HARISH-CHANDRA SERIES IN FINITE UNITARY GROUPS AND CRYSTAL GRAPHS

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Abstract. The distribution of the unipotent modules (in non-defining prime characteristic) of the finite unitary groups into Harish-Chandra series is investigated. We formulate a series of conjectures relating this distribution with the crystal graph of an integrable module for a certain quantum group. Evidence for our conjectures is presented, as well as proofs for some of their consequences for the crystal graphs involved. In the course of our work we also generalize Harish-Chandra theory for some of the finite classical groups, and we introduce their Harish-Chandra branching graphs.

1. Introduction

Harish-Chandra theory provides a means of labelling the simple modules of a finite group G of Lie type in non-defining characteristics, including 0. The set of simple modules of G (up to isomorphism) is partitioned into disjoint subsets, the Harish-Chandra series, each arising from a cuspidal simple module of a Levi subgroup of G. Inside each series, the modules are classified by the simple modules of an Iwahori-Hecke algebra arising from the the cuspidal module which representing the series.

This yields, however, a rather indirect labelling of the simple modules, as it requires the classification of the cuspidal simple modules. Moreover, for each of these, the corresponding Iwahori-Hecke algebra has to be computed and its simple modules have to be classified. This program has been completed successfully by Lusztig for modules over fields of characteristic 0 (see [38]). For modules over fields of positive characteristic, only partial results are known.

In some cases a different labelling of the simple modules of G is known. This arises from Lusztig's classification of the simple modules

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in characteristic 0, together with sufficient knowledge of Brauer's theory of decomposition numbers. This applies in particular to the general linear groups $G = \mathrm{GL}_n(q)$ and the general unitary groups $G = \mathrm{GU}_n(q)$, where the unipotent modules (in any non-defining characteristic) are labelled by partitions of n. For characteristic 0 this result is due to Lusztig and Srinivasan [39], for prime characteristic it follows from work of Dipper [3] and Geck [10]. In these cases it is natural to ask how to determine the partition of the unipotent modules into Harish-Chandra series from these labels of the unipotent modules, i.e. from the partitions of n.

By work of Dipper and Du (see [5, Section 4]), this can be done for the general linear groups. First attempts to find a similar description for the unitary groups are described in [13]. It turned out, however, that this is possible only in a favourable case, the case of linear characteristic (see [21, Corollary 8.11] in conjunction with the above mentioned results by Dipper and Du). The general description of the Harish-Chandra series of the unitary groups and other classical groups is still open.

In this paper we present a series of conjectures which, when true and proved, will solve generalized versions of this problem, at least for large characteristics.

Let us now describe our main results and conjectures. As above, G denotes a finite group of Lie type, viewed as group with a split BN-pair of characteristic p. We also let ℓ be a prime different from p. In this introduction, by a simple module for G we will always mean an absolutely simple module over a field of characteristic 0 or ℓ. In Section 2 we introduce a generalization of Harish-Chandra theory if G is a unitary, symplectic and odd dimensional orthogonal groups. Thus the Weyl group of G, as group with a BN-pair, is of type B. Instead of using all Levi subgroups for Harish-Chandra induction, we restrict to what we call pure Levi subgroups: those that arise from a connected subset of the Dynkin diagram of G which is either empty or else contains the first node ajacent to the double edge. This way we obtain more cuspidal modules, which we call weakly cuspidal. All main results of Harish-Chandra theory remain valid in this more general context. In particular, we obtain a distribution of the simple modules into weak Harish-Chandra series (Proposition 2.3). The usual Harish-Chandra series are unions of weak Harish-Chandra series. In characteristic 0, the two notions coincide for unipotent modules, as a Levi subgroup having a unipotent cuspidal module is pure by Lusztig's classification.

In Section 3 we prove some results on the endomorphism ring of a Harish-Chandra induced weakly cuspidal module. Theorem 3.2 states that, under some mild restrictions, this endomorphism ring is in fact an Iwahori-Hecke algebra of type B. Some information about the parameters of this algebra are also given. For example, if a simple weakly cuspidal module in characteristic ℓ lies in a block containing an ordinary cuspidal module, then the parameters of the two Iwahori-Hecke algebras are related through reduction modulo ℓ.

In Section 4 we define the Harish-Chandra branching graph for the unipotent modules of the classical groups considered. This graph records the socle composition factors of Harish-Chandra induced unipotent modules, very much in the spirit of Kleshchev's branching rules for modules of symmetric groups (see [33, 34, 35, 36], in particular [34, Theorem 0.5]).

Section 5 contains our conjectures. These are restricted to the case of the unitary groups. We thus let $G = \mathrm{GU}_n(q)$ from now on and we write e for the multiplicative order of $-q$ in a field of characteristic ℓ. Following [21, Definition 5.3], we call ℓ linear for G, if e is even. For our conjectures, however, we assume that e is odd and larger than 1, so that in particular ℓ is non-linear for G. (The case $e = 1$, i.e. $\ell \mid q+1$ has been settled in [14].) Conjecture 5.4 concerns the relation between Harish-Chandra series of ordinary modules and those in characteristic ℓ. It predicts that if two unipotent modules of G, labelled by the partitions λ and μ, respectively, lie in the same weak Harich-Chandra series, then λ and μ have the same 2-core, i.e. the ordinary unipotent modules labelled by these two partitions also lie in the same Harish-Chandra series. In this sense the ℓ-modular Harish-Chandra series (of unipotent modules) form a refinement of the ordinary Harish-Chandra series. According to Conjecture 5.5, the e-core of λ should be a 2-core, if λ labels a weakly cuspidal unipotent module. This amounts to the assertion that if a unipotent ℓ-block contains a weakly cuspidal module, then the block also contains an ordinary cuspidal module (not necessarily unipotent). Conjecture 5.7 relates the Harish-Chandra branching graphs with crystal graphs arising from canonical bases in submodules of Fock spaces of level 2, which are acted on by the quantum group $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$. This is in analogy to the case of Kleshchev's branching graph in characteristic p, which is isomorphic to the crystal graph of a Fock space of level 1 with an action of the quantum group $\mathcal{U}_v'(\widehat{\mathfrak{sl}_p})$ (see [33, 34, 35, 36]). The conjecure is also put in perspective by the results of Shan [40] on the branching rules on the category $\mathcal{O}$ of the cyclotomic rational double affine Hecke algebras. Finally, Conjecture 5.8 is just a weaker form of Conjecture 5.7. Its statement gives an algorithm to compute the distribution of the unipotent modules in characteristic ℓ

into weak Harish-Chandra series from the combinatorics of the crystal graph involved. In our conjectures we assume that ℓ is large enough (compared to n), without specifying any bound. In the computed examples, $\ell > n$ is good enough.

In Section 6 we collect our evidence for the conjectures. In Theorem 6.2 we prove that Conjecture 5.8 holds for some subgraphs of the Harish-Chandra branching graph and the crystal graph, respectively. It is a generalization of the main result of Geck [11] for principal series to other ordinary Harish-Chandra series. Similarly, Theorem 6.6 asserts that parts of our conjectures hold for blocks of weight 1, i.e. blocks with cyclic defect groups. We also compute the parameters of the Iwahori-Hecke algebra corresponding to a weakly cuspidal module under the assumption that Conjecture 5.5 holds true (Proposition 6.3). Finally, the truth of Conjectures 5.7 and 5.8 implies an isomorphism of certain connected components of crystal graphs with different parameters. This is discussed in 6.4.

In Section 7 we prove that the consequences implied by the conjectures for the crystal graphs are indeed true. This adds more evidence to our conjectures. Conjecture 5.8 implies that a weakly cuspidal module is labelled by a partition which gives rise to a highest weight vertex in the crystal graph. Such partitions can be characterized combinatorially (see [28]). We prove in Theorem 7.6 that the corresponding e-core is indeed a 2-core, as predicted by Conjecture 5.5. In [13, Theorem 8.3] we had proved that the unipotent module of G labelled by the partition (1^n) is cuspidal if and only if ℓ divides n or $n-1$. We prove that the anologous statement holds for corresponding vertices of the crystal graph (Proposition 7.5). Another consequence is stated in Corollary 7.7. Suppose that λ labels a weakly cuspidal module of G and that the 2-core of λ is different from λ and contains more than one node. Then there is a particular e-hook of λ such that the partition λ' obtained from λ by removing this e-hook also labels a weakly cuspidal module, and the two weakly cuspidal modules should give rise to isomorphic Harish-Chandra branching graphs. This is remarkable as n and $n-e$ have different parities and the modules of $G = \mathrm{GU}_n(q)$ and $\mathrm{GU}_{n-e}(q)$ are not directly related via Harish-Chandra induction. We prove in Theorem 7.8 that, as predicted in 6.4, the two connected components in question are isomorphic (as unlabelled) graphs. A further consequence of our conjectures is stated in Corollary 7.9: non-isomorphic composition factors of the socles of modules Harish-Chandra induced from $G = \mathrm{GU}_n(q)$ to $\mathrm{GU}_{n+2}(q)$, lie in different ℓ-blocks.

Let us finally comment on the history of this paper. First notes of the second author date back to 1993, following the completion of [13].

There, a general conjecture for the distribution of the simple modules of a unitary group into Harish-Chandra series for the linear prime case was presented. This conjecture was later verified in [21]. A further conjecture of [13] for the case that ℓ divides $q+1$ was proved in [14]. The conjectures in [13] were based on explicit decomposition matrices of unipotent modules of $\mathrm{GU}_n(q)$, computed by Gunter Malle. These decomposition matrices were completely known in the linear prime case for $n \leq 10$ and published in [13]. At that time, the information in the non-linear prime case was less comprehensive. Much more complete versions of these decomposition matrices and the distribution of the unipotent modules into Harish-Chandra series are now available by the recent work [7] of Dudas and Malle.

Since the publication of [13], many attempts have been made to find the combinatorial pattern behind the Harish-Chandra series of the unitary groups. The breakthrough occurred in 2009, when the second and last author shared an office during a special program at the Isaac Newton Institute in Cambridge. The paper [11] by Geck and some other considerations of the second author suggested that the simple modules of certain Iwahori-Hecke algebras of type B should label some unipotent modules of the unitary groups. The paper [15] by Geck and the third author on canonical basic sets then proposed the correct labelling by Uglov bipartitions. This set of bipartitions is defined through a certain crystal graph, called $\mathcal{G}_{\mathbf{c},e}$ below. The two authors compared their results on these crystal graphs on the one hand, and on the known Harish-Chandra distribution on the other hand. Amazingly, the two results matched.

2. A generalization of Harish-Chandra theory

Here we introduce a generalization of Harish-Chandra theory for certain families of classical groups by restricting the set of Levi subgroups.

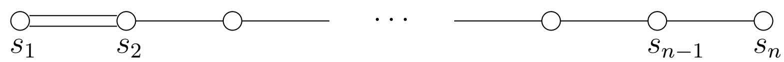
2.1. Let q be a power of the prime p. For a non-negative integer n let $G := G_n := G_n(q)$ denote one of the following classical groups, where we label the cases according to the (twisted) Dynkin type of the groups:

- $({}^2A_{2n-1})$**:** $\mathrm{GU}_{2n}(q)$,
- $({}^2A_{2n})$**:** $\mathrm{GU}_{2n+1}(q)$,
- (B_n)**:** $\mathrm{SO}_{2n+1}(q)$,
- (C_n)**:** $\mathrm{Sp}_{2n}(q)$.

(We interpret $\mathrm{GU}_0(q)$ and $\mathrm{Sp}_0(q)$ as the trivial group.)

If $n \geq 1$, the group G is a finite group with a split BN-pair of characteristic p, satisfying the commutator relations. In these cases, the Weyl group W of G is a Coxeter group of type B_n, and we number the set $S = \{s_1, \ldots, s_n\}$ of fundamental reflections of W according to the following scheme.

(1)



2.2. A subset $I \subseteq S$ is called *left connected*, if it is of the form $I = \{s_1, s_2, \ldots, s_r\}$ for some $0 \leq r \leq n$. The corresponding standard Levi subgroup L_I of G is denoted by $L_{r,n-r}$. A Levi subgroup L of G is called *pure*, if it is conjugate in N to a standard Levi subgroup L_I with I left connected. The set of all pure Levi subgroups of G is denoted by $\mathcal{L}^*$, whereas $\mathcal{L}$ denotes the set of all N-conjugates of all standard Levi subgroups of G. If $L \in \mathcal{L}^*$, a *pure Levi subgroup of* L is an element $M \in \mathcal{L}^*$ with $M \leq L$.

Notice that the set of N-conjugacy classes in $\mathcal{L}^*$ is linearly ordered in the following sense. Let $L, M \in \mathcal{L}^*$. Then $|L| < |M|$ if and only if there is $x \in N$ such that ${}^xL \leq M$. In particular, $|L| = |M|$ if and only if L and M are conjugate in N.

Put $\delta := 2$, if $G_n(q) = \mathrm{GU}_n(q)$, and $\delta := 1$, otherwise. Then the standard Levi subgroup $L_{r,n-r}$ of G has structure

$$L_{r,n-r} \cong G_r(q) \times \mathrm{GL}_1(q^\delta) \times \cdots \times \mathrm{GL}_1(q^\delta)$$

with $n-r$ factors $\mathrm{GL}_1(q^\delta)$, and with a natural embedding of the direct factors of $L_{r,n-r}$ into G.

Lemma. *Let I and J be two left connected subsets of S, and let $x \in D_{IJ}$, where $D_{IJ} \subseteq W$ denotes the set of distinguished double coset representatives with respect to the parabolic subgroups W_I and W_J of W. Then ${}^xI \cap J$ is left connected.*

Proof. We identify W with the set of permutations π of $\{\pm i \mid 1 \leq i \leq n\}$ satisfying $\pi(-i) = -\pi(i)$ for all $1 \leq i \leq n$. If $J = \emptyset$, there is nothing to prove. Thus assume that $J = \{s_1, \ldots, s_r\}$ for some $1 \leq r \leq n$. Then W_J is the stabilizer of the subset $\{\pm i \mid 1 \leq i \leq r\}$ and all the singletons not in this set. It follows that ${}^xW_I \cap W_J$ is the stabilizer of a set $\{\pm i \mid i \in Z\}$ and all the singletons not in this set, where $Z \subseteq \{1, \ldots, r\}$.

On the other hand, if $J' := {}^xI \cap J$, then ${}^xW_I \cap W_J = W_{J'}$, as $x \in D_{IJ}$. This implies that J' is left connected, as otherwise $W_{J'}$ would not be a stabilizer as above. $\square$

Götz Pfeiffer has informed us of a different proof of the above result, using the descent algebra of W. Pfeiffer's proof also applies to Weyl groups of type A and D.

