Finite groups of Lie type and their representations – Lecture III

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Contents

CLASSIFICATION OF REPRESENTATIONS: RECOLLECTION

Let G be a finite group and k an algebraically closed field with $\mathrm{char}(k) = \ell \geq 0$.

1. There are only finitely many irreducible k-representations of G up to equivalence.
2. Classify all irreducible representations of G.
3. Describe all irreducible representations of all finite simple groups.

In the following, unless otherwise said, let G be a finite reductive group of characteristic p.

In Lecture 2 we have considered the situation $\ell = p$.

In this lecture we will mainly, but not exclusively, investigate the case $\ell = 0$.

LEVI SUBGROUPS: RECOLLECTION

Recall that there is a distinguished class of subgroups of G, the parabolic subgroups.

One way to describe them is through the concept of split BN-pairs of characteristic p.

A parabolic subgroup P has a Levi decomposition $P = LU$, where $U = O_p(P) \lhd P$ is the unipotent radical of P, and L a Levi complement of U in P, i.e. L is a Levi subgroup of G.

Levi subgroups of G resemble G; in particular, they are again groups of Lie type.

Inductively, we may use the representations of the Levi subgroups to obtain information about the representations of G.

This is the idea behind Harish-Chandra theory.

HARISH-CHANDRA INDUCTION

Assume from now on that $\ell \neq p$.

Let L be a Levi subgroup of G, and M a kL-module.

View M as a kP-module via $\pi : P \to L$
($a.v := \pi(a).v$ for $v \in M$, $a \in P$).

Put

$$R_L^G(M) := \left\{ f : G \to M \mid a.f(b) = f(ab) \text{ for all } a \in P, b \in G \right\}.$$

(Modular forms.)

$R_L^G(M)$ is a kG-module, called Harish-Chandra induced module.
[Action of G: $g.f(b) := f(bg)$, $g, b \in G$, $f \in R_L^G(M)$.]

$R_L^G(M)$ is independent of the choice of P with $P \to L$.
[Lusztig, 1977 ($\ell = 0$);
Dipper-Du, 1993; Howlett-Lehrer, 1994 ($\ell > 0$)].

CENTRALISER ALGEBRAS

With L and M as before, put

$$\mathcal{H}(L, M) := \mathrm{End}_{kG}(R_L^G(M)).$$

$\mathcal{H}(L, M)$ is the centraliser algebra (or Hecke algebra) of the kG-module $R_L^G(M)$, i.e., $\mathcal{H}(L, M) =$

$$\left\{\gamma \in \mathrm{End}_k(R_L^G(M)) \mid \gamma(g.f) = g.\gamma(f) \text{ for all } g \in G, f \in R_L^G(M)\right\}.$$

$\mathcal{H}(L, M)$ is used to analyse the submodules and quotients of $R_L^G(M)$.

Iwahori's example (1964)

Suppose that $\mathrm{char}(k) = 0$.

Let $G = \mathrm{GL}_n(q)$, $L = T$, the group of diagonal matrices, M the trivial kL-module. Then

$$\mathcal{H}(L, M) = \mathcal{H}_{k,q}(S_n),$$

the Iwahori-Hecke algebra over k with parameter q associated to the Weyl group S_n of G (Iwahori).

Presentation of $\mathcal{H}_{k,q}(S_n)$ (as k-algebra):

$$\left\langle T_1, \ldots, T_{n-1} \mid \text{ braid relations }, T_i^2 = q1_k + (q-1)T_i \right\rangle_{k\text{-algebra}}.$$

Braid relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n-2).$$

HARISH-CHANDRA CLASSIFICATION

Let V be a simple kG-module.

V is called cuspidal, if V is **not** a **submodule** of $R_L^G(M)$ for some **proper** Levi subgroup L of G.

Harish-Chandra theory (HC-induction, cuspidality) yields the following classification.

THEOREM (HARISH-CHANDRA (1968), LUSZTIG ('70S) ($\ell = 0$), GECK-H.-MALLE (1996) ($\ell > 0$))

$$\{V \mid V \text{ simple } kG\text{-module}\} / \text{isomorphism}$$
$$\updownarrow$$
$$\left\{ (L, M, \theta) \;\middle|\; \begin{array}{c} L \text{ Levi subgroup of } G \\ M \text{ simple, cuspidal } kL\text{-module} \\ \theta \text{ irred. } k\text{-rep'n of } \mathcal{H}(L, M) \end{array} \right\} / \text{conjugacy}$$

PROBLEMS IN HARISH-CHANDRA THEORY

The above theorem leads to the three tasks:

1. Determine the cuspidal pairs (L, M).
2. For each of these, "compute" $\mathcal{H}(L, M)$.
3. Classify the irreducible k-representations of $\mathcal{H}(L, M)$.

State of the art in case $\ell = 0$ (Lusztig):

- Cuspidal simple kG-modules arise from étale cohomology groups of Deligne-Lusztig varieties.
- $\mathcal{H}(L, M)$ is an Iwahori-Hecke algebra (Lusztig, Howlett-Lehrer) corresponding to a Coxeter group, namely $W_G(L, M) := (N_G(L, M) \cap N)L/L$ (the N from the BN-pair).
- $\mathcal{H}(L, M) \cong kW_G(L, M)$ (Tits deformation theorem).

