Projective Summands in Tensor Products of Simple Modules of Finite Dimensional Hopf Algebras

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1 Introduction

A classical theorem of Burnside states that, if V is a module for the complex group algebra $\mathbb{C}G$, faithful for the finite group G, then each irreducible $\mathbb{C}G$-module is isomorphic to a composition factor of some tensor power $V^{\otimes n}$. Some fifty years later, Rieffel proved a Hopf algebra version of Burnsides theorem [17]. More recently, Passman and Quinn simplified and amplified the work of Rieffel in [16]. The starting point of our paper is a special case of these results: We study the tensor powers of the semisimple quotient of a finite dimensional Hopf algebra.

Let H be a finite dimensional Hopf algebra over a field k. We show that H contains a unique maximal nilpotent Hopf ideal $J_w(H)$ contained in $J(H)$, the Jacobson radical of H. We give various characterizations of $J_w(H)$, for example $J_w(H) = \mathrm{Ann}_H((H/J(H))^{\otimes n})$ for all large enough n. The smallest positive integer n with this property is denoted by $l_w(H)$. This also equals the smallest number n such that $(H/J(H))^{\otimes n}$ contains every projective indecomposable $H/J_w(H)$-module as a direct summand.

Let $l'_w(H)$ denote the minimal n such that the tensor product of n suitably chosen simple H-modules contains the projective cover of the trivial $H/J_w(H)$-

module as a direct summand. Then obviously $l'_w(H) \leq l_w(H)$. We show that these two numbers are in fact equal.

In Section 2 we introduce $J_w(H)$ and derive some of its properties. In Section 3 we define projective homomorphisms between H-modules, using a trace map. The fact that $l'_w(H)$ equals $l_w(H)$ is proved with the help of projective homomorphisms. These are also used to obtain various reciprocity laws for tensor products of simple H-modules and their projective indecomposable direct summands. In Section 4 we discuss some consequences of our general results in case $H = kG$ is a group algebra of a finite group G and k is a field of characteristic p. Here, $J_w(H) = H(kO_p(G))^+ = (kO_p(G))^+H$, where $(kO_p(G))^+$ is the augmentation ideal of the group algebra $kO_p(G)$. We obtain reduction theorems relating $l_w(kG)$ to $l_w(kN)$ for normal subgroups N of G. Using a consequence of the classification of the finite simple groups, it is easy to show that $l_w(kG) \leq 2$ if G is simple and $\mathrm{char}(k) > 3$. We also give an example of a solvable group G with $l_w(kG) = 3$.

Throughout this paper we work over a field k. Unless otherwise stated, all algebras, coalgebras and Hopf algebras are defined over k. Hom means Hom_k, and $\otimes$ means $\otimes_k$. All modules over algebras are left modules. For any k-vector space V, we write $V^{\otimes n}$ for the n-fold tensor power $V \otimes V \otimes \cdots \otimes V$. We will use [15] and [19] as basic references for Hopf algebras.

2 The Wedge of the Jacobson Radical of Finite Dimensional Hopf Algebras

The exposition in this section is inspired by [16] and [17]. Let C be a coalgebra over k with comultiplication Δ. The *wedge* of two subspaces X and Y of C is defined by

$$X \wedge Y = \mathrm{Ker}(C \xrightarrow{\Delta} C \otimes C \xrightarrow{\pi_X \otimes \pi_Y} C/X \otimes C/Y),$$

where π_X and π_Y are the canonical quotient maps. It has the following properties (see [15] or [19]):

$$X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y),$$

$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z).$$

One defines $\wedge^1 X = X$, and $\wedge^{n+1} X = (\wedge^n X) \wedge X$, for all $n \geq 1$.

Suppose now that H is a finite dimensional Hopf algebra over k with comultiplication Δ, counit ε and antipode S. Let M and N be H-modules. Then $\mathrm{Hom}(M, N)$ is an H-module with action given by

$$(h \cdot f)(m) := \Sigma\, h_1 \cdot (f(S(h_2) \cdot m)),$$

where $h \in H$, $f \in \mathrm{Hom}(M, N)$ and $m \in M$. Let k_ε denote the trivial H-module k. Then for any H-module M, the dual vector space $M^* := \mathrm{Hom}(M, k_\varepsilon)$ is an H-module. The above formula for the action of H on M^* simplifies to

$(h \cdot f)(m) := f(S(h)m)$ for $h \in H$, $f \in M^*$ and $m \in M$. Let $P(M)$ denote the projective cover of a finite dimensional H-module M throughout this section.

Let $J = \mathrm{Jac}(H)$ be the Jacobson radical of H. Since $J \subseteq \mathrm{Ker}(\varepsilon)$, we have $J = \wedge^1 J \supseteq \wedge^2 J \supseteq \wedge^3 J \supseteq \cdots$. Hence there is a positive integer n such that $\wedge^n J = \wedge^{n+1} J$. Put

$$l_w(H) := \min\{n > 0 \mid \wedge^n J = \wedge^{n+1} J\},$$

and

$$J_w(H) := \bigcap_{n>0} \wedge^n J = \wedge^{l_w(H)} J.$$

Lemma 2.1 *The following statements hold for any integer $n \geq 1$:*

(1) *$\wedge^n J$ is an ideal of H and $\wedge^n J = \mathrm{Ann}_H((H/J)^{\otimes n})$.*

(2) *$S(\wedge^n J) = \wedge^n J$.*

(3) *$I \subseteq \wedge^n J$ for all coideals I of H with $I \subseteq J$.*

Proof. (1) If I_1 and I_2 are ideals of H, then for any $h \in H$ we have

$$\begin{aligned} h \cdot (H/I_1 \otimes H/I_2) = \{0\} \quad &\Leftrightarrow \quad \Delta(h) \in I_1 \otimes H + H \otimes I_2 \\ &\Leftrightarrow \quad h \in I_1 \wedge I_2. \end{aligned}$$

Hence $I_1 \wedge I_2$ is an ideal of H and equal to $\mathrm{Ann}_H(H/I_1 \otimes H/I_2)$. Then (1) follows by induction on n.

(2) Since S is an antimorphism of the algebra H, one gets $S(J) = J$. On the other hand, S is also a coalgebra antimorphism of H. Hence for any $h \in H$, $h \cdot ((H/J)^{\otimes n}) = 0$ if and only if $S(h) \cdot ((H/J)^{\otimes n}) = \{0\}$. This shows that $S(\wedge^n J) = \wedge^n J$.

(3) Let I be a coideal of H with $I \subseteq J$. Then $\Delta(I) \subseteq H \otimes I + I \otimes H$ and hence $I \subseteq I \wedge I$, which implies that $I = \wedge^1 I \subseteq \wedge^2 I \subseteq \wedge^3 I \subseteq \cdots$. It follows that $I \subseteq \wedge^n I \subseteq \wedge^n J$. Q.E.D.