Proposition. *Let L, M be pure Levi subgroups of G, and let $x \in N$. Then ${}^xL \cap M$ is a pure Levi subgroup of G.*

Proof. We may assume that $L = L_I$ and $M = L_J$ for $I, J \subseteq S$ left connected. As ${}^xL_I \cap L_J$ is conjugate in N to ${}^yL_I \cap L_J$, where $y \in D_{IJ}$, we may also assume that $x \in D_{IJ}$. Then ${}^xI \cap J$ is left connected by the lemma. This completes the proof. □

2.3. Let k be a field of characteristic $\ell \neq p \geq 0$, such that k is a splitting field for all subgroups of G. We write kG-mod for the category of finite-dimensional kG-modules. It is known that Harish-Chandra philosophy for kG carries over to the situation where $\mathcal{L}$ is replaced by $\mathcal{L}^*$. The first ideas in this direction go back to Grabmeier's thesis [19], who replaced Green correspondence in symmetric groups by a generalized Green correspondence with respect to Young subgroups. Further developments are due to Dipper and Fleischmann [6]. A comprehensive treatment including several new aspects can be found in [2, Chapter 1]. The crucial ingredient in this generalization is Proposition 2.2.

Let $L \in \mathcal{L}$. We write R_L^G and ${}^*R_L^G$ for Harish-Chandra induction from kL-mod to kG-mod and Harish-Chandra restriction from kG-mod to kL-mod, respectively. For $X \in kL$-mod we put

$$H_k(L, X) := \mathrm{End}_{kG}(R_L^G(X))$$

for the endomorphism algebra of $R_L^G(X)$.

Let $X \in kG$-mod. We say that X is *weakly cuspidal*, if ${}^*R_L^G(X) = 0$ for all $G \neq L \in \mathcal{L}^*$. A pair (L, X) with $L \in \mathcal{L}^*$ and X a weakly cuspidal simple kL-module is called a *weakly cuspidal pair*. Let (L, X) be a weakly cuspidal pair. Then the *weak Harish-Chandra series* defined by (L, X) consists of the simple kG-modules which are isomorphic to submodules of $R_L^G(X)$. If $Y \in kG$-mod lies in the weak Harish-Chandra series defined by (L, X), then $L \in \mathcal{L}^*$ is minimal with ${}^*R_L^G(Y) \neq 0$, and X is a composition factor of ${}^*R_L^G(Y)$.

We collect a few important facts about weak Harish-Chandra series.

Proposition. *Let (L, X) be a weakly cuspidal pair.*

(a) *Write*

$$R_L^G(X) = Y_1 \oplus \cdots \oplus Y_r$$

with indecomposable modules Y_i, $1 \leq i \leq r$. Then each Y_i has a simple head Z_i, which is also isomorphic to the socle of Y_i. Moreover, $Y_i \cong Y_j$, *if and only if $Z_i \cong Z_j$. The Harish-Chandra series defined by (L, X) consists of the kG-modules isomorphic to the Z_i.*

(b) *The weak Harish-Chandra series partition the set of isomorphism types of the simple kG-modules.*

(c) *The weak Harish-Chandra series defined by (L, X) is contained in a usual Harish-Chandra series, and thus every usual Harish-Chandra series is partitioned into weak Harish-Chandra series.*

Proof. It follows from [2, Theorems 1.20(iv), 2.27] that $H_k(L, X)$ is a symmetric k-algebra (notice that the cited results are also valid in our situation where $\mathcal{L}$ is replaced by $\mathcal{L}^*$). This implies the statements of (a) (see, e.g. [2, Theorem 1.28]).

The proof of (b) is analogous to the proof in the usual Harish-Chandra theory.

To prove (c), let $M \in \mathcal{L}$, and let $Z \in kM$-mod be cuspidal (in the usual sense) such that X occurs in the socle of $R_M^L(Z)$. Then $R_L^G(X)$ is a submodule of $R_L^G(R_M^L(Z)) \cong R_M^G(Z)$, and thus every simple module in the socle of $R_L^G(X)$ also occurs in the socle of $R_M^G(Z)$ and hence in the usual Harish-Chandra series defined by (M, Z). $\square$

2.4. Let (L, X) be a weakly cuspidal pair. The following proposition gives information about those composition factors of $R_L^G(X)$ that do not lie in the weak Harish-Chandra series defined by (L, X). The corresponding result for usual Harish-Chandra series is implicitly contained in [23, Lemma 5.7] (see the remarks in [13, (2.2)]). Since this result is particularly relevant in the definition of the Harish-Chandra branching graph, and since it is not explicitly formulated in [23, Lemma 5.7], and wrongly stated in [12, Proposition 2.11(b)], we give a proof here.

Proposition. *Let (L, X) be a weakly cuspidal pair, and let Y be a composition factor of $R_L^G(X)$. Suppose that Y lies in the weak Harish-Chandra series defined by (M, Z), a weakly cuspidal pair.*

Then there is $x \in N$ such that ${}^xL \leq M$. If ${}^xL = M$, then $Z \cong {}^xX$. In particular, if Y does not lie in the weak Harish-Chandra series defined by (L, X), then $|L| < |M|$.

Proof. Let $P(Z)$ denote the projective cover of Z. We have

$$0 \neq [P(Z), {}^*R_M^G(Y)] = [R_M^G(P(Z)), Y],$$

the inequality arising from the fact that Z is a composition factor of ${}^*R_M^G(Y)$, the equation arising from adjointness. As Y is a composition factor of $R_L^G(X)$, we obtain

$$0 \neq [R_M^G(P(Z)), R_L^G(X)] = \sum_{x \in D_{M,L}} [P(Z), R^M_{M\cap {}^xL}({}^*R^{{}^xL}_{M\cap {}^xL}({}^xX))].$$

(Here, $D_{M,L} \subseteq N$ denotes a suitable set of representatives for double cosets with respect to parabolic subgroups of G with Levi complements M and L, respectively.) Thus there is $x \in D_{M,L}$ such that $[P(Z), R^M_{M\cap {}^xL}({}^*R^{{}^xL}_{M\cap {}^xL}({}^xX))] \neq 0$. As (L, X) is a weakly cuspidal pair, so is $({}^xL, {}^xX)$. It follows that $M \cap {}^xL = {}^xL$, and thus ${}^xL \leq M$. If ${}^xL = M$, we obtain $[P(Z), {}^xX] \neq 0$, hence our claim. □

2.5. If $\mathrm{char}(k) = 0$, a kG-module is *unipotent*, if it is simple and its character is unipotent. If $\ell > 0$, a kG-module is *unipotent*, if it is simple and its Brauer character (with respect to a suitable ℓ-modular system) is a linear combination of unipotent characters (restricted to ℓ'-elements).

As $\mathcal{L}^* \subseteq \mathcal{L}$, every cuspidal kG-module X is weakly cuspidal. The converse is not true, as the following example shows. Let $G = \mathrm{GU}_6(q)$ and suppose that $\ell > 6$ and divides $q^2 - q + 1$. The Levi subgroup $L = \mathrm{GL}_3(q^2)$ (a Levi complement of the stabilizer of a maximal isotropic subspace of the natural vector space of G), contains a cuspidal unipotent kL-module X by [13, Theorem 7.6]. By applying [14, Lemma 3.16] and [20, Proposition 2.3.5] we find that $R_L^G(X)$ is indecomposable. Let Y denote the unique head composition factor of $R_L^G(X)$ (see [14, Theorem 2.4]). By construction, Y is not cuspidal, but weakly cuspidal. (The kG-module Y has label 2^3 in the notation of [7, Table 8]).

Now suppose that $\ell = 0$. Then a weakly cuspidal unipotent kG-module is cuspidal. Indeed, $\mathrm{GL}_n(q^\delta)$ has a cuspidal unipotent module over k only if $n = 1$. In particular, if $L \in \mathcal{L}$ has a cuspidal unipotent module over k, then $L \in \mathcal{L}^*$. If X is a weakly cuspidal unipotent kG-module and $L \in \mathcal{L}$ is minimal with ${}^*R_L^G(X) \neq 0$, every constituent of ${}^*R_L^G(X)$ is cuspidal. Thus $L \in \mathcal{L}^*$ and hence, as X is weakly cuspidal, $L = G$.

3. The endomorphism algebra of Harish-Chandra induced weakly cuspidal modules

In important special cases the endomorphism algebras $H_k(L, X)$ of weakly cuspidal pairs (L, X) are Iwahori-Hecke algebras. The result applies in particular when X is unipotent.

We keep the notation of Section 2, except that we assume that $n \geq 1$ here. Thus if $G = G_n(q)$ is one of the groups introduced in 2.1, then G

has a split BN-pair of rank n. Let ℓ be a prime not dividing q. We choose an ℓ-modular system $(K, \mathcal{O}, k)$ such that K is large enough for G. That is, $\mathcal{O}$ is a complete discrete valuation ring with field of fractions K of characteristic 0, and residue class field k of characteristic ℓ. Moreover, K is a splitting field for all subgroups of G.

3.1. Put $r := n-1$ and $L := L_{r,1} \in \mathcal{L}^*$. Thus $L = M \times T$ with $M \cong G_r(q)$ and $T \cong \mathrm{GL}_1(q^\delta)$. (In case $n = 1$, either M is the trivial group, or cyclic of order $q+1$ if $G = \mathrm{GU}_3(q)$.) Let P denote the standard parabolic subgroup of G with Levi complement L and let U denote its unipotent radical. We have $|W_G(L)| = 2$ and we let $s \in N_G(L)$ denote an inverse image of the involution in $W_G(L)$. We choose s of order 2 if G is unitary or orthogonal, and of order 4 with $s^2 \in T$ if G is symplectic, and such that s centralizes M. (Such an s always exists.)

Let R be one of the rings K, $\mathcal{O}$, or k. As M is an epimorphic image of P, we get a surjective homomorphism $\pi : RP \to RM$. Consider the element

$$y := \sum_{\substack{u,u' \in U \\ su'su \in P}} su'su \in RP. \tag{2}$$

Then $z := \pi(y) \in Z(RM)$ as s centralizes M.

Lemma. *With the above notation, $z = (q-1)z'$ for some $z' \in Z(RM)$. In case G is a unitary group, we have $z' = 1 + (q+1)z''$ for some $z'' \in Z(RM)$.*

Proof. We first claim that $T \cong \mathrm{GL}_1(q^\delta)$ acts on

$$\mathcal{U} := \{(u', u) \in U \times U \mid su'su \in P\}$$

by

$$x.(u', u) := (sxs^{-1}u'sx^{-1}s^{-1}, xux^{-1}), \quad x \in T, (u', u) \in \mathcal{U}.$$

Indeed,

$$s(sxs^{-1}u'sx^{-1}s^{-1})s(xux^{-1})s = (s^2xs^{-2})su'sus(s^{-1}x^{-1}s) \tag{3}$$

for $x \in T, (u', u) \in \mathcal{U}$. As s normalizes T, the claim follows. Now $\pi(x) = 1$ for $x \in T$ and thus (3) implies $\pi(su'sus) = \pi(sv'sv)$ if $(u', u), (v', v) \in \mathcal{U}$ are in the same T-orbit.

The claims in the arguments below can be verified by a direct computation in G. Suppose that G is a unitary or symplectic group. For each $1 \neq u \in Z(U)$ there is a unique $u' \in Z(U)$ such that $(u', u) \in \mathcal{U}$. For every such pair we have $\pi(su'sus) = 1$. The elements $(u', u) \in \mathcal{U}$ with $u \notin Z(U)$ lie in regular T-orbits, as T acts fixed point freely on

$U \setminus Z(U)$ by conjugation. This implies our result, as $|Z(U)| = q$ and $|T| = q^\delta - 1$. Now suppose that G is an orthogonal group. Then T acts with regular orbits on $U\setminus\{1\}$, hence on $\mathcal{U}$, again implying our result. $\square$

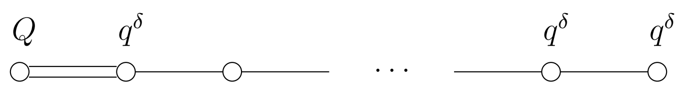
3.2. Let R be one of K or k. If X is an indecomposable RG-module, we let ω_X denote the central character of RG determined by the block containing X.

Let r be an integer with $0 \leq r \leq n$ and put $m := n - r$. Let $L := L_{r,m} \in \mathcal{L}^*$ denote the standard Levi subgroup of $G = G_n(q)$ isomorphic to $G_r(q) \times \mathrm{GL}_1(q^\delta)^m$. Write M and T for the direct factors of L isomorphic to $G_r(q)$ and $\mathrm{GL}_1(q^\delta)^m$, respectively. Let X be a weakly cuspidal simple RM-module, extended trivially to an RL-module.

For $R = K$ and X cuspidal, the following result is due to Lusztig (see [37, Section 5]).

Theorem. *With the above notation, $H_R(L, X)$ is an Iwahori-Hecke algebra corresponding to the Coxeter group of type B_m, with parameters as in the following diagram.*

(4)



The parameter Q is determined as follows. Let U and z be as in 3.1, applied to G_{r+1}. Put $\gamma := \omega_X(z) \in R$ and let $\xi \in R$ be a solution of the quadratic equation

$$x^2 - \gamma\, x - |U| = 0.$$

Then

$$Q = \frac{\xi\gamma}{|U|} + 1.$$

Moreover, the following statements hold.

(a) *Suppose that $R = k$ and that X lies in a block containing a cuspidal KM-module Y. If $\hat{Q}$ is the parameter of $H_K(L, Y)$ associated to the leftmost node of the diagram (4), then Q is the reduction modulo ℓ of $\hat{Q}$.*

(b) *If $R = k$ and $\ell \mid q - 1$, then $Q = 1$.*

(c) *If $R = k$ and $\ell \mid q + 1$, then $Q = -1$.*

Proof. First notice that we have $W_G(L, X) = W_G(L)$, and that $W_G(L)$ is isomorphic to a subgroup of W and a Coxeter group of type B_m (see [25]). We also have

$$\dim_R(H_R(L, X)) = |W_G(L)|.$$

Put $\mathcal{N}(L) := (N_G(L) \cap N)L$ (recall that G has a BN-pair), so that $W_G(L) = \mathcal{N}(L)/L$. Then $\mathcal{N}(L) = M \times C$ with $T \leq C$ and $C/T \cong W_G(L)$. In particular, we may view X as an $R\mathcal{N}(L)$-module on which C acts trivially.

The parameters not corresponding to the leftmost node of (4) can now be computed exactly as in the case where X is cuspidal and unipotent (see [14, Proposition 4.4]).

To determine Q we may assume that $m = 1$. Thus $G = G_{r+1}(q)$ and $L \cong M \times \mathrm{GL}_1(q^2)$. We are thus in the situation of 3.1 and make use of the notation introduced there. Then $H := H_R(L, X)$ is 2-dimensional over R with basis elements B_1 and B_s, where B_1 is the unit element of H and B_s is defined as follows. We may realize $R_L^G(X)$ as

$$R_L^G(X) = \{f : G \to X \mid f(hg) = h.f(g), \text{ for all } h \in P, g \in G\}.$$

Then B_s is defined by

$$B_s(f)(g) := \frac{1}{|U|} \sum_{u \in U} f(sug), \qquad f \in R_L^G(X), g \in G,$$

as $s \in C$ acts trivially on X. We have $B_s^2 = \zeta B_1 + \eta B_s$ with $\zeta = 1/|U|$, and η such that the element y of (2) acts as the scalar $|U|\eta$ on X. This is proved exactly as in [26, Proposition 3.14].

