

$$d_{\nu, \mu} := [K \otimes_{\mathcal{O}} \hat{X}_i : Y_{\nu}].$$

By the result of Geck [17], $d_{\nu, \mu} \neq 0$ implies that either $\nu = \mu$ and $d_{\mu, \mu} = 1$, or else that ν is strictly smaller than μ in the dominance order on partitions. The latter implies that $n(\mu) < n(\nu)$ (see e.g. [21, Exercise 5.6]). Thus $d_{\nu, \mu} \neq 0$ implies that $\nu = \mu$ or $n(\mu) < n(\nu)$. Now

$$d_{\nu, \mu} = [K \otimes_{\mathcal{O}} \hat{X}_i : Y_{\nu}] = [F_K(K \otimes_{\mathcal{O}} \hat{X}_i) : F_K(Y_{\nu})] = [S_k^{\nu} : M^{\mu}]$$

by Brauer reciprocity applied to $(H_K(L, Y), H_{\mathcal{O}}(L, \hat{X}), H_k(L, X))$, as $F_K(Y_{\nu}) = S_K^{\nu}$ and $F_{\mathcal{O}}(\hat{X}_i)$ is the lift of the projective cover of $F_k(X_{\mu}) = M^{\mu}$. Now $n(\nu) = \mathbf{a}(\nu)$ by Lemma 5.4. Thus $[S_k^{\mu} : M^{\mu}] = 1$ and $[S_k^{\nu} : M^{\mu}] \neq 0$ implies that $\nu = \mu$ or $\mathbf{a}(\mu) < \mathbf{a}(\nu)$. Hence $\bar{\Lambda}^0$ is a canonical basic set for $H_k(L, X)$ as claimed.

It now follows from [19, Lemma 5.2 and Example 5.6] that $\bar{\Lambda}^0$ equals the set $\mathcal{U}_{\mathbf{s}, e}$ with $\mathbf{s} = (s + (1 - e)/2, 0)$, thus proving our assertion. \square

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