Corollary 2.2 *Let $n \geq 1$. Then the following statements are equivalent:*

(1) *$\wedge^n J$ is a Hopf ideal of H.*

(2) *$\wedge^n J$ is a coideal of H.*

(3) *$\wedge^n J \subseteq (\wedge^n J) \wedge (\wedge^n J)$.*

(4) *$\wedge^n J = \wedge^{n+1} J$.*

Proof. This follows from Lemma 2.1. Q.E.D.

Corollary 2.3 *The following statements hold:*

(1) *$J_w(H)$ is the maximal nilpotent Hopf ideal, which is also the maximal coideal of H contained in J.*

(2) *$l_w(H/J_w(H)) = l_w(H)$ and $J_w(H/J_w(H)) = \{0\}$.*

(3) *$l_w(H) = 1$ if and only if $J_w(H) = J$.*

Proof. This follows from Lemma 2.1. Q.E.D.

Corollary 2.4 *The following statements are equivalent:*

(1) $J_w(H) = \{0\}$.

(2) *J contains no nonzero Hopf ideal.*

(3) *$(H/J)^{\otimes l_w(H)}$ is faithful as an H-module.*

(4) *$(H/J)^{\otimes n}$ is faithful as an H-module for any $n \geq l_w(H)$.*

(5) *There exists an n such that $(H/J)^{\otimes n}$ is faithful as an H-module. In this case, $n \geq l_w(H)$.*

(6) *Every projective indecomposable H-module is isomorphic to a direct summand of $(H/J)^{\otimes l_w(H)}$.*

(7) *Every projective indecomposable H-module is isomorphic to a direct summand of $(H/J)^{\otimes n}$ for any $n \geq l_w(H)$.*

(8) *There exists an n such that every projective indecomposable H-module is isomorphic to a direct summand of $(H/J)^{\otimes n}$. In this case, $n \geq l_w(H)$.*

Proof. This follows from Lemma 2.1, Corollary 2.3, and [6, Proposition 1.3]. Q.E.D.

We next introduce certain invariants closely related to $l_w(H)$ and collect some of their properties.

Definition 2.5 (1) If $J_w(H) = \{0\}$, let

$$l'_w(H) := \min\{n > 0 \mid (H/J)^{\otimes n} \text{ contains } P(k_\varepsilon) \text{ as a direct summand}\}$$

and

$$l''_w(H) := \min\{n > 0 \mid (H/J)^{\otimes n} \text{ has a nonzero projective direct summand}\}.$$

(2) In general, let $l'_w(H) := l'_w(H/J_w(H))$ and $l''_w(H) := l''_w(H/J_w(H))$.

Proposition 2.6 (1) *We have $l_w(H) = 1$ if and only if $l'_w(H) = 1$.*

(2) *We have $l'_w(H) \leq l_w(H) \leq l'_w(H) + 1$ and $l''_w(H) \leq l'_w(H) \leq l''_w(H) + 1$.*

(3) *If H has a simple, projective module, then $l'_w(H) \leq 2$ and $J_w(H) = \{0\}$.*

Proof. (1) We may assume that $J_w(H) = \{0\}$. If $l_w(H) = 1$ then H is semisimple by Corollary 2.3(3). Hence $l'_w(H) = 1$, too. Now suppose that $l'_w(H) = 1$. By Definition 2.5, $P(k_\varepsilon)$ is a direct summand of H/J. It follows that k_ε is a projective H-module. Hence $\varepsilon : H \to k_\varepsilon$ is split as an H-module epimorphism, and consequently, $H = \int_H^l \oplus \mathrm{Ker}(\varepsilon)$ as H-modules, where $\int_H^l$ is the space of left integrals of H. It follows that $\varepsilon(\int_H^l) \neq \{0\}$, which is equivalent to the statement that H is semisimple (see [15] or [19]). Corollary 2.3(3) implies that $l_w(H) = 1$.

(2) Let V be a projective H-module. Then $V^* \otimes V$ is also projective by the Fundamental Theorem of Hopf modules (see [15, Theorem 1.9.4]). Since the evaluation map $V^* \otimes V \to k_\varepsilon$, $f \otimes v \mapsto \langle f, v \rangle$ is an H-module epimorphism, $V^* \otimes V$ contains $P(k_\varepsilon)$ as a direct summand. Hence $l'_w(H) \leq l''_w(H) + 1$. Furthermore, for any simple H-module M, $P(k_\varepsilon) \otimes M$ contains $P(M)$ as a direct summand. Thus $l_w(H) \leq l'_w(H) + 1$.

(3) The hypothesis says $l''_w(H) = 1$, so that the first assertion follows from (2). Corollary 2.4 implies that $J_w(H) = \{0\}$. Q.E.D.

Note that by Corollary 2.3(3), $l_w(H) = 1$ if and only if $H/J_w(H)$ is semisimple. We shall show in the next section that if H is not semisimple, then $l_w(H) = l'_w(H)$ and $l''_w(H) = l'_w(H) - 1$. We shall also prove a converse to Part (3) of the above proposition.

3 Trace

Throughout this section we assume that H is a finite dimensional Hopf algebra over the field k with comultiplication Δ, counit ε and antipode S. Let J denote the Jacobson radical $\mathrm{Jac}(H)$ of H. Throughout this and the next section, let k_ε denote the trivial H-module k, and let $P(M)$ and $I(M)$ denote the projective cover and injective envelope of a finite dimensional H-module M, respectively. Suppose that Λ is a *left integral* of H, i.e., $\Lambda \in H$ with $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$. Moreover, let λ be a *right integral* of the dual Hopf algebra H^*, i.e., $\lambda \in H^*$ with $\lambda f = f(1)\lambda$ for all $f \in H^*$. In addition, assume that $\lambda(\Lambda) = 1$. Note that such a pair (Λ, λ) always exist (see [19]).

Note that H^* is a left H-module under the action $\rightharpoonup$ given by

$$\langle h \rightharpoonup f, x \rangle := (h \rightharpoonup f)(x) := f(xh), \quad h, x \in H, \quad f \in H^*,$$

and that H^* is isomorphic to the regular module H via

$$\theta_\lambda : H \to H^*, h \mapsto (h \rightharpoonup \lambda).$$

The following lemma, which goes back to [12], is fundamental. We prove it for completeness.

Lemma 3.1 *Let $\{h_1, h_2, \cdots, h_n\}$ be a k-basis of H and let $\{h^1, h^2, \cdots, h^n\}$ be the dual basis in H^*. Then:*

(1) $\Sigma\lambda(\Lambda_1)S(\Lambda_2) = \Sigma\lambda(\Lambda_1)\Lambda_2 = 1$.