Now y acts in the same way on X as $z = \pi(y)$. Since X is absolutely irreducible, $z \in Z(RM)$ acts by the scalar $\omega_X(z)$. Thus $|U|\eta = \omega_X(z) = \gamma$. Put

$$T_s := \xi B_s, \quad T_1 := B_1.$$

Then

$$T_s^2 = QT_1 + (Q-1)T_s$$

with $Q = \xi\eta + 1$. This gives our first claim.

To prove (a), put $\hat{\gamma} := \omega_Y(z)$, and let $\hat{\xi}$ be a solution of $x^2 - \hat{\gamma}x - |U| = 0$. Observe that $\hat{\gamma}, \hat{\xi} \in \mathcal{O}$. Then the reduction modulo ℓ of $\hat{\gamma}$ equals $\omega_X(z)$, and the reduction modulo ℓ of $\hat{\xi}$ is a solution of $x^2 - \gamma x - |U| = 0$. Thus the reduction modulo ℓ of $\hat{Q} := \hat{\xi}\hat{\eta} + 1$ equals $\xi\eta + 1 = Q$ and (a) is proved.

Suppose now that $R = k$. If $\ell \mid q - 1$, we have $\gamma = 0$ by Lemma 3.1 and thus $Q = 1$. If G is unitary and $\ell \mid q + 1$, we have $\gamma = -2$, again by Lemma 3.1. Also, $|U|$ is an odd power of q, i.e. $|U| = -1$ in k, hence $\xi = -1$ and $Q = -1$. This completes our proof. $\square$

4. The Harish-Chandra branching graph

In this section we fix a prime power q of p and a prime $\ell \neq p$. We also let k denote an algebraically closed field of characteristic ℓ.

4.1. For $n \in \mathbb{N}$, we let $G := G_n := G_n(q)$ denote one of the groups of 2.1. Recall that G_n is naturally embedded into G_{n+1}, by embedding G_n into the pure Levi subgroup $L_{n,1} \cong G_n \times \mathrm{GL}_1(q^\delta)$ of G_{n+1}. By iterating, we obtain an embedding of G_n into G_{n+m} for every $m \in \mathbb{N}$.

By kG-mod^u we denote the full subcategory of kG-mod consisting of the modules that have a filtration by unipotent kG-modules. By the result of Broué and Michel [1], and by [22], kG-mod^u is a direct sum of blocks of kG. The above embedding of G_n into G_{n+m} yields a functor

$$R_n^{n+m} : kG_n\text{-mod}^u \to kG_{n+m}\text{-mod}^u,$$

defined by

$$R_n^{n+m}(X) := R_{L_{n,m}}^{G_{n+m}}(\mathrm{Infl}_{G_n}^{L_{n,m}}(X)), \quad X \in kG_n\text{-mod}^u,$$

where $\mathrm{Infl}_{G_n}^{L_{n,m}}(X)$ denotes the trivial extension of X to $L_{n,m} \cong G_n \times \mathrm{GL}_1(q^\delta)^m$. The adjoint functor

$$^*R_n^{n+m} : kG_{n+m}\text{-mod}^u \to kG_n\text{-mod}^u,$$

is given by

$$R_n^{n+m}(X) := \mathrm{Res}_{G_n}^{L_{n,m}}(^*R_{L_{n,m}}^{G_{n+m}}(X)), \quad X \in kG_{n+m}\text{-mod}^u.$$

Let $\mathcal{R}_n := \mathcal{R}_n(q)$ denote the Grothendieck group of kG_n-mod^u, and put

$$\mathcal{R} := \mathcal{R}(q) := \bigoplus_{n\in\mathbb{N}} \mathcal{R}_n.$$

For an object $X \in kG_n$-mod^u, we let $[X]$ denote its image in $\mathcal{R}_n$.

4.2. The *(twisted) Dynkin type* of G is one of the symbols $^2A_\iota$ with $\iota \in \{0,1\}$, B or C, where $\mathrm{GU}_r(q)$ has twisted Dynkin type $^2A_\iota$ with $\iota \equiv (r \bmod 2)$.

The *Harish-Chandra branching graph* $\mathcal{G}_{\mathcal{D},q,\ell}$ corresponding to q, ℓ and the (twisted) Dynkin type $\mathcal{D}$ is the directed graph whose vertices are the elements $[X]$, where X is a simple object in kG_n-mod^u for some $n \in \mathbb{N}$. Thus the vertices of $\mathcal{G}_{\mathcal{D},q,\ell}$ are the standard basis elements of $\mathcal{R}$. We say that the a vertex $[X]$ has *rank* n, if $[X] \in \mathcal{R}_n$. Let $[X]$ and $[Y]$ be vertices in $\mathcal{G}_{\mathcal{D},q,\ell}$. Then there is a directed edge from $[X]$ to $[Y]$ if and only if there is $n \in \mathbb{N}$ such that $[X]$ has rank n and $[Y]$ has rank $n+1$, and such that Y is a head composition factor of $R_n^{n+1}(X)$. A vertex in $\mathcal{G}_{\mathcal{D},q,\ell}$ is called a *source vertex*, if it has only outgoing edges.

As every unipotent kG-module is self dual, Y is a head composition factor of $R_n^{n+1}(X)$ if and only if Y is in the socle of $R_n^{n+1}(X)$. By adjunction, Y is a head composition factor of $R_n^{n+1}(X)$ if and only if X is in the socle of ${}^*R_n^{n+1}(Y)$, and Y is in the socle of $R_n^{n+1}(X)$ if and only if X is a head composition factor of ${}^*R_n^{n+1}(Y)$.

An example for part of a Harish-Chandra branching graph is displayed in Table 1, where the vertices are represented by their labels. This can be proved with the help of the decomposition matrices computed in [7] plus some ad hoc arguments.

4.3. We have the following relation with the weak Harish-Chandra series of G.

Proposition. *Let $[X]$ be a vertex of rank n of $\mathcal{G}_{\mathcal{D},q,\ell}$. Then $[X]$ is a source vertex if and only if $X \in kG_n$-mod^u is weakly cuspidal.*

Suppose that X is weakly cuspidal and let $m \in \mathbb{N}$. View X as a module of $L_{n,m}$ via inflation. Then a simple object $Y \in kG_{n+m}$-mod^u lies in the weak $(L_{n,m}, X)$ Harish-Chandra series, if and only if there is a directed path from $[X]$ to $[Y]$ in $\mathcal{G}_{\mathcal{D},q,\ell}$.

Proof. Clearly, X is weakly cuspidal if $n = 0$. Assume that $n \geq 1$. Then X is weakly cuspidal if and only if ${}^*R_{n-1}^n(X) = 0$, which is the case if and only if $[X]$ is a source vertex.

Assume now that X is weakly cuspidal, let $m \in \mathbb{N}$ and let $[Y]$ be a vertex of rank $n + m$. Suppose there is a path from $[X]$ to $[Y]$. We proceed by induction on m to show that Y occurs in the head of $R_n^{n+m}(X)$. If $m = 0$, there is nothing to prove. So assume that $m > 0$ and that the claim has been prove for $m - 1$. Let $[Z]$ be a vertex of rank $n+m-1$ that occurs in a path from $[X]$ to $[Y]$. By induction, Z is a head composition factor of $R_n^{n+m-1}(X)$. By exactness, $R_{n+m-1}^{n+m}(Z)$ is a quotient of $R_{n+m-1}^{n+m}(R_n^{n+m-1}(X)) \cong R_n^{n+m}(X)$. As Y is a quotient of $R_{n+m-1}^{n+m}(Z)$, we are done.

Suppose now that Y occurs in the head of $R_n^{n+m}(X)$. We proceed by induction on m to show that there is a path from $[X]$ to $[Y]$, the cases $m \leq 1$ being trivial. As Y is isomorphic to a quotient of $R_n^{n+m}(X) \cong R_{n+m-1}^{n+m}(R_n^{n+m-1}(X))$, there is a composition factor Z of $R_n^{n+m-1}(X)$ such that Y is a quotient of $R_{n+m-1}^{n+m}(Z)$. In particular, there is an edge from $[Z]$ to $[Y]$. If Z occurs in the head of $R_n^{n+m-1}(X)$, there is a path from $[X]$ to $[Z]$ by induction, and we are done. Aiming at a contradiction, assume that Z does not occur in the head of $R_n^{n+m-1}(X)$. Then Z does not lie in the weak Harish-Chandra series of G_{n+m-1} defined by $(L_{n,m-1}, X)$. It follows from Proposition 2.4 that Z lies in the weak Harish-Chandra series defined by

$(L_{n',n-n'+m-1}, X')$ for some $n < n'$ and some weakly cuspidal module X'. In particular, Y lies in this weak Harish-Chandra series. This contradiction completes our proof. □

5. Conjectures

Here we formulate a series of conjectures about the ℓ-modular Harish-Chandra series and the Harish-Chandra branching graph for the unitary groups.

5.1. As always, we let q denote a power of a prime p, and we fix a prime ℓ different from p. The multiplicative order of $-q$ modulo ℓ is denoted by $e := e(q, \ell)$. Thus e is the smallest positive integer such that ℓ divides $(-q)^e - 1$.

For a non-negative integer n we let $G := \mathrm{GU}_n(q)$ be the unitary group of dimension n. Also, $(K, \mathcal{O}, k)$ denotes an ℓ-modular system such that K is large enough for G and with k algebraically closed.

5.2. The set of partitions of a non-negative integer n is denoted by $\mathcal{P}_n$ and we write $\lambda \vdash n$ if $\lambda \in \mathcal{P}_n$. We put $\mathcal{P} := \cup_{n \in \mathbb{N}} \mathcal{P}_n$. Let $\lambda \in \mathcal{P}$. Then $\lambda_{(2)}$ and $\lambda^{(2)}$ denote the 2-core and the 2-quotient of λ, respectively. (As in [9, Section 1], the 2-quotient is determined via a β-set for λ with an odd number of elements, where we use the term β-set in its original sense of being a finite set of non-negative integers as introduced in [30, p. 77f].) For a non-negative integer t we write $\Delta_t := (t, t-1, \ldots, 1)$ for the triangular partition of $t(t+1)/2$. Then $\lambda_{(2)} = \Delta_t$ for some $t \in \mathbb{N}$. Suppose that $\lambda^{(2)} = (\mu^1, \mu^2)$. We then put $\bar{\lambda}^{(2)} := (\mu^1, \mu^2)$ if t is even, and $\bar{\lambda}^{(2)} := (\mu^2, \mu^1)$, otherwise. If $\mu = (\mu^1, \mu^2)$ is a bipartition, we let $\Phi_t(\mu)$ denote the unique partition λ with $\lambda_{(2)} = \Delta_t$ and $\bar{\lambda}^{(2)} = (\mu^1, \mu^2)$ (see [30, Theorem 2.7.30]).

The set of bipartitions of n is denoted by $\mathcal{P}_n^{(2)}$, and we put $\mathcal{P}^{(2)} := \cup_{n \in \mathbb{N}} \mathcal{P}_n^{(2)}$. Finally, we write $\mu \vdash_2 n$ if $\mu \in \mathcal{P}_n^{(2)}$.

5.3. By a result of Lusztig and Srinivasan [39], the unipotent KG-modules are labelled by partitions of n. We write Y_λ for the unipotent KG-module labelled by $\lambda \in \mathcal{P}_n$. Let λ and μ be partitions of n. It follows from the main result of Fong and Srinivasan [8, Theorem (7A)], that Y_λ and Y_μ lie in the same ℓ-block of G, if and only if λ and μ have the same e-core. The *e-weight* and the *e-core* of the ℓ-block containing Y_λ are, by definition, the e-weight and the e-core of λ, respectively.

It was shown by Geck in [10] that if the Y_λ, $\lambda \vdash n$, are ordered downwards lexicographically, the corresponding matrix of ℓ-decomposition numbers is square and upper unitriangular. This defines a labelling of the unipotent kG-modules by partitions of n, and we write X_μ for the unipotent kG-module labelled by $\mu \in \mathcal{P}_n$. Thus X_μ is determined by the following two conditions. Firstly, X_μ occurs exactly once as a composition factor in a reduction modulo ℓ of Y_μ, and secondly, if X_μ is a composition factor in a reduction modulo ℓ of Y_ν for some $\nu \in \mathcal{P}_n$, then $\nu \leq \mu$.

5.4. Our first conjecture asserts a compatibility between ordinary and modular Harish-Chandra series.

Conjecture. *Let $\mu, \nu \in \mathcal{P}_n$. If X_μ and X_ν lie in the same weak Harish-Chandra series of kG-modules, then μ and ν have the same 2-core, i.e. Y_μ and Y_ν lie in the same Harish-Chandra series of KG-modules. (In other words, the partition of $\mathcal{P}_n$ arising from the weak ℓ-modular Harish-Chandra series is a refinement of the partition of $\mathcal{P}_n$ arising from the ordinary Harish-Chandra series.)*

5.5. We also conjecture that a weakly cuspidal unipotent module can only occur in an ℓ-block of G which contains a cuspidal simple KG-module (not necessarily unipotent). In fact, if e is odd, a unipotent ℓ-block $\mathbf{B}$ contains a cuspidal simple KG-module if and only if the e-core of $\mathbf{B}$ is a 2-core. This can be seen as follows. Suppose first that the e-core of $\mathbf{B}$ is the 2-core Δ_s. Put $m' := s(s+1)/2$. Let x be an ℓ-element in G with $C := C_G(x) = (q^e+1)^w \times \mathrm{GU}_{m'}(q)$, where $(q^e+1)^w$ denotes a direct product of w factors of the cyclic group of order q^e+1 (and $n = we+m'$). Let Z denote the cuspidal unipotent KC-module labelled by Δ_s, and let Y be the simple KG-module corresponding to Z under Lusztig's Jordan decomposition. Then Y is cuspidal by [37, 7.8.2], and Y lies in $\mathbf{B}$ by [8, Theorem (7A) and Proposition (4F)]. Conversely, suppose that $\mathbf{B}$ contains some cuspidal simple KG-module Y. Then Y determines a unipotent KC-module, where C is the centralizer in G of some ℓ-element. Let $\mu \in \mathcal{P}$ be the partition labelling Z. Then μ is a 2-core, and in turn, the e-core of μ is a 2-core as well. As the e-core of μ equals the e-core of $\mathbf{B}$, again by [8, Theorem (7A) and Proposition (4F)], our claim follows.

Conjecture. *Let $\lambda \in \mathcal{P}_n$. If X_λ is weakly cuspidal, then the e-core of λ is a 2-core.*

It follows from [21, Corollary 8.8] that if e is even, then X_λ is cuspidal if and only if λ is a 2-core. (In this case, λ also is an e-core.)

Assuming that Conjecture 5.5 holds, the parameter Q of a weakly cuspidal unipotent kG-module X_λ of G can be computed from the e-core of λ by Corollary 6.3 below.

5.6. To present our next conjectures, we first have to introduce the Fock space of level 2 and its corresponding crystal graph. The results summarized below are due to Jimbo, Misra, Miwa and Okado [31] and Uglov [41]. For a detailed exposition see also [16, Chapter 6].

A *charged bipartition* is a pair $(\mu, \mathbf{c})$, written as $|\mu, \mathbf{c}\rangle$ with $\mu \in \mathcal{P}^{(2)}$ and $\mathbf{c} \in \mathbb{Z}^2$. Fix $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2$, and let v denote an indeterminate. The Fock space (of level 2) and charge $\mathbf{c}$ is the $\mathbb{Q}(v)$-vector space

$$\mathcal{F}_{\mathbf{c}} := \bigoplus_{m \in \mathbb{N}} \bigoplus_{\mu \vdash_2 m} \mathbb{Q}(v)|\mu, \mathbf{c}\rangle.$$

Assume that $e \geq 2$. There is an action of the quantum group $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$ on $\mathcal{F}_{\mathbf{c}}$ such that $\mathcal{F}_{\mathbf{c}}$ is an integrable $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$-module and $|\mu, \mathbf{c}\rangle$ is a weight vector for every $m \in \mathbb{N}$ and $\mu \vdash_2 m$. Moreover, $|(-,-), \mathbf{c}\rangle$ is a highest weight vector and $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e}).|(-,-), \mathbf{c}\rangle$ is isomorphic to $V(\Lambda(\mathbf{c}))$, the simple highest weight module with weight $\Lambda(\mathbf{c}) = \Lambda_{c_1 \bmod e} + \Lambda_{c_2 \bmod e}$, where the Λ_i $0 \leq i \leq e-1$ denote the fundamental weights of $\widehat{\mathfrak{sl}_e}$. We write $\mathcal{F}_{\mathbf{c},e}$ when we view $\mathcal{F}_{\mathbf{c}}$ as a $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$-module.

There is a crystal graph $\mathcal{G}_{\mathbf{c},e}$ describing the canonical basis of $\mathcal{F}_{\mathbf{c},e}$. The vertices of $\mathcal{G}_{\mathbf{c},e}$ are all charged bipartitions $|\mu, \mathbf{c}\rangle$, $\mu \vdash_2 m$, $m \in \mathbb{N}$. There is a directed, coloured edge $|\mu, \mathbf{c}\rangle \xrightarrow{i} |\nu, \mathbf{c}\rangle$ if and only if ν is obtained from μ by adding a good i-node, where the colours i are in the range $0 \leq i \leq e-1$. The associated Kashiwara operator $\tilde{f}_i$ acts on $\mathcal{G}_{\mathbf{c},e}$ by mapping the vertex $|\mu, \mathbf{c}\rangle$ to $|\nu, \mathbf{c}\rangle$ if and only if there is an edge $|\mu, \mathbf{c}\rangle \xrightarrow{i} |\nu, \mathbf{c}\rangle$, and to 0, otherwise (see e.g. [16, 6.1]).

Let us now describe, following [16], how to compute the good i-nodes of $|\mu, \mathbf{c}\rangle$, and thus the graph $\mathcal{G}_{\mathbf{c},e}$, algorithmically. A *node* of $\mu = (\mu^1, \mu^2)$ is a triple (a, b, j), where (a, b) is a node in the Young diagram of μ^j, for $j = 1, 2$. A node γ of μ is called addable (respectively removable) if $\mu \cup \{\gamma\}$ (respectively $\mu \backslash \{\gamma\}$) is still a bipartition. The content of $\gamma = (a, b, j)$ is the integer $\mathrm{cont}(\gamma) = b - a + c_j$. The residue of γ is the element of $\{0, 1, \ldots, e-1\}$ defined by $\mathrm{res}(\gamma) = \mathrm{cont}(\gamma) \bmod e$. For $0 \leq i \leq e-1$, γ is called an i-node if $\mathrm{res}(\gamma) = i$.

Fix $i \in \{0, 1, \ldots, e-1\}$, and define an order on the set of addable and removable i-nodes of μ by setting

$$\gamma \prec_{\mathbf{c}} \gamma' \text{ if } \begin{cases} \mathrm{cont}(\gamma) < \mathrm{cont}(\gamma') \text{ or} \\ \mathrm{cont}(\gamma) = \mathrm{cont}(\gamma') \quad \text{and} \quad j > j'. \end{cases}$$

Sort these set of nodes according to $\prec_{\mathbf{c}}$, starting from the smallest one. Encode each addable (respectively removable) i-node by the letter A (respectively R), and delete recursively all occurences of consecutive letters RA. This yields a word of the form $A^{\alpha_i} R^{\beta_i}$, which is called the reduced i-word of μ. Note that by Kashiwara's crystal theory [32, Section 4.2], we have the following expression for the weight of the vector $|\mu, \mathbf{c}\rangle$:

$$\mathrm{wt}(\mu, \mathbf{c}) = \sum_{i=0}^{e-1} (\alpha_i - \beta_i)\Lambda_i. \tag{5}$$

Let γ be the rightmost addable (respectively leftmost removable) i-node in the reduced i-word of μ. Then γ is called the *good addable* (respectively *good removable*) i-node of μ.

Each connected component of $\mathcal{G}_{\mathbf{c},e}$ is isomorphic to the crystal of a simple highest weight module of $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$, whose highest weight vector is the unique source vertex of the component. The *rank* of a vertex $|\mu, \mathbf{c}\rangle$ of $\mathcal{G}_{\mathbf{c},e}$ is m, if $\mu \vdash_2 m$. We write $\mathcal{G}^{\leq m}_{\mathbf{c},e}$ for the induced subgraph of $\mathcal{G}_{\mathbf{c},e}$ containing the vertices of rank at most m.

As an example, the graph $\mathcal{G}^{\leq 3}_{(0,0),3}$ is displayed in Table 3.

5.7. Let t be a non-negative integer, put $r := t(t+1)/2$ and $\iota := r \pmod 2 \in \{0, 1\}$. Then $K\mathrm{GU}_r(q)$ has a unipotent cuspidal module Y, and $(\mathrm{GU}_r(q), Y)$ determines a Harish-Chandra series of unipotent $K\mathrm{GU}_{r+2m}(q)$-modules for every $m \in \mathbb{N}$. Recall from 4.2 that $\mathcal{G}_{^2A_\iota,q,\ell}$ denotes the Harish-Chandra branching graph corresponding to q, ℓ and the groups $\mathrm{GU}_{2n+\iota}(q)$. As we are dealing exclusively with unitary groups in this section, we shall replace the index $^2A_\iota$ by ι in the symbol for the graph. The vertices of $\mathcal{G}_{\iota,q,\ell}$ correspond to the isomorphism classes of the unipotent $k\mathrm{GU}_{2n+\iota}(q)$-modules, where n runs through the set of positive integers. We may thus label the vertices of $\mathcal{G}_{\iota,q,\ell}$ by the set $\cup_{n\in\mathbb{N}}\mathcal{P}_{2n+\iota}$.

To formulate our next conjecture, we assume that Conjecture 5.4 holds. Under this assumption, the induced subgraph of $\mathcal{G}_{\iota,q,\ell}$ whose vertices are labelled by the set of partitions with 2-core Δ_t, is a union of connected components of $\mathcal{G}_{\iota,q,\ell}$. We write $\tilde{\mathcal{G}}^t_{\iota,q,\ell}$ for the graph with vertices $\mathcal{P}^{(2)}$, and a directed edge $\mu \to \nu$, if and only if there is a directed edge in $\mathcal{G}_{\iota,q,\ell}$ between the vertices labelled by $\Phi_t(\mu)$ and $\Phi_t(\nu)$. If $\mu \vdash_2 m$ is a vertex of $\tilde{\mathcal{G}}^t_{\iota,q,\ell}$, the *rank* of this vertex is m. For a non-negative integer d we let $\tilde{\mathcal{G}}^{t,\leq d}_{\iota,q,\ell}$ denote the induced subgraph of $\tilde{\mathcal{G}}^t_{\iota,q,\ell}$ containing the vertices of rank at most d.

Conjecture. *Let the notation be as above. Assume that e is odd and put $\mathbf{c} := (t + (1-e)/2, 0)$. Then there is an integer $b := b(\ell)$ such that $\tilde{\mathcal{G}}^{t,\leq b}_{\iota,q,\ell}$ equals $\mathcal{G}^{\leq b}_{\mathbf{c},e}$, if the colouring of the edges of the latter graph is neglected.*

5.8. As the Harish-Chandra series of unipotent kG-modules can be read off from the Harish-Chandra branching graph by Proposition 4.3, the truth of Conjecture 5.7 would give an algorithm to determine the partition of the kG-modules into weak Harish-Chandra series from the labels of the modules, at least if ℓ is large enough. In particular, the question of whether X_λ is weakly cuspidal, can be read off from λ.

Conjecture. *Let $\lambda \in \mathcal{P}$ and let $t \in \mathbb{N}$ such that $\lambda_{(2)} = \Delta_t$. Let $\mu = \bar{\lambda}^{(2)}$ (see 5.2). Assume that ℓ is large enough, that e is odd and put $\mathbf{c} := (t + (1-e)/2, 0)$.*

Then X_λ is weakly cuspidal, if and only if $|\mu, \mathbf{c}\rangle$ is a source vertex in $\mathcal{G}_{\mathbf{c},e}$.

Suppose that X_λ is weakly cuspidal and let $\rho \in \mathcal{P}$. Then X_ρ lies in the weak Harish-Chandra series defined by X_λ, if and only if $\rho_{(2)} = \lambda_{(2)} = \Delta_t$, and $|\bar{\rho}^{(2)}, \mathbf{c}\rangle$ lies in the connected component of $\mathcal{G}_{\mathbf{c},e}$ containing $|\mu, \mathbf{c}\rangle$, i.e. $|\bar{\rho}^{(2)}, \mathbf{c}\rangle$ is obtained from $|\mu, \mathbf{c}\rangle$ by adding a sequence of good nodes.

6. Some evidence

Here we present the evidence for our conjectures. Keep the notation of Section 5. We also assume that e is odd and larger than 1 in this section.

6.1. Conjecture 5.7 holds for $e = 3, 5$ and the groups $\mathrm{GU}_n(q)$ for $n \leq 10$, if $\ell > n$. In these cases, most of the decomposition numbers and the Harish-Chandra series have been computed by Dudas and Malle [7]. The Harish-Chandra branching graphs can be determined from this information using some additional arguments. The corresponding crystal graphs can be computed with the GAP3 programs written by one of the authors (see [27]).

Conjecture 5.8 holds for $n = 12$ and $e = 3$ if $\ell \geq 13$.

6.2. There are cases where Conjecture 5.8 is known to be true.

Theorem. *Let $0 \leq t < (e-1)/2$ be an integer, put $r := t(t+1)/2$ and let $\lambda := \Delta_t$.*

Let $m \in \mathbb{N}$, put $n := r + 2m$ and $G := \mathrm{GU}_n(q)$. Then

$$L := L_{r,m} \cong \mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$$

is a pure Levi subgroup of G and X_λ is a cuspidal unipotent kL-module.

If ℓ is large enough, the unipotent kG-module X_ρ lies in the Harish-Chandra defined by (L, X_λ) if and only if

$$\bar{\rho}^{(2)} \in \Phi_{e,m}^{(t+(1-e)/2,0)},$$

where $\Phi_{e,m}^{(t+(1-e)/2,0)}$ *denotes the set of Uglov bipartitions of* m. *(See* [15, Definition 4.4]*; the Uglov bipartitions are simply the bipartitions labelling the vertices of the connected component of the crystal graph containing* $|(-,-),\mathbf{c}\rangle$*.)*

Proof. The cuspidal unipotent KG-module Y_λ of $\mathrm{GU}_r(q)$ reduces irreducibly to the unipotent kG-module X_λ (see [13, Theorem 6.10]). In particular, X_λ is cuspidal.

Let $\hat{X}_\lambda$ denote the (unique) $\mathcal{O}L$-lattice in Y_λ. The endomorphism algebra $H_{\mathcal{O}}(L,\hat{X}_\lambda)$ is an Iwahori-Hecke algebra over $\mathcal{O}$ of type B_m with parameters q^{2t+1} and q^2. By a result of Dipper [4, Theorem 4.9], the ℓ-modular decomposition matrix of $H_{\mathcal{O}}(L,\hat{X}_\lambda)$ is embedded into the decomposition matrix of the unipotent KG-modules as a submatrix.

By our assumption, ℓ does not divide the order of L and thus X_λ and $\hat{X}_\lambda$ are projective. It follows that $R_L^G(\hat{X}_\lambda)$ is projective. The corresponding columns of the decomposition matrix of $\mathcal{O}G$ are exactly the columns of the decomposition matrix of $H_{\mathcal{O}}(L,\hat{X}_\lambda)$. Let $\hat{Z}$ be an indecomposable summand of $R_L^G(\hat{X}_\lambda)$ and let Y_ρ be a composition factor of $K\otimes_{\mathcal{O}}\hat{Z}$ with ρ maximal. Then X_ρ equals the head of $k\otimes_{\mathcal{O}}\hat{Z}$ and thus lies in the Harish-Chandra series defined by (L,X_λ). Every element of this series arises in this way.

To proceed, we will make use of the notion of a *canonical basic set* as defined in [16, Definition 3.2.1]. Applying the results of [11, Section 3], we obtain the following facts. Firstly, the Iwahori-Hecke algebra $H_k(L,X_\lambda)$ has a canonical basic set with respect to Lusztig's a-function on $H_k(L,X_\lambda)$ (see [16, p. 13]), if ℓ is large enough. Secondly, this canonical basic set agrees with the canonical basic set of a suitable specialization of a generic Iwahori-Hecke algebra to an Iwahori-Hecke algebra $H_K^{(2e)}$ of type B_m, whose parameters are powers of a $2e$th root of unity. The canonical basic set of $H_k(L,X_\lambda)$ (or rather of the algebra $H_K^{(2e)}$), is determined in [15, Theorem 5.4, Example 5.6]. The elements of this canonical basic set are labelled by the set of Uglov m-bipartitions.