(2) $\Sigma_{i=1}^n h^i \otimes h_i = \Sigma\theta_\lambda(\Lambda_1) \otimes S(\Lambda_2)$.

Proof. (1) For any $f \in H^*$ we have $f(\Sigma\lambda(\Lambda_1)\Lambda_2) = \Sigma\lambda(\Lambda_1)f(\Lambda_2) = (\lambda f)(\Lambda) = f(1)\lambda(\Lambda) = f(1)$. It follows that $\Sigma\lambda(\Lambda_1)\Lambda_2 = 1$, and also that $\Sigma\lambda(\Lambda_1)S(\Lambda_2) = 1$.

(2) Since H is finite dimensional, $H^* \otimes H \cong \mathrm{Hom}(H, H)$. Hence we may regard both sides of the equation in (2) as elements of $\mathrm{Hom}(H, H)$. Let $x \in H$.

Then

$$\begin{aligned}\Sigma\theta_\lambda(\Lambda_1)(x)S(\Lambda_2) &= \Sigma\lambda(x\Lambda_1)S(\Lambda_2)\\ &= \Sigma\lambda(x_1\Lambda_1)S(\Lambda_2)S(x_2)x_3\\ &= \Sigma\lambda(\Lambda_1)S(\Lambda_2)x\\ &= x = \Sigma_{i=1}^n h^i(x)h_i.\end{aligned}$$

Hence $\Sigma_{i=1}^n h^i \otimes h_i = \Sigma\theta_\lambda(\Lambda_1)\otimes S(\Lambda_2)$. Q.E.D.

Note that k is a Hopf subalgebra of H. Let V be a k-vector space. Then we have the induced H-module $V{\uparrow}_k^H = H \otimes V$ and the coinduced H-module $V{\Uparrow}_k^H = \mathrm{Hom}(H,V)$. One can easily check that the natural k-linear isomorphism

$$\begin{aligned}\sigma : H^*\otimes V &\rightarrow \mathrm{Hom}(H,V) = V{\Uparrow}_k^H\\ f\otimes v &\mapsto (h\mapsto f(h)v)\end{aligned}$$

is an H-module isomorphism, where the H-action on $H^*\otimes V$ is given by $\rightharpoonup$ only acting on the left tensor factor H^*. Hence we have an H-module isomorphism

$$\Theta_\lambda = \sigma\circ(\theta_\lambda\otimes\mathrm{id}) : V{\uparrow}_k^H = H\otimes V \rightarrow \mathrm{Hom}(H,V) = V{\Uparrow}_k^H$$

$$\Theta_\lambda(h\otimes v)(h') = \lambda(h'h)v,\quad h,h'\in H,\quad v\in V.$$

Let $\{h_1, h_2, \cdots, h_n\}$ be a k-basis of H and let $\{h^1, h^2, \cdots, h^n\}$ be the dual basis in H^*. Then it is easy to show that $\sigma^{-1} : \mathrm{Hom}(H,V) \rightarrow H^*\otimes V$ is given by $\sigma^{-1}(f) = \Sigma_{i=1}^n h^i\otimes f(h_i)$. It follows from Lemma 3.1(2) that $\sigma^{-1}(f) = \Sigma\theta_\lambda(\Lambda_1)\otimes f(S(\Lambda_2))$ for any $f\in\mathrm{Hom}(H,V)$. Thus we get the following corollary.

Corollary 3.2 *Let V be a k-vector space. Then $V{\uparrow}_k^H \cong V{\Uparrow}_k^H$ as H-modules. An isomorphism $\Theta_\lambda : V{\uparrow}_k^H = H\otimes V \rightarrow \mathrm{Hom}(H,V) = V{\Uparrow}_k^H$ is given by*

$$\Theta_\lambda(h\otimes v)(h') = \lambda(h'h)v,\quad h,h'\in H,\quad v\in V.$$

The inverse $\Theta_\lambda^{-1} : V{\Uparrow}_k^H = \mathrm{Hom}(H,V) \rightarrow H\otimes V = V{\uparrow}_k^H$ is given by

$$\Theta_\lambda^{-1}(f) = \Sigma\Lambda_1\otimes f(S(\Lambda_2)),\quad f\in\mathrm{Hom}(H,V).$$

Corollary 3.3 *Let M be an H-module and let V be a k-vector space. Then we have k-linear isomorphisms:*

(1) $T : \mathrm{Hom}(M,V) \rightarrow \mathrm{Hom}_H(M, V{\uparrow}_k^H) = \mathrm{Hom}_H(M, H\otimes V)$ *given by*
$$T(f)(m) = \Sigma\Lambda_1\otimes f(S(\Lambda_2)m),\ f\in\mathrm{Hom}(M,V),\ m\in M.$$

(2) $F : \mathrm{Hom}(V,M) \rightarrow \mathrm{Hom}_H(V{\uparrow}_k^H, M) = \mathrm{Hom}_H(H\otimes V, M)$ *given by*
$$F(f)(h\otimes v) = hf(v),\ f\in\mathrm{Hom}(V,M),\ h\in H,\ v\in V.$$

Proof. This follows from the Nakayama relations (see, e.g., [3, Proposition 2.8.3]) and Corollary 3.2. Q.E.D.

Let M be any H-module. Then the subspace of *invariants* is defined by

$$M^H := \{m\in M \mid hm = \varepsilon(h)m,\quad \forall h\in H\}.$$

Thus M^H is the unique maximal trivial H-submodule of M and there is a trace map

$$\widehat{\Lambda}\colon M \to M^H, \quad m \mapsto \Lambda m, \quad m \in M.$$

If V is another H-module, then $\mathrm{Hom}(M,V)$ is an H-module and

$$\mathrm{Hom}(M,V)^H = \mathrm{Hom}_H(M,V).$$

We thus have a trace map $\widehat{\Lambda}\colon \mathrm{Hom}(M,V) \to \mathrm{Hom}_H(M,V)$, whose image we denote by $\mathrm{Hom}_H(M,V)_1$. By definition, $\widehat{\Lambda}(f)(m) = (\Lambda \cdot f)(m) = \Sigma\Lambda_1 f(S(\Lambda_2)m)$ for $f \in \mathrm{Hom}(M,V)$ and $m \in M$. The elements of $\mathrm{Hom}_H(M,V)_1$ are called *projective homomorphisms.*

Proposition 3.4 *Let M and V be H-modules. Let $\phi \in \mathrm{Hom}(M,V)$. Then $\phi \in \mathrm{Hom}_H(M,V)_1$ if and only if there exist a projective H-module P, maps $\psi_1 \in \mathrm{Hom}_H(M,P)$ and $\psi_2 \in \mathrm{Hom}_H(P,V)$ such that $\phi = \psi_2 \circ \psi_1$.*