The simple $H_K(L,Y_\lambda)$-modules correspond to the simple constituents of $R_L^G(Y_\lambda)$. Arrange the latter by lexicographically decreasing labels. By [17, Theorem 3.7] and the results of Lusztig summarized in [16, 2.2.12], this ordering corresponds to the ordering of the simple modules of $H_K(L,Y_\lambda)$ via Lusztig's a-function. Through the embedding of the decomposition matrix of $H_{\mathcal{O}}(L,\hat{X}_\lambda)$, the members of the canonical basic set thus correspond to the composition factors of $R_L^G(Y_\lambda)$ which are at the top of their respective columns in the decomposition matrix of $\mathcal{O}G$. As these top composition factors label the kG-modules in the Harish-Chandra series of kG defined by (L, X_λ), our claim follows. $\square$

Theorem 6.2 is true without the assumption that $t < (e-1)/2$ if Conjecture 5.4 holds. Indeed, in this case every unipotent kG-module in the (L, X_λ)-series is labelled by a partition with 2-core Δ_t. Let $\widehat{P(X_\lambda)} \in \mathcal{O}L$-mod denote the projective cover of X_λ. Again by [4, Theorem 4.9], the decomposition matrix of $R_L^G(\widehat{P(X_\lambda)})$ contains the decomposition matrix of $H_{\mathcal{O}}(L, Y_\lambda)$ as a submatrix (with a row of the latter labelled by $\mu \vdash_2 m$ corresponding to a row of the former labelled by $\Phi_t(\mu)$). Let $\hat{Z}$ be an indecomposable summand of $R_L^G(\widehat{P(X_\lambda)})$ such that the head of $k \otimes_{\mathcal{O}} \hat{Z}$ lies in the Harish-Chandra series defined by (L, X_λ). Put $Z := K \otimes_{\mathcal{O}} \hat{Z}$, and let Y_ρ be a unipotent composition factor of Z with ρ maximal. Then X_ρ is the head of $k \otimes_{\mathcal{O}} \hat{Z}$, and hence $\rho_{(2)} = \Delta_t$. It follows as in the proof above that $\bar{\rho}^{(2)} \in \Phi_{e,m}^{(t+(1-e)/2,0)}$.

6.3. Provided Conjecture 5.5 is true, we can compute the parameters of $H_k(L, X)$ for weakly cuspidal pairs (L, X). We use the notation of Theorem 3.2 in the following.

Proposition. *Suppose that X lies in a kM-block $\mathbf{B}$ whose e-core equals the 2-core Δ_s for some $s \geq 0$. Then $Q = q^{2s+1}$.*

Proof. By the results summarized in 5.5, the block $\mathbf{B}$ contains a cuspidal simple KM-module Y. By Theorem 3.2(a), the parameter Q is equal to the corresponding parameter of the Iwahori-Hecke algebra $H_K(L, Y)$. By the results of Lusztig [37, Section 5], we have $Q = q^{2s+1}$. $\square$

6.4. If the Conjectures 5.7 and 5.8 are true, Proposition 6.3 implies a compatibility between certain connected components of the crystal graph.

Suppose that X_λ is weakly cuspidal, that $\lambda_{(2)} = \Delta_t$, and that the e-core of λ equals Δ_s. (The e-core of λ should be a 2-core by Conjecture 5.5.)

Put $r := t(t+1)/2$ and suppose that $n = r + 2m$ and let L denote the pure Levi subgroup of $\mathrm{GU}_n(q)$ isomorphic to $\mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$. By Theorem 3.2 and Proposition 6.3, we have that $H_k(L, X_\lambda)$ is an Iwahori-Hecke algebra of type B_m with parameters q^{2s+1} and q^2. According to [15], the irreducible modules of this Hecke algebra are labelled by $\Phi_e^{(s+(1-e)/2,0)} = \cup_{m\geq 0} \Phi_{e,m}^{(s+(1-e)/2,0)}$. By the generalization of [14, Theorem 2.4] to weakly cuspidal modules, the elements of the

(L, X_λ)-Harish-Chandra series of kG are labelled by these bipartitions (see also Proposition 2.3(a)).

On the other hand, by Conjecture 5.8, this Harish-Chandra series should also be labelled by the set of bipartitions arising from $\bar{\lambda}^{(2)}$ by adding a sequence of good nodes with respect to the charge $(t + (1 - e)/2, 0)$.

The compatibility of the two labellings is guaranteed by Theorem 7.8 below.

6.5. We give an example for the phenomenon discussed above. Suppose that $e = 3$ and let $L := \mathrm{GU}_4(q) \times \mathrm{GL}_1(q^2)^m$. Then the Steinberg kL-module $X_{(1^4)}$ is cuspidal. As the 2-core of (1^4) is trivial we have $t = 0$. According to Conjecture 5.7, the connected component of the Harish-Chandra branching graph beginning in (1^4) should coincide, up to some rank depending on ℓ, with the component of the crystal graph corresponding to $e = 3$ and charge $(-1, 0)$ containing the bipartition $(-, 1^2)$.

The Iwahori-Hecke $H_k(L, X_{(1^4)})$ is of type B_m with parameters q^3 and q^2, as $s = 1$. Its simple modules are labelled by the Uglov-bipartitions corresponding to $e = 3$ and charge $(0, 0)$.

6.6. For blocks of e-weight 1 (for the notions of e-core and e-weight of a unipotent ℓ-block of G see 5.3), Conjecture 5.5 is true.

Theorem. *Let* $\mathbf{B}$ *be a unipotent* ℓ*-block of* $\mathrm{GU}_n(q)$ *of* e*-weight* 1*. Then* $\mathbf{B}$ *contains a weakly cuspidal* kG*-module, if and only if the* e*-core of* $\mathbf{B}$ *is a* 2*-core.*

Proof. Suppose first that the e-core of $\mathbf{B}$ is a 2-core. Then $\mathbf{B}$ contains a cuspidal simple KG-module by the results recalled in 5.5. In particular, $\mathbf{B}$ contains a cuspidal unipotent kG-module.

Now suppose that the e-core of $\mathbf{B}$ is not a 2-core. Let $s(\mathbf{B})$ denote the Scopes number of $\mathbf{B}$ (see [24, 7.2] for the definition of $s(\mathbf{B})$). Our assumption implies that $s(\mathbf{B}) \geq 1$. Indeed, consider an e-abacus diagram (in the sense of [30, p. 78f] or [9, Section 1]) for the e-core of $\mathbf{B}$. Since the latter is not a 2-core, there is $0 \leq i \leq e - 1$ such that the number of beads on string i is at least one larger than the number of beads on string $i - 2$, if $2 \leq i \leq e - 1$, and at least two larger than the number of beads on string $e - 2$ or $e - 1$, if $i = 0$ or 1, respectively. This exactly means $s(\mathbf{B}) \geq 1$. The Reduction Theorem and its consequence [24, Theorems 7.10, 8.1] now imply that every projective kG-module of $\mathbf{B}$ is obtained from Harish-Chandra induction of a projective kG-module of $\mathrm{GU}_{n-2}(q) \times \mathrm{GL}_1(q^2)$. In particular, $\mathbf{B}$ contains no weakly cuspidal kG-module. $\square$

6.7. We now determine all partitions $\mu \in \mathcal{P}$ of e-weight 1 such that X_μ is weakly cuspidal. For $0 \leq t \leq (e-1)/2$ let

$$\mu_{t,e} := (t, t-1, \ldots 3, 2, 1^{e+1}),$$

and for $0 \leq t < (e-1)/2$ let

$$\nu_{t,e} := (t+2, t+1, \ldots, 3, 2, 1^{e-2t-2}).$$

(we understand $\mu_{0,e} = 1^e$ and $\mu_{1,e} = 1^{e+1}$). For $t = (e-1)/2$, we also put $\nu_{t,e} := \mu_{t,e}$.

Proposition. *Let $\mu \in \mathcal{P}_n$ have e-weight 1. Then X_μ is weakly cuspidal if and only if $n = t(t+1)/2 + e$ for some $0 \leq t \leq (e-1)/2$ and $\mu \in \{\mu_{t,e}, \nu_{t,e}\}$.*

Proof. Let $\mathbf{B}$ denote the unipotent ℓ-block of G containing X_μ.

Assume first that X_μ is weakly cuspidal. Then, by Theorem 6.6, the e-core of μ is a 2-core, Δ_t, say. In particular, $n = t(t+1)/2 + e$. As Δ_t is an e-core, we have $0 \leq t \leq (e-1)/2$.

By [9, (6A)], the partitions $\mu_{t,e}$ and $\nu_{t,e}$ label the unipotent KG-modules in $\mathbf{B}$ connected to the exceptional vertex of the Brauer tree of $\mathbf{B}$ (there is only one such if $t = (e-1)/2$).

Assume that $\mu \notin \{\mu_{t,e}, \nu_{t,e}\}$. Let $\mu' \in \{\mu_{t,e}, \nu_{t,e}\}$ such that Y_μ and $Y_{\mu'}$ lie on the same side of the exceptional vertex in the Brauer tree of $\mathbf{B}$. Then μ and μ' have the same 2-core Δ_s, say, again by [9, (6A)]. If $\mu' = \mu_{t,e}$, we clearly have $s < t$, and thus Δ_s is an e-core. If $\mu' = \nu_{t,e}$, then $s = t+2$, and Δ_s is an e-core if $e \geq 2t+5$, and of e-weight 1 if $e = 2t+3$. In the latter case, $n = t(t+1)/2 + (2t+3) = (t+2)(t+3)/2 = |\Delta_s|$, and thus $\mu = \Delta_s = \nu_{t,e}$, a contradiction. Thus in any case Δ_s is an e-core, and so X_{Δ_s} is projective. Using [9, (6A)] once more, we find that X_μ lies in the Harish-Chandra series defined by (L, X_{Δ_s}), where L is the pure standard Levi subgroup of G corresponding to $\mathrm{GU}_{|\Delta_s|}(q)$. In particular, X_μ is not weakly cuspidal, contradicting our assumption.

Now assume that μ is one of $\mu_{t,e}$ or $\nu_{t,e}$. Then the e-core of μ equals Δ_t, and X_μ corresponds to the edge of the Brauer tree linking Y_μ with the exceptional vertex. By the results summarized in 5.5, the exceptional vertex labels cuspidal simple KG-modules. Thus X_μ is cuspidal. This completes our proof. □

More evidence for our conjectures is given in the next section where we prove some consequences of our conjectures for the crystal graph.

7. Some properties of the crystal graph

The conjectures formulated in Section 5 imply some combinatorial properties of the crystal graphs involved. In this final section we prove some of these properties. Throughout this section we let e and t be non-negative integers with e odd and larger than 1. (Contrary to previous usage, the letter k no longer denotes a field, but just an integer.)

7.1. Following [16, 6.5.17], we define a 1-runner abacus to be a subset $\mathfrak{A}$ of $\mathbb{Z}$ such that $-j \in \mathfrak{A}$ and $j \notin \mathfrak{A}$ for all $j \geq n$ and some $0 \neq n \in \mathbb{N}$. Let $\mathfrak{A}$ be a 1-runner abacus. We enumerate the elements of $\mathfrak{A}$ by $a_1, a_2, \ldots$ with $a_1 > a_2 > \cdots$. The elements of $\mathbb{Z} \setminus \mathfrak{A}$ are called *the holes* of $\mathfrak{A}$. If we define λ_j to be the number of holes of $\mathfrak{A}$ less than a_j, $j = 1, 2, \ldots$, then $\lambda := (\lambda_1, \lambda_2, \ldots)$ is *the partition associated to* $\mathfrak{A}$. The *charge* of $\mathfrak{A}$ is the integer $a_1 - \lambda_1$. Let n be a positive integer such that $\{-j \mid j \geq n\} \subseteq \mathfrak{A}$. Then the number of elements of $\mathfrak{A}$ larger than $-n$ equals n plus the charge of $\mathfrak{A}$. Moreover, a β-set for λ, in the sense of [29, p. 2], is obtained by adding a constant d to the elements of $\mathfrak{A} \setminus \{-j \mid j \geq n\}$ to make them all non-negative. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be 1-runner abaci with associated partitions λ and λ' and charges c and c'. Then $\mathfrak{A} = \mathfrak{A}'$ if and only if $\lambda = \lambda'$ and $c = c'$. Also, if $\mathfrak{A} \subseteq \mathfrak{A}'$ and $|\mathfrak{A}' \setminus \mathfrak{A}| = 1$, then $c' = c + 1$.

By a *symbol* we mean a pair $\mathfrak{B} := (\mathfrak{B}^1, \mathfrak{B}^2)$ of 1-runner abaci. The components $\mathfrak{B}^1$ and $\mathfrak{B}^2$ are also called the *first* and *second row* of $\mathfrak{B}$, respectively. If μ^i and c_i are the partition associated to $\mathfrak{B}^i$ and the charge of $\mathfrak{B}^i$, respectively, $i = 1, 2$, we also write $\mathfrak{B} = \mathfrak{B}(\mu, \mathbf{c})$ with $\mu = (\mu^1, \mu^2)$ and $\mathbf{c} = (c_1, c_2)$. Let $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2$ and let $\mu \in \mathcal{P}^{(2)}$ be a bipartition. Then $\mathfrak{B}(\mu, \mathbf{c})$ can be computed as follows (see [28, 2.2]). Let $\mu = (\mu^1, \mu^2)$ with $\mu^i = (\mu^i_j)_{j \geq 1}$ and $\mu^i_j \geq \mu^i_{j+1} \geq 0$ for $j \geq 1$ and $i = 1, 2$. Then $\mathfrak{B}(\mu, \mathbf{c}) = (\mathfrak{B}(\mu, \mathbf{c})^1, \mathfrak{B}(\mu, \mathbf{c})^2)$ with $\mathfrak{B}(\mu, \mathbf{c})^i := \mathfrak{B}(\mu, \mathbf{c})^i_j$, where $\mathfrak{B}(\mu, \mathbf{c})^i_j := \mu^i_j - j + c_i + 1$ for $i = 1, 2$ and $j \geq 1$.

7.2. Put $\mathbf{c} = (t + (1 - e)/2, 0)$ and let $\mu = (\mu^1, \mu^2)$ be a bipartition. To $\mathfrak{B}(\mu, \mathbf{c})$ we associate the 1-runner abacus

$$\mathfrak{A}_e(\mu, \mathbf{c}) := \{2j + e \mid j \in \mathfrak{B}(\mu, \mathbf{c})^1\} \cup \{2j \mid j \in \mathfrak{B}(\mu, \mathbf{c})^2\}.$$

In order to determine the partition associated to $\mathfrak{A}_e(\mu, \mathbf{c})$, choose an even positive integer $n = 2m$ such that $\{-j \mid j \geq n - 1\} \subseteq \mathfrak{A}_e(\mu, \mathbf{c})$ and put

$$\bar{\mathfrak{A}} := \{x + n \mid x \in \mathfrak{A}_e(\mu, \mathbf{c}), x \geq -n\}.$$

Then $\bar{\mathfrak{A}}$ is a β-set for the partition associated to $\mathfrak{A}_e(\mu, \mathbf{c})$ with $0, 1 \in \bar{\mathfrak{A}}$. Let

$$\bar{\mathfrak{A}}^1 := \{(x-1)/2 \mid x \in \bar{\mathfrak{A}}, x \text{ odd}\}$$

and

$$\bar{\mathfrak{A}}^2 := \{x/2 \mid x \in \bar{\mathfrak{A}}, x \text{ even}\}.$$

Then

$$\bar{\mathfrak{A}}^1 = \{j + (e-1)/2 + m \mid j \in \mathfrak{B}(\mu, \mathbf{c})^1, j \geq -m - (e-1)/2\}$$

and

$$\bar{\mathfrak{A}}^2 = \{j + m \mid j \in \mathfrak{B}(\mu, \mathbf{c})^2, j \geq -m\}.$$

In particular, $\bar{\mathfrak{A}}^i$ is a β-set for μ^i, $i = 1, 2$ and $|\bar{\mathfrak{A}}^1| = |\bar{\mathfrak{A}}^2| + t$. The latter equality follows from the remarks in the first paragraph of 7.1.