Proof. Suppose that $\phi \in \mathrm{Hom}_H(M,V)_1$. Then there is an $f \in \mathrm{Hom}(M,V)$ such that $\phi = \widehat{\Lambda}(f)$. Let $P := V{\downarrow}_k^H{\uparrow}_k^H = H \otimes V{\downarrow}_k$. Then P is a projective (in fact a free) H-module. Let $\psi_1 := T(f)$ and $\psi_2 : P = H \otimes V \to V$, $h \otimes v \mapsto hv$, where T is as in Corollary 3.3. Then $\psi_1 \in \mathrm{Hom}_H(M,P)$ and $\psi_2 \in \mathrm{Hom}_H(P,V)$. Now let $m \in M$. Then we have $(\psi_2 \circ \psi_1)(m) = \psi_2(T(f)(m)) = \Sigma\Lambda_1 f(S(\Lambda_2)m) = \widehat{\Lambda}(f)(m) = \phi(m)$.

To prove the converse, note that for any projective H-module P there is a k-vector space W such that P is a direct summand of $W{\uparrow}_k^H$. Hence we may assume that there is a k-vector space W and H-module morphisms $\psi_1 : M \to W{\uparrow}_k^H$ and $\psi_2 : W{\uparrow}_k^H \to V$ such that $\phi = \psi_2 \circ \psi_1$. By Corollary 3.3 there exist $\phi_1 \in \mathrm{Hom}(M,W)$ with $\psi_1 = T(\phi_1)$, and $\phi_2 \in \mathrm{Hom}(W,V)$ with $\psi_2 = F(\phi_2)$. Then for any $m \in M$, $\phi(m) = (\psi_2 \circ \psi_1)(m) = F(\phi_2)(T(\phi_1)(m)) = F(\phi_2)(\Sigma\Lambda_1 \otimes \phi_1((S\Lambda_2)m)) = \Sigma\Lambda_1\phi_2(\phi_1((S\Lambda_2)m)) = \Sigma\Lambda_1(\phi_2 \circ \phi_1)((S\Lambda_2)m) = (\Lambda \cdot (\phi_2 \circ \phi_1))(m)$. Hence $\phi = \Lambda \cdot (\phi_2 \circ \phi_1) \in \mathrm{Hom}(M,V)_1$. Q.E.D.

Corollary 3.5 *Let M and V be finite-dimensional H-modules and let $\phi \in \mathrm{Hom}_H(M,V)$. Then the following statements are equivalent:*

(1) $\phi \in \mathrm{Hom}_H(M,V)_1$.

(2) *ϕ factors through the projective cover of V.*

(3) *ϕ factors through the injective envelope of M.*

Proof. Since a finite dimensional Hopf algebra is a Frobenius algebra, an H-module is projective if and only if it is injective. By using Proposition 3.4 the corollary can be proven in the same way as that of [11, Lemma II.2.7]. Q.E.D.

Corollary 3.6 *Let M be a finite-dimensional H-module and let V be a simple H-module. Then the following statements hold:*

(1) $\mathrm{Hom}_H(M,V)_1 \neq \{0\}$ *if and only if $P(V)$ is a direct summand of M.*

(2) $\mathrm{Hom}_H(V,M)_1 \neq \{0\}$ *if and only if $I(V)$ is a direct summand of M.*

Proof. If $\mathrm{Hom}_H(M,V)_1 \neq \{0\}$, then it follows from Corollary 3.5 that there are H-module morphisms $\phi_1 : M \to P(V)$ and $\phi_2 : P(V) \to V$ such that $\phi_2 \circ \phi_1 \neq 0$. Since V is simple and $P(V)$ is the projective cover of V, ϕ_1 must be surjective. It follows that $P(V)$ is a direct summand of M. Conversely, if $P(V)$ is a direct summand of M, then there is an H-module epimorphism $\phi_1 : M \to P(V)$. Of course, there is an H-module epimorphism $\phi_2 : P(V) \to V$. Hence $\phi_2 \circ \phi_1 \neq 0$, and it follows from Corollary 3.5 that $\mathrm{Hom}_H(M,V)_1 \neq \{0\}$. This proves (1). Part (2) follows with a dual argument. Q.E.D.

Theorem 3.7 *Let M, V and W be H-modules. Then*

(1) $\mathrm{Hom}(W \otimes M, V) \cong \mathrm{Hom}(W, \mathrm{Hom}(M,V))$ *as H-modules and*

(2) $\mathrm{Hom}_H(W \otimes M, V)_1 \cong \mathrm{Hom}_H(W, \mathrm{Hom}(M,V))_1$.

If, in addition, M or V is finite dimensional, then

(3) $\mathrm{Hom}(W \otimes M, V) \cong \mathrm{Hom}(W, V \otimes M^*)$ *as H-modules and*

(4) $\mathrm{Hom}_H(W \otimes M, V)_1 \cong \mathrm{Hom}_H(W, V \otimes M^*)_1$.

Proof. Parts (1) and (2) are straightforward. Parts (3) and (4) follow from Parts (1), (2) and [10, Proposition III.5.2].

Note that Property (4) has been proved by Landrock and Michler in case of group algebras (see [11, Theorem II.6.10]). Q.E.D.

Corollary 3.8 *Let V_1 and V_2 be simple H-modules. Then*

(1) *$P(k_\varepsilon)$ is isomorphic to a direct summand of $V_1 \otimes V_2$ if and only if $V_1 \cong V_2^*$ and V_1 is projective.*

(2) *$I(k_\varepsilon)$ is isomorphic to a direct summand of $V_1 \otimes V_2$ if and only if $V_1^* \cong V_2$ and V_1 is projective.*

(3) *$P(k_\varepsilon)$ is isomorphic to a direct summand of $V_1 \otimes V_2$ if and only if $I(k_\varepsilon)$ is a direct summand of $V_2 \otimes V_1$.*

(4) *If S^2 is inner, then $P(k_\varepsilon)$ is isomorphic to a direct summand of $V_1 \otimes V_2$ if and only if $I(k_\varepsilon)$ is a direct summand of $V_1 \otimes V_2$.*

Proof. (1) By Corollary 3.6 and Theorem 3.7(4) we have:

$$\begin{aligned} P(k_\varepsilon) \text{ is a direct summand of } V_1 \otimes V_2 &\Leftrightarrow Hom_H(V_1 \otimes V_2, k_\varepsilon)_1 \neq \{0\} \\ &\Leftrightarrow \mathrm{Hom}_H(V_1, V_2^*)_1 \neq \{0\} \\ &\Leftrightarrow P(V_2^*) \text{ is a direct summand of } V_1 \\ &\Leftrightarrow V_1 \cong V_2^* \text{ and } V_1 \text{ is projective.} \end{aligned}$$

(2) For any H-module V, we define an H-action on V^* as follows:

$$(h \triangleright f)(v) := f(S^{-1}(h)v), \quad h \in H, \quad f \in V^*, \quad v \in V.$$

Denote the H-module $(V^*, \triangleright)$ by V°. One can easily check that $V^{*\circ} \cong V \cong V^{\circ *}$ as H-modules when V is finite dimensional. Since H is a Frobenius algebra, a finite dimensional H-module V is projective if and only if V^* is projective if and only if V° is projective. Hence by Corollary 3.6 and Theorem 3.7(4) we have

$$\begin{aligned}
I(k_\varepsilon) \text{ is a direct summand of } V_1 \otimes V_2 \quad &\Leftrightarrow \quad \mathrm{Hom}_H(k_\varepsilon, V_1 \otimes V_2)_1 \neq \{0\} \\
&\Leftrightarrow \quad \mathrm{Hom}_H(k_\varepsilon, V_1 \otimes V_2^{\circ *})_1 \neq \{0\} \\
&\Leftrightarrow \quad \mathrm{Hom}_H(V_2^\circ, V_1)_1 \neq \{0\} \\
&\Leftrightarrow \quad P(V_1) \text{ is a direct summand of } V_2^\circ \\
&\Leftrightarrow \quad V_1 \cong V_2^\circ \text{ and } V_1 \text{ is projective} \\
&\Leftrightarrow \quad V_1^* \cong V_2 \text{ and } V_1 \text{ is projective}
\end{aligned}$$

(3) This follows from (1) and (2).

(4) If S^2 is inner, then $V^{**} \cong V$ for any finite dimensional H-module V, i.e., $V^\circ \cong V^*$. Now (4) follows from (1) and (2). Q.E.D.

Corollary 3.9 *Let $V_1, \cdots, V_r$ be simple H-modules with $r \geq 2$. Then $P(k_\varepsilon)$ is a direct summand of $V_1 \otimes \cdots \otimes V_r$ if and only if $P(V_r^*)$ is a direct summand of $V_1 \otimes \cdots \otimes V_{r-1}$.*

Proof. By Theorem 3.7(4), $\mathrm{Hom}_H(V_1 \otimes \cdots \otimes V_r, k_\varepsilon)_1 \cong \mathrm{Hom}_H(V_1 \otimes \cdots \otimes V_{r-1}, V_r^*)_1$. This implies the result by Corollary 3.6. Q.E.D.

We can now prove the results mentioned at the end of Section 2.

Corollary 3.10 *Suppose that $J_w(H) \neq J$. Then $l'_w(H) = 2$ if and only if $H/J_w(H)$ has a simple and projective module.*

Proof. This follows from Corollary 3.8 and Proposition 2.6(1). Q.E.D.

Corollary 3.11 (1) *We have $l_w(H) = l'_w(H)$.*

(2) *Suppose that $l'_w(H) > 1$. Then $l''_w(H) = l'_w(H) - 1$.*

Proof. (2) This is immediate by Corollary 3.9.

(1) We first show, using the argument of Alperin in the proof of [1, Theorem 2], that $l_w(H) \leq l''_w(H) + 1$. For this it suffices to prove that if P is a nonzero projective H-module and if V is a simple H-module, then there is a simple H-module V_1 such that $P(V)$ is a direct summand of $V_1 \otimes P$. Indeed, let V_1 be a simple submodule of $V \otimes P^*$. By Theorem 3.7(3) we have $\mathrm{Hom}_H(V_1, V \otimes P^*) \cong \mathrm{Hom}_H(V_1 \otimes P, V)$ (since the isomorphism given there is an isomorphism of H-modules). Since $V_1 \otimes P$ is projective and V is simple this implies that $P(V)$ is a direct summand of $V_1 \otimes P$.

If $l'_w(H) = 1$, then $l_w(H) = l'_w(H)$ by Proposition 2.6(1). We may thus assume that $l'_w(H) > 1$. By Part (2) we find that

$$l_w(H) \leq l''_w(H) + 1 = l'_w(H) \leq l_w(H),$$

proving our assertion. Q.E.D.

The following result is needed in the proof of the reciprocity laws of Corollary 3.14.

Lemma 3.12 *Let M be a finite-dimensional H-module and let V be a simple H-module. If k is a splitting field for H, then*

(1) $\dim_k \mathrm{Hom}_H(M,V)_1$ *equals the multiplicity of $P(V)$ as a direct summand of M, and*

(2) $\dim_k \mathrm{Hom}_H(V,M)_1$ *equals the multiplicity of $I(V)$ as a direct summand of M.*

Proof. Suppose $M = P_1 \oplus P_2 \oplus \cdots \oplus P_n \oplus W$ such that $P_i \cong P(V)$ for $1 \le i \le n$, and such that $P(V)$ is not a direct summand of W. Let $\pi_i : M \to P_i$ be the projection corresponding to the above decomposition, and let $\sigma_i : P_i \to P(V)$ be an H-module isomorphism for $1 \le i \le n$. Let $\tau : P(V) \to V$ be a fixed H-module epimorphism. Let $\phi_i = \tau \circ \sigma_i \circ \pi_i$, $1 \le i \le n$. Then ϕ_i is an H-module epimorphism from M to V, and $\phi_i \in \mathrm{Hom}_H(M,V)_1$ by Corollary 3.5. Clearly, $\phi_i(W) = \{0\}$ and if $i \ne j$ then $\phi_i(P_j) = \{0\}$. Suppose that $\Sigma_{i=1}^n \alpha_i \phi_i = 0$ for some $\alpha_i \in k$. Then for any j we have $\{0\} = (\Sigma_{i=1}^n \alpha_i \phi_i)(P_j) = \alpha_j \phi_j(P_j) = \alpha_j \phi_j(M) = \alpha_j V$, which implies that $\alpha_j = 0$. It follows that $\{\phi_i \mid 1 \le i \le n\}$ is linearly independent over k.