Lemma. *The partition associated to $\mathfrak{A}_e(\mu, \mathbf{c})$ equals $\Phi_t(\mu)$.*

Proof. Use the notation introduced above. Then $|\bar{\mathfrak{A}}| = (|\bar{\mathfrak{A}}^1| + |\bar{\mathfrak{A}}^2|) \equiv t \,(\mathrm{mod}\ 2)$. Thus $\bar{\mathfrak{A}}$ is a β-set for the partition with 2-core Δ_t, and 2-quotient (computed with respect to a β-set with an odd number of elements) (μ^2, μ^1) if t is odd, and (μ^1, μ^2) if t is even. This implies our claim. $\square$

7.3. Let $\mathbf{c} = (t + (1-e)/2, 0)$ and let $\mu \in \mathcal{P}^{(2)}$. We are interested in the operation of deleting e-hooks from $\Phi_t(\mu)$. On $\mathfrak{A}_e(\mu, \mathbf{c})$, this amounts to replacing an element $y \in \mathfrak{A}_e(\mu, \mathbf{c})$ with $y - e \notin \mathfrak{A}_e(\mu, \mathbf{c})$ by $y - e$. If y is odd, this replacement corresponds to the operation of deleting $j = (y-e)/2$ from $\mathfrak{B}(\mu, \mathbf{c})^1$ and inserting j into $\mathfrak{B}(\mu, \mathbf{c})^2$. If y is even, this replacement corresponds to the operation of deleting $j = y/2$ from $\mathfrak{B}(\mu, \mathbf{c})^2$ and inserting $j - e$ into $\mathfrak{B}(\mu, \mathbf{c})^1$. This leads to the following operations on symbols, to which we refer as *elementary operations*.

(a) Delete an element j in the first row, which is not in the second row, and insert j in the second row.
(b) Delete an element j in the second row, such that $j - e$ is not in the first row, and insert $j - e$ in the first row.

Iterating the two operations we end up with a symbol for which no such operation is possible. Even though the resulting symbol does not depend on the order in which we perform these operations, we decide to do the former operation first if possible, and always take the largest possible j so that each step in the algorithm is well defined. This gives the following elementary operations in a more restrictive sense.

(a′) Delete the largest element j in the first row, which is not in the second row, and insert j in the second row.

(b′) If every element in the first row is contained in the second row, delete the largest element j in the second row, such that $j-e$ is not in the first row, and insert $j-e$ in the first row.

Proposition. *Put* $\lambda := \Phi_t(\mu)$. *Let* $\mu' = ((\mu')^1, (\mu')^2) \in \mathcal{P}^{(2)}$ *and* $\mathbf{c}' \in \mathbb{Z}^2$ *such that* $\mathfrak{B}(\mu', \mathbf{c}')$ *is obtained from* $\mathfrak{B}(\mu, \mathbf{c})$ *by an elementary operation of type (a) or (b).*

Applying this elementary operation corresponds to removing an e-hook from λ. *Denote by* λ' *the resulting partition, and let* t' *be such that* $\lambda'_{(2)} = \Delta_{t'}$. *Suppose that* $\tilde{\mu} = (\tilde{\mu}^1, \tilde{\mu}^2)$ *is the bipartition such that* $\Phi_{t'}(\tilde{\mu}) = \lambda'$.

Then $t' = t + 2$, *if the elementary operation applied is of type (b). If the elementary operation applied is of type (a), then*

$$t' = \begin{cases} t-2, & \text{if } t \geq 2, \\ 0, & \text{if } t = 1, \\ 1, & \text{if } t = 0. \end{cases}$$

Moreover,

$$\tilde{\mu} = \begin{cases} \mu', & \text{if } t \text{ and } t' \text{ have the same parity,} \\ ((\mu')^2, (\mu')^1), & \text{otherwise,} \end{cases}$$

Proof. Consider a β-set $\bar{\mathfrak{A}}$ for $\Phi_t(\mu)$ as constructed in 7.2. An elementary operation results in replacing an element x of $\bar{\mathfrak{A}}$ by $x-e$ yielding the β-set $\bar{\mathfrak{A}}'$ for λ'. (Notice that $\bar{\mathfrak{A}}'$ is constructed from $\mathfrak{B}(\mu', \mathbf{c}')$ in the same way as $\bar{\mathfrak{A}}$ from $\mathfrak{B}(\mu, \mathbf{c})$.) Moreover, x is even or odd, if the elementary operation is of type (b) or (a), respectively. In the former case, the number of odd elements of $\bar{\mathfrak{A}}$ increases by 1, and thus $t' = t+2$. In the latter case, the number of odd elements of $\bar{\mathfrak{A}}$ decreases by 1. Hence $t' = t-2$ if $t \geq 2$, $t' = 0$ if $t = 1$, and $t' = 1$ if $t = 0$.

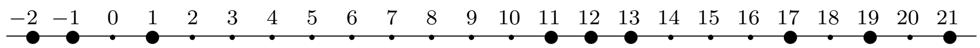
If the parity of t is the same as that of t', then the constructions of $\Phi_t(\mu)$ and of $\Phi_{t'}(\tilde{\mu})$ are the same, namely we have $\lambda^{(2)} = (\mu^1, \mu^2)$ and $(\lambda')^{(2)} = (\tilde{\mu}^1, \tilde{\mu}^2)$ (respectively $\lambda^{(2)} = (\mu^2, \mu^1)$ and $(\lambda')^{(2)} = (\tilde{\mu}^2, \tilde{\mu}^1)$) if t is even (respectively odd). Therefore, one can read off $\tilde{\mu}$ directly on the symbol $\mathfrak{B}(\mu', \mathbf{c}')$ (or on the β-sets $\bar{\mathfrak{A}}^i$, $i = 1, 2$). It follows that $\tilde{\mu} = \mu'$.

On the contrary, if t and t' have different parities (say, without loss of generality, t even and t' odd), then the construction of $\Phi_{t'}(\tilde{\mu})$ requires a permutation, unlike that of $\Phi_t(\mu)$. Therefore, one needs to permute the components of the bipartition one reads off $\mathfrak{B}(\mu', \mathbf{c}')$, i.e. $\tilde{\mu} = ((\mu')^2, (\mu')^1)$ □

As an example, consider the bipartition $\mu = ((5^3, 4^2), (6))$, let $e = 3$ and $t = 5$. Then $\mathbf{c} = (4, 0)$ and

$$\mathfrak{B}(\mu, \mathbf{c}) = \begin{pmatrix} \cdots & -2 & -1 & 6 & & & & \\ \cdots & -2 & -1 & 4 & 5 & 7 & 8 & 9 \end{pmatrix}.$$

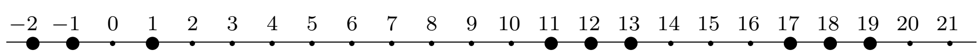
The associated 1-runner abacus $\mathfrak{A}_3(\mu, \mathbf{c})$ can be represented as follows:



With the notation of 7.2, taking $n = 2$ we obtain the β-set $\bar{\mathfrak{A}} = \{0, 1, 3, 13, 14, 15, 19, 21, 23\}$ for the partition $\lambda := \Phi_5(\mu)$ associated to $\mathfrak{A}_3(\mu, \mathbf{c})$. We also have $\bar{\mathfrak{A}}^1 = \{0, 1, 6, 7, 9, 10, 11\}$ and $\bar{\mathfrak{A}}^2 = \{0, 7\}$, wich are β-sets for $\mu^1 = (5^3, 4^2)$ and $\mu^2 = (6)$ respectively. Notice that $\lambda = (15, 14, 13, 10^3, 1)$. An elementary operation of type (a′) on the symbol yields

$$\mathfrak{B}(\mu', \mathbf{c}') = \begin{pmatrix} \cdots & -2 & -1 & 6 & 9 & & \\ \cdots & -2 & -1 & 4 & 5 & 7 & 8 \end{pmatrix},$$

with $\mu' = ((5^2, 4^2), (8, 6))$ and $\mathbf{c}' = (3, 1))$. The 1-runner abacus $\mathfrak{A}_3(\mu', \mathbf{c}')$ cna be pictured as follows:



We obtain $\bar{\mathfrak{A}}' = \{0, 1, 3, 13, 14, 15, 19, 20, 21\}$, again using $n = 2$. Next, $(\bar{\mathfrak{A}}')^1 = \{0, 1, 6, 7, 9, 10\}$ and $(\bar{\mathfrak{A}})^2 = \{0, 7, 10\}$, wich are β-sets for $(5^2, 4^2)$ and $(8, 6)$ respectively. The partition associated to $\mathfrak{A}_3(\mu', \mathbf{c}')$ is $\lambda' = (13^3, 10^3, 1)$ which is obtained from λ by removing a 3-hook. We have $\lambda'_{(2)} = \Delta_3$, i.e. $t' = 3$, and $\Phi_3(\mu') = \lambda'$.

7.4. In the following we will make use of the notion of an e-period of a symbol (see [28, Definition 2.2]) and the concept of totally periodic symbols (see [28, Definition 5.4]). Let $|\mu, \mathbf{c}\rangle$ be a charged bipartition. In our special situation, an e-period of $\mathfrak{B}(\mu, \mathbf{c})$ is a sequence $(i_1, k_1), (i_2, k_2), \ldots, (i_e, k_e)$ of pairs of integers with $2 \geq k_1 \geq k_2 \geq \cdots \geq k_e \geq 1$ such that $\mathfrak{B}(\mu, \mathbf{c})^{k_l}_{i_l} = m - l + 1$ for some integer m. Moreover, m is the largest element in $\mathfrak{B}(\mu, \mathbf{c})^1 \cup \mathfrak{B}(\mu, \mathbf{c})^2$, and if $m - l + 1 \in \mathfrak{B}(\mu, \mathbf{c})^1$ for some $1 \leq l \leq e$, then $k_l = 1$. Suppose that $\mathfrak{B}(\mu, \mathbf{c})$ has an e-period $(i_1, k_1), (i_2, k_2), \ldots, (i_e, k_e)$. Then this e-period is unique and the entries $\mathfrak{B}(\mu, \mathbf{c})^{k_l}_{i_l}$ of $\mathfrak{B}(\mu, \mathbf{c})$ are called the elements of the period. Removing these elements from $\mathfrak{B}(\mu, \mathbf{c})$, we obtain the symbol $\mathfrak{B}(\mu', \mathbf{c}')$ corresponding to a charged bipartition $|\mu', \mathbf{c}'\rangle$ which may or may not have an e-period. If iterating this procedure ends up in a symbol $\mathfrak{B}(\nu, \mathbf{d})$

such that ν is the empty bipartition, then $\mathfrak{B}(\mu, \mathbf{c})$ is called totally periodic.

By [28, Theorem 5.9], the symbol $\mathfrak{B}(\mu, \mathbf{c})$ is totally periodic, if and only if $|\mu, \mathbf{c}\rangle$ is a highest weight vertex of $\mathcal{G}_{\mathbf{c},e}$. If $\mathfrak{B}(\mu, \mathbf{c})$ is totally periodic, then for each entry j in $\mathfrak{B}(\mu, \mathbf{c})$, there is a symbol $\mathfrak{B}'$, obtained from $\mathfrak{B}(\mu, \mathbf{c})$ by removing a sequence of e-periods, and an e-period $(i_1, k_1), \ldots, (i_e, k_e)$ of $\mathfrak{B}'$, such that $j = (\mathfrak{B}')_{i_l}^{k_l}$ for some $1 \leq l \leq e$. By a slight abuse of terminology, we say that j is contained in the period $(i_1, k_1), \ldots, (i_e, k_e)$ of $\mathfrak{B}$.

Let $\mathfrak{B}'$ denote the symbol obtained from $\mathfrak{B}(\mu, \mathbf{c})$ by applying an elementary operation.

Lemma. *If $\mathfrak{B}(\mu, \mathbf{c})$ is totally periodic, so is $\mathfrak{B}'$.*

Proof. Suppose first that $\mathfrak{B}'$ is obtained from $\mathfrak{B}(\mu, \mathbf{c})$ by an elementary operation (a). Moving j from row 1 to row 2 transforms the period $(i_1, k_1), \ldots, (i_e, k_e)$ containing j into a period $(i'_1, k'_1), \ldots, (i'_e, k'_e)$ such that $(\mathfrak{B}')_{i'_l}^{k'_l} = \mathfrak{B}(\mu, \mathbf{c})_{i_l}^{k_l}$ for all l. In particular, $\mathfrak{B}'$ is also totally periodic.

Suppose now that $\mathfrak{B}'$ is obtained from $\mathfrak{B}(\mu, \mathbf{c})$ by an elementary operation (b). Deleting j from row 2 and inserting $j - e$ in row 1 transforms the period $(i_1, k_1), \ldots, (i_e, k_e)$ containing j into a period $(i'_1, k'_1), \ldots, (i'_e, k'_e)$ such that $(\mathfrak{B}')_{i'_l}^{k'_l} = \mathfrak{B}(\mu, \mathbf{c})_{i_l}^{k_l} - 1$ for all $l < e-1$ and $(\mathfrak{B}')_{i'_e}^{k'_e} = \mathfrak{B}(\mu, \mathbf{c})_{i_e}^{k_e} - e$. In particular, $\mathfrak{B}'$ is also totally periodic. $\square$

7.5. Let $G = \mathrm{GU}_n(q)$, and let ℓ and e be as in 5.1. In [13, Theorem 8.3] we have proved that $X_{(1^n)}$ is cuspidal if and only if e is odd and divides n or $n-1$. This is consistent with Conjecture 5.8, as will be shown below. Let $\lambda = (1^n)$. Then the 2-core of λ equals Δ_t with $t = 0$ if n is even, and $t = 1$ if n is odd. Also $\bar{\lambda}^{(2)} = (-, 1^m)$ with $m = \lfloor n/2 \rfloor$; notice that $n = 2m + t$.

Proposition. *Let $e \geq 3$ be an odd integer, let $m \in \mathbb{N}$ and $t \in \{0, 1\}$. Put $\mathbf{c} := (t + (1-e)/2, 0)$.*

Then the vertex $|(-, 1^m), \mathbf{c}\rangle$ of $\mathcal{G}_{\mathbf{c},e}$ is a highest weight vertex, if and only if $e \mid 2m + t$ or $e \mid 2m + t - 1$.