On the other hand, let $\phi \in \mathrm{Hom}_H(M,V)_1$. It follows from Corollary 3.5 that there is an H-module morphism $\psi : M \to P(V)$ such that $\phi = \tau \circ \psi$. And then $\phi|_W = \tau \circ (\psi|_W)$. Since $P(V)$ is not a direct summand of W, it follows from Corollaries 3.5 and 3.6 that $\phi|_W = 0$. Since $P(V) \cong P_i$ is a projective cover of V and k is a splitting field for H, $\mathrm{Hom}_H(P_i,V) \cong \mathrm{Hom}_H(P(V),V) \cong \mathrm{Hom}_H(P(V)/\mathrm{rad}P(V),V) \cong k$. Now $\mathrm{Hom}_H(P_i,V) \ni \phi_i|_{P_i} = \tau \circ \sigma_i \ne 0$, and hence there is an $\alpha_i \in k$ such that $\phi|_{P_i} = \alpha_i \phi_i|_{P_i}$ for any $1 \le i \le n$. We claim that $\phi = \Sigma_{i=1}^n \alpha_i \phi_i$. In fact, we have proved that $\phi(w) = \Sigma_{i=1}^n \alpha_i \phi_i(w) = 0$ for all $w \in W$. Let $m \in P_j$. Then $(\Sigma_{i=1}^n \alpha_i \phi_i)(m) = \Sigma_{i=1}^n \alpha_i \phi_i(m) = \alpha_j \phi_j(m) = \phi(m)$. This shows the claimed equation. From the above discussion it is easy to see that if the multiplicity of $P(V)$ as a direct summand of M is infinite then $\dim_k \mathrm{Hom}_H(M,V)_1 = \infty$. This completes the proof of (1). One can prove (2) by a dual argument. Q.E.D.

Corollary 3.13 *Let V_1, V_2 and V_3 be simple H-modules. Assume that k is a splitting field for H. Then the following numbers are equal:*

(1) *The multiplicity of $P(V_3)$ as a direct summand of $V_1 \otimes V_2$.*

(2) *The multiplicity of $I(V_1)$ as a direct summand of $V_3 \otimes V_2^*$.*

Proof. This follows from Lemma 3.12 and Theorem 3.7(4). Q.E.D.

Recall from [5] or [15] that a Hopf algebra H is *almost cocommutative* if the antipode S of H is bijective and if there exists an invertible element $R \in H \otimes H$ such that for all $h \in H$,

$$\Delta^{op}(h) := \Sigma h_2 \otimes h_1 = R\Delta(h)R^{-1}.$$

It is well-known (see, e.g., [15]), that if H is almost cocommutative then S^2 is inner and $V \otimes W \cong W \otimes V$ for any H-modules V and W.

Corollary 3.14 *Assume that k is a splitting field for H and that H is almost cocommutative. Let V_1, V_2, and V_3 be simple H-modules. Then the following numbers are equal:*

(1) *The multiplicity of $P(V_3)$ as a direct summand of $V_1 \otimes V_2$.*

(2) *The multiplicity of $P(V_2^*)$ as a direct summand of $V_1 \otimes V_3^*$.*

(3) *The multiplicity of $P(V_1^*)$ as a direct summand of $V_2 \otimes V_3^*$.*

(4) *The multiplicity of $I(V_1)$ as a direct summand of $V_2^* \otimes V_3$.*

(5) *The multiplicity of $I(V_2)$ as a direct summand of $V_1^* \otimes V_3$.*

(6) *The multiplicity of $I(V_3^*)$ as a direct summand of $V_1^* \otimes V_2^*$.*

Proof. This follows from Lemma 3.12, Theorem 3.7(4), and the proof of Corollary 3.8. Q.E.D.

4 The Case of Group Algebras

In this section we apply and extend our previous result to the case of group algebras of finite groups. Thus let G be a finite group and k a field. We also fix a prime number p.

The first part of the following result is well known (see, e.g., [17, Corollary 2] or [16, p. 329]).

Lemma 4.1 *Put $H = kG$ and let I be a Hopf ideal of H. Then there is a normal subgroup N of G such that $I = HK^+ = K^+H$, and hence $H/I \cong k(G/N)$ as Hopf algebras, where $K = kN$ is a normal Hopf subalgebra of H, $K^+ = K \cap H^+$, $H^+ = \mathrm{Ker}(\varepsilon)$.*

Furthermore, if $\mathrm{char}(k) = p > 0$, then I is nilpotent if and only if N is a p-group.

Proof. Let $\pi : H \to H/I$ be the natural projection. Since π is a Hopf algebra epimorphism and G is a k-basis in H, $H/I = k\overline{G}$, where $\overline{G} = \{\pi(g) \mid g \in G\} = G(H/I)$. The map π induces a group epimorphism by restriction $\pi|_G : G \to \overline{G}$. Let $N = \mathrm{Ker}(\pi|_G)$. Then N is a normal subgroup and $\pi|_G$ induces a group isomorphism $G/N \cong \overline{G}$, and the composition $H \xrightarrow{\pi} H/I = k\overline{G} \cong k(G/N)$ is induced by the natural projection of groups $G \to G/N$. It follows that $\mathrm{Ker}(\pi) = HK^+ = K^+H$, where $K = kN$.

If I is nilpotent, then K^+ is a nilpotent ideal of kN which implies that N is a p-group. Conversely, if N is a p-group, then N acts trivially on every simple kG-module and thus $K^+ \subseteq J$. Hence $I \subseteq J$ is nilpotent. Q.E.D.

Recall that $O_p(G)$ denotes the largest normal p-subgroup of G.

Corollary 4.2 *Suppose that* $\mathrm{char}(k) = p > 0$*. Put* $H = kG$*. Then* $J_w(H) = H(kO_p(G))^+ = (kO_p(G))^+H$*, and hence* $H/J_w(H) \cong k(G/O_p(G))$*. Consequently* $O_p(G) = \{1\}$ *if and only if* $J_w(H) = \{0\}$*.*

Proof. By Corollary 2.3(1), $J_w(H)$ is the maximal nilpotent Hopf ideal of H. The result follows from Lemma 4.1. Q.E.D.

Note that Corollary 2.4(6), together with Corollary 4.2 provide an alternative proof for a theorem of Alperin [1, Theorem 1], namely that every indecomposable projective kG-module is irreducibly generated if $O_p(G) = \{1\}$.

Suppose that $\mathrm{char}(k) = p$ and let F be an extension field of k. Then we can naturally identify FG with $F \otimes kG$. Hence $J_w(FG) = F \otimes J_w(kG) = FJ_w(kG)$ by Corollary 4.2. It is well-known (see, e.g., [9, Theorem 1.5]), that $\mathrm{Jac}(FG) = F \otimes \mathrm{Jac}(kG)$. By induction on n one can easily check that $\wedge^n \mathrm{Jac}(FG) = F \otimes \wedge^n \mathrm{Jac}(kG)$ for all $n \geq 1$. It follows from Corollary 4.2 that $l_w(FG) = l_w(kG)$. This in turn implies $l_w(kG) = l_w(k'G)$ for any two fields k and k' of the same characteristic.

Definition 4.3 We define

$$h_p(G) := l_w(kG),$$

where k is a field of characteristic p. (By the preceding remarks this is independent of the chosen field.)

As an application of the ideas developed so far we prove a well known result (see [16, Corollary 8]). Recall from [2] that a Hopf algebra H is said to have the *Chevalley property* if the tensor product of any two simple H-modules is semisimple. This is equivalent to the statement that the Jacobson radical $J = \mathrm{Jac}(H)$ of H is a Hopf ideal (see [13] or Lemma 2.1(1) and Corollary 2.2).

Theorem 4.4 *Suppose that* $\mathrm{char}(k) = p > 0$*. Then the following statements are equivalent:*

(1) $H = kG$ *has the Chevalley property.*

(2) *Any Sylow* p*-subgroup of* G *is normal in* G*.*

(3) $O_p(G)$ *is a Sylow* p*-subgroup of* G*.*

(4) $h_p(G) = 1$*.*

Proof. By Lemma 4.1 and Corollary 4.2, $J_w(H) = H(kO_p(G))^+ = (kO_p(G))^+H$ and $H/J_w(H) \cong k\overline{G}$ as Hopf algebras, where $\overline{G} := G/O_p(G)$. Now by Corollary 2.3 we have

$$\begin{aligned} J \text{ is a Hopf ideal} &\Leftrightarrow J = J_w(H) \\ &\Leftrightarrow H/J_w(H) \text{ is semisimple} \\ &\Leftrightarrow k\overline{G} \text{ is semisimple} \\ &\Leftrightarrow p \text{ does not divide } |\overline{G}| = [G : O_p(G)] \\ &\Leftrightarrow O_p(G) \text{ is a Sylow } p\text{-subgroup of } G. \end{aligned}$$

This shows the equivalence of (1) and (3). The equivalences of (1) and (4) and of (2) and (3) are obvious. Q.E.D.

Using the classification of the finite simple groups we can determine $h_p(G)$ for such groups provided p is at least 5.

Corollary 4.5 *Let G be a finite, non-abelian simple group and let p be a prime. If $p \geq 5$, then $h_p(G) \leq 2$.*

Proof. Let k be algebraically closed with $\mathrm{char}(k) = p \geq 5$. By the classification of the finite simple groups, a finite non-abelian simple group must be one of the following: (i) an alternating group A_n with $n \geq 5$; (ii) a finite simple group of Lie type; (iii) a sporadic simple group. If G is as in case (i), then it follows from [8] that kG has a simple and projective module. If G is as in case (ii), by [14, Theorem 5.1], kG has a simple and projective module. Using GAP [7], Thomas Breuer has checked that kG has a simple and projective module if G is a sporadic simple group. Thus in any case, kG has a simple and projective module.

It follows from Proposition 2.6 and Corollary 3.11(1) that $h_p(G) = l_w(kG) = l'_w(kG) \leq 2$. Q.E.D.

We next derive two reduction theorems relating the invariant $h_p(G)$ to normal subgroups and central factor groups of G, respectively.

Theorem 4.6 *Let N be a normal subgroup of G. Then the following holds.*

(1) $h_p(N) \leq h_p(G)$.

(2) *If $p \nmid [G : N]$, then $h_p(N) = h_p(G)$.*

Proof. Clearly, $O_p(N) = N \cap O_p(G)$. Let $\overline{G} := G/O_p(G)$ and $\overline{N} := N/O_p(N)$. Then $\overline{N} \cong NO_p(G)/O_p(G)$ is a normal subgroup of $\overline{G}$. Moreover, $[G : N] = [G : NO_p(G)]\,[NO_p(G) : N] = [\overline{G} : \overline{N}]\,[O_p(G) : O_p(N)]$. Hence if $p \nmid [G : N]$ then $p \nmid [\overline{G} : \overline{N}]$. Thus we may assume that $O_p(G) = \{1\}$ and $O_p(N) = \{1\}$ without loss of generality.

Assume that $\mathrm{char}(k) = p$ and put $H := kG$ and $L := kN$. Then L is a Hopf subalgebra of H and $J_w(H) = J_w(L) = \{0\}$. Let $J(H)$ and $J(L)$ denote the Jacobson radicals of H and L respectively. Then $J(L) = L \cap J(H)$ since N is a normal subgroup of G. Denote the wedge in L by $\wedge_L^n$ and the one in H by $\wedge_H^n$.

(1) Since L is a Hopf subalgebra of H and $J(L) = L \cap J(H)$, one easily checks that $\wedge_L^n J(L) = L \cap \wedge_H^n J(H)$ for all $n \geq 1$. It follows that $h_p(N) \leq h_p(G)$.

(2) Assume that $p \nmid [G : N]$. Then $V{\uparrow}^H$ is semisimple for any simple L-module V (see, e.g., [9, Theorem VII.9.4]). Put $n := l'_w(kN)$ and let $V_1, \ldots, V_n$ be simple kN-modules such that $V_1 \otimes \cdots \otimes V_n$ contains the projective cover Q of the trivial kN-module as a direct summand. This implies that $V_1{\uparrow}^H \otimes \cdots \otimes V_n{\uparrow}^H$ contains the projective cover of the trivial kG-module as a direct summand. Since $V_i{\uparrow}^H$ is semisimple for all i, it follows that $l'_w(kG) \leq n$. Hence $h_p(G) = l_w(kG) = l'_w(kG) \leq l'_w(kN) = l_w(kN) = h_p(N)$. Q.E.D.

Proposition 4.7 *Let $Z(G)$ denote the center of G, let Z be a subgroup of $Z(G)$ and put $\overline{G} := G/Z$. If $O_p(G) = \{1\}$ then $O_p(\overline{G}) = \{1\}$ and $h_p(G) \leq h_p(\overline{G})$.*

Proof. Assume that $O_p(G) = \{1\}$. Let $K/Z = O_p(\overline{G}) = O_p(G/Z)$, and let T be a Sylow p-subgroup of K. Since $p \nmid |Z|$ and Z is a subgroup of $Z(G)$, $K = T \times Z$, and hence $T = O_p(K)$. It follows that T is normal in G, which implies $T = \{1\}$ since $O_p(G) = \{1\}$. Hence $O_p(\overline{G}) = \{1\}$.

Assume that $\mathrm{char}(k) = p$. Then the canonical group epimorphism $G \to \overline{G}$ induces an algebra epimorphism $kG \to k\overline{G}$. In turn, this induces an algebra epimorphism $kG/J(kG) \to k\overline{G}/J(k\overline{G})$ which is also a homomorphism of kG-modules. Note that the projective cover $P(k_\varepsilon)$ of the trivial $k\overline{G}$-module k_ε is also the projective cover of the trivial kG-module k_ε. Hence if $(k\overline{G}/J(k\overline{G}))^{\otimes n}$ contains $P(k_\varepsilon)$ as a $k\overline{G}$-module direct summand for some $n \geq 1$, then $(kG/J(kG))^{\otimes n}$ contains $P(k_\varepsilon)$ as a kG-module direct summand. This shows that $l'_w(kG) \leq l'_w(k\overline{G})$. The assertion now follows from Corollary 3.11(1). Q.E.D.