Proof. The proof proceeds by induction on m, the case $m = 0$ being clear. Assume that $m > 0$ and let $s, s' \in \{(e-1)/2, (e-3)/2\}$ with $s \neq s'$. The symbol $\mathfrak{B}$ of $|(-, 1^m), (-s, 0)\rangle$ equals

$$\mathfrak{B} = \begin{pmatrix} \cdots & -m & 2-m & 3-m & \cdots & -s+1 & -s+2 & \cdots & 1 \\ \cdots & -m & 1-m & 2-m & \cdots & -s & & & \end{pmatrix}.$$

Let $\mathfrak{B}'$ be the symbol obtained by removing the e-period from $\mathfrak{B}$. If $m < e-1$, we find

$$\mathfrak{B}' = \begin{pmatrix} \cdots & -(e-1) & \cdots & -m & 2-m & \cdots & -s \\ \cdots & -(e-1) & & & & & \end{pmatrix},$$

and if $m = e-1$, we have

$$\mathfrak{B}' = \begin{pmatrix} \cdots & -(e-1) & -(e-3) & \cdots & -s \\ \cdots & -(e-1) & & & \end{pmatrix}.$$

In the latter two cases, $\mathfrak{B}'$ does not have an e-period and thus $\mathfrak{B}$ is not totally periodic. On the other hand, e does not divide one of $2m-1$, $2m$, or $2m+1$, as $1 \le m \le e-1$.

If $m \ge e$, then

$$\mathfrak{B}' = \begin{pmatrix} \cdots & -m & 2-m & 3-m & \cdots & -e+2 & -e+3 & \cdots - s \\ \cdots & -m & 1-m & 2-m & \cdots & -e+1 & & \end{pmatrix}.$$

Thus $\mathfrak{B}'$ is the symbol of $|(-, 1^{m-s-1}), (-s', 0)\rangle$. Now $\mathfrak{B}$ is totally e-periodic if and only if $\mathfrak{B}'$ is totally e-periodic. By induction, $\mathfrak{B}'$ is totally e-periodic if and only if $e \mid 2m-2s-2$ or $e \mid 2m-2s-3$ in case $s' = (e-1)/2$, respectively if and only if $e \mid 2m-2s-2$ or $e \mid 2m-2s-1$ in case $s' = (e-3)/2$. Suppose first that $s' = (e-1)/2$. Then $s = (e-3)/2$ and thus $2m-2s-2 = 2m+1-e$. The claim follows. The other case works analogously. □

7.6. Let $|\mu, \mathbf{c}\rangle$ be a charged bipartition, put $\mathfrak{B} := \mathfrak{B}(\mu.\mathbf{c})$ and $\mathfrak{B}^k := \mathfrak{B}(\mu, \mathbf{c})^k$ for $k = 1, 2$.

Lemma. *Suppose that $\mathfrak{B}$ is totally e-periodic, that $\mathfrak{B}^1 \subseteq \mathfrak{B}^2$ and that $j - e \in \mathfrak{B}^1$ for all $j \in \mathfrak{B}^2$ with $j \ge m$ for some $m \in \mathbb{Z}$.*

Then for $k = 1, 2$ we have $j-1 \in \mathfrak{B}^k$ for all $j \in \mathfrak{B}^k$ with $j \ge m-e+1$.

Proof. Let $j \in \mathfrak{B}^k$ with $j-1 \notin \mathfrak{B}^k$. Then $j-1 \notin \mathfrak{B}^1$ and the period of $\mathfrak{B}$ containing j ends in j. The first element in this period is $j+e-1$, and $j+e-1 \in \mathfrak{B}^2$. As $j-1 = j+e-1-e$, it follows that $j+e-1 < m$, hence our claim. □

Put $\mathbf{c} = (t+(1-e)/2, 0)$. If Conjecture 5.8 is true, the highest weight vectors of the crystal graph $\mathcal{G}_{\mathbf{c},e}$ label the weakly cuspidal unipotent $\mathrm{GU}_n(q)$-modules for large enough primes ℓ with $e = e(q,\ell)$. More explicitly, a weakly cuspidal $\mathrm{GU}_n(q)$-module X_λ with $\lambda_{(2)} = \Delta_t$ should be labelled by the highest weight vector $|\bar{\lambda}^{(2)}, \mathbf{c}\rangle$. Moreover, if X_λ is weakly cuspidal, the e-core of λ should be a 2-core by Conjecture 5.5.

Recall that λ with $\lambda_{(2)} = \Delta_t$ and $\bar{\lambda}^{(2)}$ are related by $\lambda = \Phi_t(\bar{\lambda}^{(2)})$.

Theorem. *Let the notation be as above. Let $\mu \in \mathcal{P}^{(2)}$ be such that $|\mu, \mathbf{c}\rangle$ is a highest weight vertex in $\mathcal{G}_{\mathbf{c},e}$. Then the e-core of $\Phi_t(\mu)$ is a 2-core.*

Proof. Starting with $\mathfrak{B}(\mu, \mathbf{c})$, we apply a sequence of elementary operations, until we reach a symbol $\mathfrak{B}'$, which does not allow any such operation. Starting with $\mathfrak{A}_e(\mu, \mathbf{c})$, the corresponding sequence of operations results in a 1-runner abacus $\mathfrak{A}'$, such that $y - e \in \mathfrak{A}'$ for all $y \in \mathfrak{A}'$. By Lemma 7.2, the partition associated to $\mathfrak{A}'$ is the e-core of $\Phi_t(\mu)$.

The symbol $\mathfrak{B}'$ is totally e-periodic by Lemma 7.4, and satisfies the assumptions of the above lemma for all $m \in \mathbb{Z}$. Hence for $k = 1, 2$, we have $j - 1 \in (\mathfrak{B}')^k$ for every $j \in (\mathfrak{B}')^k$. This implies that $x - 2 \in \mathfrak{A}'$ for all $x \in \mathfrak{A}'$. In particular, the partition associated to $\mathfrak{A}'$ is a 2-core. $\square$

We now sketch a different proof of the above theorem. Consider, for $s \in \mathbb{Z}$, the space of semi-infinite wedge products $\Lambda^{s+\infty/2}$, as it is defined in [41, §4]. We do not need the precise definition of this space here but we need to know that there are three ways to index the elements of its basis ("the semi-infinite ordered wedges"):

- by the set of elements denoted by $|\lambda, s\rangle$ where $\lambda \in \mathcal{P}$;
- by the set of elements denoted by $|\mu, \mathbf{c}\rangle$, where $\mu \in \mathcal{P}^{(2)}$ and $\mathbf{c} = (c_1, c_2) \in \mathbb{Z}^2$ is such that $c_1 + c_2 = s$. The way to pass from $|\lambda, s\rangle$ to $|\mu, \mathbf{c}\rangle$ is purely combinatorial;
- by the set of elements denoted by $|\lambda^{(e)}, \mathbf{c}^{(e)}\rangle$ where $\lambda^{(e)}$ is the e-quotient of λ and $\mathbf{c}^{(e)} = (c_1, \dots, c_e) \in \mathbb{Z}^e$ satisfies $\sum_{i=1}^{e} = s$ and parametrizes the e-core of λ.

Setting $u := -v^{-1}$, we have three actions of the algebras $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$, $\mathcal{U}'_u(\widehat{\mathfrak{sl}_2})$ and another algebra $\mathcal{H}$ (the Heisenberg algebra) on the space $\Lambda^{s+\infty/2}$. Moreover these three actions commute and we have the following decomposition (see [41, Theorem 4.8]):

$$\Lambda^{s+\infty/2} = \bigoplus_{\mathbf{c} \in A_e^2(s)} \mathcal{U}'_v(\widehat{\mathfrak{sl}_e}).\mathcal{H}.\mathcal{U}'_u(\widehat{\mathfrak{sl}_2}).|(-,-), \mathbf{c}\rangle,$$

where $A_e^2(s)$ is the set of elements $\mathbf{c} \in \mathbb{Z}^2$ such that $c_1 - c_2 \leq e$ and $c_1 + c_2 = s$. In addition, if we fix $\mathbf{c}$, the associated Fock space of level 2 is a $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-submodule of $\Lambda^{s+\infty/2}$ (that is the actions are compatible).

Let $i \in \{0, 1\}$. Denote by E_i and F_i the Chevalley operators of $\mathcal{U}'_u(\widehat{\mathfrak{sl}_2})$. Regarding the action of E_i on the set of charged bipartitions following Uglov's work, we see that $|\mu, \mathbf{d}\rangle$ appears in the expansion of $E_i.|\lambda, \mathbf{c}\rangle$ if and only if the symbol of $|\mu, \mathbf{d}\rangle$ is obtained from the symbol of $|\lambda, \mathbf{c}\rangle$ by one of the two elementary operations (a) and (b) described in 7.3. This thus gives an algebraic interpretation of these transformations on symbols. Moreover, combining this interpretation with some properties of the crystal of $\Lambda^{s+\infty/2}$ (see [41, § 4.3]) leads to an alternative proof of the above theorem.

7.7. For a highest weight vertex $|\mu, \mathbf{c}\rangle$, write $B(\mu, \mathbf{c})$ for the connected component of $\mathcal{G}_{\mathbf{c},e}$ containing μ. General crystal theory (see [32] for instance) ensures that $B(\mu, \mathbf{c}) \simeq B(\nu, \mathbf{d})$ as soon as $|\mu, \mathbf{c}\rangle$ and $|\nu, \mathbf{d}\rangle$ are both highest weight vertices and $\mathrm{wt}(\nu, \mathbf{c}) = \mathrm{wt}(\mu, \mathbf{d})$. Moreover, by the characterization (5), the weights of $|\mu, \mathbf{c}\rangle$ and $|\nu, \mathbf{d}\rangle$ coincide if these two charged bipartitions have the same reduced i-word for all $0 \leq i \leq e-1$.

From now on, let $|\mu, \mathbf{c}\rangle$ be a highest weight vertex in $\mathcal{G}_{\mathbf{c},e}$. Let $|\mu', \mathbf{c}'\rangle$ be a charged bipartition such that $\mathfrak{B}(\mu', \mathbf{c}')$ is the symbol obtained from $\mathfrak{B}(\mu, \mathbf{c})$ by applying one of the elementary operations described in 7.3 (a′), (b′). By Lemma 7.2, this implies in particular that $\Phi_t(\mu)$ is not an e-core.

Lemma. *Under the above hypothesis, $|\mu', \mathbf{c}'\rangle$ is a highest weight vertex and there is a crystal isomorphism $B(\mu, \mathbf{c}) \simeq B(\mu', \mathbf{c}')$.*

Proof. By Lemma 7.4, we know that $\mathfrak{B}(\mu', \mathbf{c}')$ is totally periodic, and thus $|\mu', \mathbf{c}'\rangle$ is a highest weight vertex by [28, Theorem 5.9]. By the discussion at the beginning of this paragraph, it remains to show that the reduced i-words of $|\mu, \mathbf{c}\rangle$ and $|\mu', \mathbf{c}'\rangle$ coincide for all $0 \leq i \leq e-1$. Denote these words by $w_i(\mu, \mathbf{c})$ and $w_i(\mu', \mathbf{c}')$. In this proof, we use for more clarity the notation $A_k(j)$ (respectively $R_k(j)$) instead of simply A (respectively R) to encode the addable (respectively removable) node of content j lying in component k of $\mathfrak{B}(\mu, \mathbf{c})$. Note that the contents of the addable and removable nodes of a bipartition are the elements $j-1$ and j, respectively, for j in the corresponding symbol (provided j encodes a non-zero part). In fact, a removable node of content $j-1$ corresponds to an element $j \in \mathfrak{B}(\mu, \mathbf{c})^k$ such that $j-1 \notin \mathfrak{B}(\mu, \mathbf{c})^k$, and an addable node of content j corresponds to an element $j \in \mathfrak{B}(\mu, \mathbf{c})^k$ such that $j+1 \notin \mathfrak{B}(\mu, \mathbf{c})^k$. Therefore, since an elementary operation affects either just one element j or just j and $j-e$, the only differences that can occur between $w_i(\mu, \mathbf{c})$ and $w_i(\mu', \mathbf{c}')$ are with letters A and R corresponding to nodes of content $j-1$, j, $j-e-1$ and $j-e$. We review the only possible changes by enumerating the cases.

Suppose first that we apply the elementary operation (a′), that is to say we move j from row 1 of $\mathfrak{B}(\mu, \mathbf{c})$ to row 2. Moreover, j is the largest element in $\mathfrak{B}(\mu, \mathbf{c})^1$ for which $j \notin \mathfrak{B}(\mu, \mathbf{c})^2$. Denote by l the largest element of $\mathfrak{B}(\mu,\mathbf{c})^2$. To begin with, assume that j is the largest element of $\mathfrak{B}(\mu,\mathbf{c})^1$.

If $j > l$, then j is the first element of its period, and thus $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$. Moreover, either

- $j > l+1$, in which case the elementary operation takes $A_1(j)$ to $A_2(j)$ and creates an occurence of $R_2(j-1)A_1(j-1)$, which cancels in the reduced i-word (for $i = j-1 \bmod e$), or
- $j = l+1$, in which case $A_1(j)$ in $\mathfrak{B}(\mu,\mathbf{c})$ becomes $A_2(j)$ in $\mathfrak{B}(\mu',\mathbf{c}')$, and $A_2(j-1)$ becomes $A_1(j-1)$.

If $j < l$, the following possibilities arise.

- If $j+1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, then again j is the first element of its period, and thus $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$. Moreover, either
  * $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, and $A_1(j)$ becomes $A_2(j)$ and $R_2(j-1)A_1(j-1)$ appears, or
  * $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, and $A_2(j-1)$ becomes $A_1(j-1)$ and $A_1(j)$ becomes $A_2(j)$.
- If $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, then either
  * $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, in which case $R_2(j)A_1(j)$ vanishes and $R_2(j-1)A_1(j-1)$ appears, or
  * $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$ and $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, in which case $A_2(j-1)$ becomes $A_1(j-1)$ and $R_2(j)A_1(j)$ vanishes, or
  * $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, in which case $R_1(j-1)$ becomes $R_2(j-1)$ and $R_2(j)A_1(j)$ vanishes, or
  * $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$ and $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, in which case j is the last element in its period; if $m \geq j+1$ is the smallest element of $\mathfrak{B}(\mu,\mathbf{c})^2$ with $m+1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, then m and $j-1$ are congruent modulo e, and $R_1(j-1)A_2(m)$ and $R_2(j)A_1(j)$ vanish.

Assume now that j is not the largest element of $\mathfrak{B}(\mu,\mathbf{c})^1$. First we consider the case that $j+1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$. The fact that $\mathfrak{B}(\mu,\mathbf{c})$ is totally periodic then implies that $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ if $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$ if $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$ and $j+1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$. We obtain the following five subcases.

- If $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$, $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, then $R_2(j)A_1(j)$ vanishes and $R_1(j-1)$ becomes $R_2(j-1)$.
- If $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$, $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, then $R_2(j)A_1(j)$ vanishes and $R_2(j-1)A_1(j-1)$ appears.
- If $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, $j+1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, then $A_1(j)$ becomes $A_2(j)$ and $R_2(j-1)A_1(j-1)$ appears.
- If $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$, then $R_2(j)A_1(j)$ vanishes and $R_2(j-1)A_1(j-1)$ appears.

- If $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^k$, $k=1,2$ and $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$, then $R_2(j)A_1(j)$ vanishes and $A_2(j-1)$ becomes $A_1(j-1)$.

If $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, then $j+1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ otherwise j would not be moved.