Theorem 4.8 *Assume that $\mathrm{char}(k) = p$. Let N be a normal subgroup G with $p \nmid |N|$ and let P be a Sylow p-subgroup of G.*

(1) *Let $M_1, M_2, \cdots, M_n$ be simple kN-modules, and put*

$$U_i := \mathrm{Stab}_G(M_i) = \{g \in G \mid {}^gM_i \cong M_i\}.$$

If $U_1 \cap U_2 \cdots \cap U_n \cap P = \{1\}$, then there exist simple kG-modules $V_1, V_2, \cdots, V_n$ such that $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ has a nonzero projective direct summand.

(2) *Assume in addition that N is abelian and has a complement in G. Assume also that k is a splitting field for all subgroups of G. Suppose that $V_1, V_2, \cdots, V_m$ are simple kG-modules such that $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ has a nonzero projective direct summand. Let M_i be a simple submodule of $V_i{\downarrow}_N$, and put $U_i := \mathrm{Stab}_G(M_i)$, $i = 1, 2, \cdots, m$. Then ${}^{y_1}U_1 \cap {}^{y_2}U_2 \cap \cdots \cap {}^{y_m}U_m \cap P = \{1\}$ for some $y_1, y_2, \cdots, y_m \in G$.*

Proof. (1) By Clifford theory (see [4, p. 259]), there are kU_i-modules W_i, $i = 1, 2, \cdots, n$, such that $V_i := W_i{\uparrow}_{U_i}^G$ is simple. Using Mackey's tensor product theorem (see [4, p. 240]), one can easily show by induction on n that $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is isomorphic to a direct sum of modules of the form

$$(({}^{x_1}W_1 \otimes {}^{x_2}W_2 \otimes \cdots \otimes {}^{x_n}W_n){\downarrow}_{{}^{x_1}U_1 \cap {}^{x_2}U_2 \cap \cdots \cap {}^{x_n}U_n}){\uparrow}^G,$$

with $x_i \in G$, $1 \leq i \leq n$. One of the direct summands occurs for $x_1 = x_2 = \cdots = x_n = 1$. Since P is a Sylow p-subgroup of G, $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ contains a nonzero projective direct summand if and only if $(V_1 \otimes V_2 \otimes \cdots \otimes V_n){\downarrow}_P$ contains one. By Mackey's subgroup theorem (see [4, p. 237]), $(V_1 \otimes V_2 \otimes \cdots \otimes V_n){\downarrow}_P$ contains a direct summand isomorphic to $((W_1 \otimes W_2 \otimes \cdots \otimes W_n){\downarrow}_{U_1 \cap U_2 \cap \cdots \cap U_n \cap P}){\uparrow}^P$, which is projective since $U_1 \cap U_2 \cap \cdots \cap U_n \cap P = \{1\}$.

(2) Since k is a splitting field for kN by assumption, the M_i are all 1-dimensional. Since N is abelian and has a complement in G, the M_i can be

extended to simple (1-dimensional) kU_i-modules W_i. As in the proof of (1), $(V_1 \otimes V_2 \otimes \cdots \otimes V_m){\downarrow_P}$ contains a nonzero projective direct summand. Hence there are elements $y_1, y_2, \cdots, y_m \in G$ such that

$$(({}^{y_1}W_1 \otimes {}^{y_2}W_2 \otimes \cdots \otimes {}^{y_m}W_m){\downarrow_{{}^{y_1}U_1 \cap {}^{y_2}U_2 \cap \cdots \cap {}^{y_m}U_m \cap P}}){\uparrow^P}$$

has a nonzero projective direct summand. Since P is a p-group, a kP-module is projective if and only if it is free. Since all the W_i are 1-dimensional, one gets ${}^{y_1}U_1 \cap {}^{y_2}U_2 \cap \cdots \cap {}^{y_m}U_m \cap P = \{1\}$. Q.E.D.

Proposition 4.9 *Let G be a finite solvable group. Suppose that, for some prime $q \neq p$, there exists an elementary abelian normal q-subgroup N of G with $C_G(N) = N$, and such that G acts irreducibly on N. Then $h_p(G) \leq 4$.*

Proof. We may assume that k is algebraically closed and that $\mathrm{char}(k) = p$. Let $\widehat{N} := \mathrm{Hom}(N, \mathbb{C}^\times)$ denote the set of irreducible $\mathbb{C}$-characters of N. Then, by assumption, $L := G/N$ acts irreducibly and faithfully on $\widehat{N}$. It follows from [18, Theorems 2.1 and 3.1] that there exist $\lambda_1, \lambda_2, \lambda_3 \in \widehat{N}$ with $\mathrm{Stab}_G(\lambda_1) \cap \mathrm{Stab}_G(\lambda_2) \cap \mathrm{Stab}_G(\lambda_3) = N$. Since $\mathrm{char}(k) \nmid |N|$, it follows that there exist simple kN-modules M_1, M_2, M_3 (with Brauer characters λ_1, λ_2, λ_3 respectively) such that $\mathrm{Stab}_G(M_1) \cap \mathrm{Stab}_G(M_2) \cap \mathrm{Stab}_G(M_3) = \{1\}$. It follows from Theorem 4.8(1) that there are simple kG-modules V_1, V_2 and V_3 such that $V_1 \otimes V_2 \otimes V_3$ has a nonzero projective direct summand. Proposition 2.6(2) and Corollary 3.11(1) now imply the desired result. Q.E.D.

We conclude with an example of a group G of the kind discussed above, which satisfies $h_2(G) = 3$. We have not been able to find a group with $h_p(G) = 4$.

Example 4.10 Let $\mathbb{F}_3$ denote the field with 3 elements and put

$$P = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \leq GL(2, \mathbb{F}_3).$$

Thus P is the dihedral group of order 8. Let N be the 2-dimensional column space over $\mathbb{F}_3$. Then P has two orbits on $N \setminus \{0\}$, namely

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}.$$

Let $G = N \rtimes P$. Then $|G| = 72$ and $O_2(G) = \{1\}$. By Theorem 4.8(2), G does not have a simple projective kG-module, but by Theorem 4.8(1) there are two simple kG-modules V_1 and V_2 such that $V_1 \otimes V_2$ contains a projective direct summand. Hence $h_2(G) = 3$.

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