- If $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$, then $R_1(j-1)$ becomes $R_2(j-1)$ and $R_2(j)$ becomes $R_1(j)$.
- The case $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$ can not occur as $\mathfrak{B}(\mu,\mathbf{c})$ is totally periodic.
- If $j-1 \notin \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, then $R_2(j)$ becomes $R_1(j)$ and $R_2(j-1)A_1(j-1)$ appears.
- If $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$ and $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, then $R_2(j)$ becomes $R_1(j)$ and $A_2(j-1)$ becomes $A_1(j-1)$.

Suppose now that we apply operation (b′), that is to say, that we delete j from $\mathfrak{B}(\mu,\mathbf{c})^2$ and insert $j-e$ in $\mathfrak{B}(\mu,\mathbf{c})^1$. This implies in particular that all elements of $\mathfrak{B}(\mu,\mathbf{c})^1$ are in $\mathfrak{B}(\mu,\mathbf{c})^2$. Again, assume first that j is the largest element of $\mathfrak{B}(\mu,\mathbf{c})^2$. As $\mathfrak{B}(\mu,\mathbf{c})$ is totally e-periodic, $j-1$ appears in $\mathfrak{B}(\mu,\mathbf{c})$, hence $j-1 \in \mathfrak{B}(\mu,\mathbf{c})^2$. Denote by l the largest element of $\mathfrak{B}(\mu,\mathbf{c})^1$. Suppose first that $j-e>l$.

- If $j-e>l+1$, then $A_2(j)$ becomes $A_1(j-e)$, and $R_1(j-1-e)A_2(j-1)$ appears.
- If $j-e=l+1$, then $A_2(j)$ becomes $A_1(j-e)$ and $A_1(j-e-1)$ becomes $A_2(j-1)$.

Now assume that $j-e<l$. Note that in this case $j-e+1 \in \mathfrak{B}(\mu,\mathbf{c})^1$. ndeed, j is the first element in the period of $\mathfrak{B}(\mu,\mathbf{c})$, and $j-e+1$ the last. As $l \geq j-e+1$ and l lies in the first row, so does $j-e+1$.

- If $j-e-1 \notin \mathfrak{B}(\mu,\mathbf{c})^1$, then $R_1(j-e)A_2(j)$ vanishes and $R_1(j-e-1)A_2(j-1)$ appears.
- If $j-e-1 \in \mathfrak{B}(\mu,\mathbf{c})^1$, then $A_1(j-e-1)$ becomes $A_2(j-1)$ and $R_1(j-e)A_2(j)$ vanishes.

Finally, assume that j is not the largest element of $\mathfrak{B}(\mu,\mathbf{c})^2$ and let j' denote the largest element of $\mathfrak{B}(\mu,\mathbf{c})^2$. Then $j'-e \in \mathfrak{B}(\mu,\mathbf{c})^1$, as our operation of type (b′) always moves the largest possible element. Hence $l \geq j'-e > j-e$. Now j is the largest element of $\mathfrak{B}(\mu,\mathbf{c})^2$ such that $j-e$ is not in $\mathfrak{B}(\mu,\mathbf{c})^1$. By Lemma 7.6, this implies that for $k=1,2$ and every $r>j-e+1$ we have $r-1 \in \mathfrak{B}(\mu,\mathbf{c})^k$ if $r \in \mathfrak{B}(\mu,\mathbf{c})^k$. Hence all integers in the interval $[j-e+1,j']$ and $[j-e+1,l]$ are contained in $\mathfrak{B}(\mu,\mathbf{c})^2$ and $\mathfrak{B}(\mu,\mathbf{c})^1$, respectively. This implies in particular that $j-e \in \mathfrak{B}(\mu,\mathbf{c})^2$ as otherwise the element $j-e+1$ of the second row must be the last element in its period. But then the element $j-e+1$

of the first row must lie in an earlier period, which is impossible. This leaves to check the following possibilities.

- If $j-e-1 \in \mathfrak{B}(\mu, \mathbf{c})^1$ it is also contained in $\mathfrak{B}(\mu, \mathbf{c})^2$, and $R_1(j-e)$ becomes $R_2(j)$ and $A_1(j-e-1)$ becomes $A_2(j-1)$.
- If $j-e-1 \in \mathfrak{B}(\mu, \mathbf{c})^2$ and $j-e-1 \notin \mathfrak{B}(\mu, \mathbf{c})^1$, then $R_1(j-e)$ becomes $R_2(j)$ and $R_1(j-e-1)A_1(j-1)$ appear.
- If $j-e-1 \notin \mathfrak{B}(\mu, \mathbf{c})^2$ and $j-e-1 \notin \mathfrak{B}(\mu, \mathbf{c})^1$, then $R_1(j-e)$ becomes $R_2(j)$ and $R_1(j-e-1)A_1(j-1)$ appear.

In each case, we see that $w_i(\mu) = w_i(\mu')$, for all $i = 1, \ldots, e-1$. □

We record a first consequence of the above lemma. Let $t' \in \mathbb{N}$ and $\tilde{\mu} \in \mathcal{P}^{(2)}$ be such that $\lambda' := \Phi_{t'}(\tilde{\mu})$ equals the partition obtained from $\lambda := \Phi_t(\mu)$ be removing the e-hook which corresponds to the elementary operation transforming $\mathfrak{B}(\mu, \mathbf{c})$ into $\mathfrak{B}(\mu', \mathbf{c}')$. (See Proposition 7.3 how to compute t' and $\tilde{\mu}$.) Suppose that t and t' have the same parity and put $\tilde{\mathbf{c}} := (t' + (1-e)/2, 0)$. Then $\tilde{\mu} = \mu'$ and $\tilde{\mathbf{c}}$ is obtained from $\mathbf{c}'$ by adding or subtracting 1 to each of its components. By definition of the crystal graph, it is clear that translating each component of the charge by some fixed integer, results in the same graph with an overall translation of the labels of the arrows. In particular, $|\tilde{\mu}, \tilde{\mathbf{c}}\rangle$ is a highest weight vertex.

Corollary. *Suppose that Conjecture 5.7 is true. Then the Harish-Chandra branching graphs corresponding to the weakly cuspidal modules X_λ and $X_{\lambda'}$ are isomorphic (up to some rank).*

Proof. It follows from the considerations preceeding the corollary, that $B(\mu, \mathbf{c})$ and $B(\tilde{\mu}, \tilde{\mathbf{c}}0$ are isomorphic up to a global shift of the arrow labels. □

This corollary shows that the validity of Conjecture 5.7 would yield a remarkable connection between the Harish-Chandra theory of unitary groups of odd and even degrees.

7.8. We finally prove a property of the crystal graph which is implied by the considerations in 6.4. Let $\lambda \in \mathcal{P}$ with $\lambda_{(2)} = \Delta_t$ and $\bar{\lambda}^{(2)} = \mu$. Put $\mathbf{c} := (t+(1-e)/2, 0)$. Assume that $|\mu, \mathbf{c}\rangle$ is a highest weight vector in $\mathcal{G}_{\mathbf{c},e}$. By Theorem 7.6, the e-core of λ is a 2-core, Δ_s, say, for some non-negative integer s. Put $\mathbf{s} := (s + (1-e)/2, 0)$.

Theorem. *With the notation introduced above, there is a graph isomorphism*

$$B(\mu, \mathbf{c}) \simeq B((-,-), \mathbf{s}),$$

up to a shift of the labels of the arrows.

Proof. We apply the algorithm used to compute the e-core of $\lambda = \Phi_t(\mu)$ described in the proof of Theorem 7.6. Applying a sequence of elementary operations of types (a′) and (b′) to $\mathfrak{B}(\mu, \mathbf{c})$, we end up with the symbol $\mathfrak{B}((-,-), \mathbf{d})$ for some charge $\mathbf{d} = (d_1, d_2)$.

We may as well apply the corresponding sequence of moves to the β-set $\bar{\mathfrak{A}}$ for $\lambda = \Phi_t(\mu)$ as constructed in 7.2. This results in a β-set $\bar{\mathfrak{A}}'$ for Δ_s. The number of odd elements of $\bar{\mathfrak{A}}$ exceeds its number of even elements by $t = c_1 - (1-e)/2 - c_2$. If the number of odd elements of $\bar{\mathfrak{A}}'$ is not smaller than the number of its even elements, the difference between the two numbers equals s. Otherwise, there are $s+1$ more even numbers in $\bar{\mathfrak{A}}'$ than odd ones. An operation of type (a′) decreases the first component of the current charge by 1 and increases the second component by 1. The corresponding move on the β-set replaces an odd number by an even one. The analogous remarks apply for elementary operations of type (b′). We thus find

$$d_1 - d_2 = s + (1-e)/2 \tag{6}$$

or

$$d_2 - d_1 = s + (1+e)/2. \tag{7}$$

By Lemma 7.7, we have a crystal isomorphism

$$B(\mu, \mathbf{c}) \simeq B((-,-), \mathbf{d}).$$

If we set $\mathbf{d}' = (d_2 - e, d_1)$, we also have a crystal isomorphism

$$B(\mu, \mathbf{c}) \simeq B((-,-), \mathbf{d}')$$

(see [16, 6.2.9, 6.2.17]). By the remark preceding Corollary 7.7, we obtain

$$B(\mu, \mathbf{c}) \simeq B((-,-), (d_1 - d_2, 0))$$

and

$$B(\mu, \mathbf{c}) \simeq B((-,-), (d_2 - e - d_1, 0))$$

up to an overall shift of the labels of the arrows. Applying Identities (6) respectively (7), we see that $s + (1-e)/2$ equals $d_1 - d_2$ in the first case and $d_2 - e - d_1$ in the second. This concludes our proof. □

Note that there should be a way to relate these elementary crystal isomorphisms with the so-called canonical crystal isomorphism of [18].

7.9. Put $\mathbf{c} := (t + (1-e)/2, 0)$. Let $\mu = (\mu^1, \mu^2)$ be a bipartition. For $0 \leq j \leq e-1$, let $\widetilde{f}_j$ denote the associated Kashiwara operator on $\mathcal{G}_{\mathbf{c},e}$ (see 5.6).

Proposition. *Let $0 \leq j_1 \neq j_2 \leq e-1$. Suppose that $\widetilde{f}_{j_i}.|\mu, \mathbf{c}\rangle \neq 0$ for $i = 1, 2$. Write $\widetilde{f}_{j_i}.|\mu, \mathbf{c}\rangle = |\nu_i, \mathbf{c}\rangle$, $i = 1, 2$. Then the e-cores of $\Phi_t(\nu_1)$ and of $\Phi_t(\nu_2)$ are distinct.*

Proof. Let $0 \leq j \leq e-1$. First note that if $\widetilde{f}_j.|\mu, \mathbf{c}\rangle \neq 0$ then

(1) $\mathfrak{B}(\mu, \mathbf{c})^1 = \mathfrak{B}(\widetilde{f}_j.\mu, \mathbf{c})^1$ and $\mathfrak{B}(\widetilde{f}_j.\mu, \mathbf{c})^2 = \mathfrak{B}(\mu, \mathbf{c})^2 \cup \{k\} \setminus \{k-1\}$ for $k \in \mathbb{Z}$ such that $k \equiv j \pmod e$, or

(2) $\mathfrak{B}(\mu, \mathbf{c})^2 = \mathfrak{B}(\widetilde{f}_j.\mu, \mathbf{c})^2$ and $\mathfrak{B}(\widetilde{f}_j.\mu, \mathbf{c})^1 = \mathfrak{B}(\mu, \mathbf{c})^1 \cup \{k\} \setminus \{k-1\}$ for $k \in \mathbb{Z}$ such that $k \equiv j \pmod e$.

We have seen in 7.6 how to compute the e-cores of $\Phi_t(\nu_i)$, $i = 1, 2$. In this procedure, some of the elements x in $\mathfrak{B}(\nu_i, \mathbf{c})$, $i = 1, 2$, must be replaced by $x - k.e$ for some $k \in \mathbb{N}$. If the e-core of $\Phi_t(\nu_1)$ equals the e-core of $\Phi_t(\nu_2)$, this implies that at the end of these procedures, we obtain the same symbols. However, this is impossible as $j_1 \not\equiv j_2 \pmod e$. □

Corollary. *Suppose that Conjecture 5.7 is true. Let X be a unipotent $k\mathrm{GU}_n(q)$-module. Then, if ℓ is large enough, any two non-isomorphic simple submodules of $R_{\mathrm{GU}_{n(q)}}^{\mathrm{GU}_{n+2(q)}}(X)$ lie in distinct ℓ-blocks.*

Proof. By Conjecture 5.7, the non-isomorphic simple submodules of $R_{\mathrm{GU}_{n(q)}}^{\mathrm{GU}_{n+2(q)}}(X)$ correspond to two distinct directed edges in a suitable crystal graph. By the Proposition, the corresponding partitions have distinct e-cores, and thus the unipotent modules labelled by these partitions are in distinct ℓ-blocks. □

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TABLE 1. The Harish-Chandra branching graph for $\mathrm{GU}_n(q)$, $1 \leq n \leq 7$ odd, $\ell \mid q^2 - q + 1$, $\ell > 7$

$$\begin{tikzcd}
 & & 1 \arrow[d] & & & & & & & \\
 & & 3 \arrow[dl] \arrow[d] \arrow[dr] & & & 1^3 \arrow[dl] \arrow[dr] & & & & \\
 & 5 \arrow[dl] \arrow[d] & 31^2 \arrow[dl] \arrow[d] & 32 \arrow[dl] \arrow[d] & 2^21 \arrow[d] \arrow[dr] & & 1^5 \arrow[d] \arrow[dr] & & & \\
7 & 51^2 & 32^2 & 52 & 3^21 & 421 & 31^4 & 2^21^3 & 1^7 & 321^2
\end{tikzcd}$$

TABLE 2. The Harish-Chandra branching graph for $\mathrm{GU}_n(q)$, $1 \leq n \leq 7$ odd, $\ell \mid q^2 - q + 1$, $\ell > 7$, cont.

$$\begin{tikzcd}
 & & 21 \arrow[dl] \arrow[dr] & & & \\
 & 41 \arrow[dl] \arrow[d] & & 21^3 \arrow[d] \arrow[dr] & & \\
61 & 43 & & 41^3 & 21^5 & 2^31
\end{tikzcd}$$

TABLE 3. The crystal graph $\mathcal{G}^{\leq 3}_{(0,0),3}$

$$
\begin{tikzcd}
 & & -.- \arrow[d] & & & & & & & \\
 & & 1.- \arrow[dl] \arrow[d] \arrow[dr] & & & -.1 \arrow[dl] \arrow[dr] & & & & \\
 & 2.- \arrow[dl] \arrow[d] & 1.1 \arrow[dl] \arrow[d] & 1^2.- \arrow[dl] \arrow[d] & -.2 \arrow[d] \arrow[dr] & & -.1^2 \arrow[d] \arrow[dr] & & & \\
3.- & 2.1 & 1^2.1 & 21.- & 1.2 & -.3 & 1.1^2 & -.21 & -.1^3 & 1^3.-
\end{tikzcd}
$$

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