EXAMPLE: $\mathrm{SL}_2(q)$

Let $G = \mathrm{SL}_2(q)$ and $\ell = 0$.

The group T of diagonal matrices is the only proper Levi subgroup; it is a cyclic group of order $q - 1$.

Put $W_G(T) := (N_G(T) \cap N)/T \; (:= N_G(\mathbf{T})/T)$.

Then $W_G(T) = \langle T, s \rangle / T$ with $s = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and so $|W_G(T)| = 2$.

Let M be a simple kT-module. Then $\dim M = 1$ and M is cuspidal, and $\dim R_T^G(M) = q + 1$ (since $[G : B] = q + 1$).

Case 1: $W_G(T, M) = \{1\}$. Then $\mathcal{H}(T, M) \cong k$ and $R_T^G(M)$ is simple.

Case 2: $W_G(T, M) = W_G(T)$. Then $\mathcal{H}(T, M) \cong kW_G(T)$, and $R_T^G(M)$ is the sum of two simple kG-modules.

DRINFELD'S EXAMPLE

The cuspidal simple $k\mathrm{SL}_2(q)$-modules have dimensions $q-1$ and $(q-1)/2$ (the latter only occur if p is odd).

How to construct these?

Consider the affine curve

$$C = \{(x, y) \in \bar{\mathbb{F}}_p^2 \mid xy^q - x^q y = 1\}.$$

$G = \mathrm{SL}_2(q)$ acts on C by linear change of coordinates.

Hence G also acts on the étale cohomology group

$$H_c^1(C, \bar{\mathbb{Q}}_r),$$

where r is a prime different from p.

It turns out that the simple $\bar{\mathbb{Q}}_r G$-submodules of $H_c^1(C, \bar{\mathbb{Q}}_r)$ are the cuspidal ones (here $k = \bar{\mathbb{Q}}_r$).

CHARACTERS

Let G be a finite group and k a field.

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be a k-representation of G.

The character afforded by $\mathfrak{X}$ is the map

$$\chi_{\mathfrak{X}} : G \to k, \quad g \mapsto \mathrm{Trace}(\mathfrak{X}(g)).$$

(This is not the same as the formal character introduced in Lecture II.)

$\chi_{\mathfrak{X}}$ is constant on conjugacy classes: a class function on G.

Equivalent k-representations have the same character.

Irreducible characters

If $\mathfrak{X}$ is irreducible, $\chi_{\mathfrak{X}}$ is called an irreducible character.

Facts

1. *If $W \leq V$ is G-invariant, then $\chi_{\mathfrak{X}} = \chi_{\mathfrak{X}_W} + \chi_{\mathfrak{X}_{V/W}}$.*
2. *There are only finitely many irreducible characters of G.*
3. *The set of irreducible characters of G is linearly independent (in $\mathrm{Maps}(G, k)$).*
4. *Every character is a sum of irreducible characters.*
5. *Two irreducible representations are equivalent, if and only if their characters are equal.*
6. *Suppose that $\mathrm{char}(k) = 0$. Then two representations are equivalent, if and only if their characters are equal.*

THE ORDINARY CHARACTER TABLE

From now on let k be algebraically closed of characteristic 0.

Put $\mathrm{Irr}(G) :=$ set of irreducible k-characters of G, $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_m\}$.

Let $g_1, \ldots, g_m$ be representatives of the conjugacy classes of G (same m as above!).

The square matrix

$$[\chi_i(g_j)]_{1 \leq i,j \leq m}$$

is called the ordinary character table of G.

An example: The alternating group A_5

Example (The character table of $A_5 \cong \mathrm{SL}_2(4)$)

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A = (1-\sqrt{5})/2, \qquad *A = (1+\sqrt{5})/2$$

$$1 \in 1a, \quad (1,2)(3,4) \in 2a, \quad (1,2,3) \in 3a,$$
$$(1,2,3,4,5) \in 5a, \quad (1,3,5,2,4) \in 5b$$

Goals and results

Aim

Describe all ordinary character tables of all finite simple groups and related finite groups.

Almost done:

1. For alternating groups: Frobenius, Schur
2. For groups of Lie type: Green, Deligne, **Lusztig**, Shoji, . . . (only "a few" character values missing)
3. For sporadic groups and other "small" groups:

Atlas of Finite Groups, Conway, Curtis, Norton, Parker, Wilson, 1986

THE GENERIC CHARACTER TABLE FOR $\mathrm{SL}_2(q)$, q EVEN

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$

$\zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \qquad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a)$ ($\mu \in \mathbb{F}_q$ a primitive $(q-1)$th root of 1)

$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \underset{\sim}{\in} C_4(b)$ ($\nu \in \mathbb{F}_{q^2}$ a primitive $(q+1)$th root of 1)

Specialising q to 4, gives the character table of $\mathrm{SL}_2(4) \cong A_5$.

Deligne-Lusztig varieties

Let r be a prime different from p and put $k := \bar{\mathbb{Q}}_r$.

Let $(\mathbf{G}, F)$ be a finite reductive group, $G = \mathbf{G}^F$.

Deligne and Lusztig (1976) construct for each pair $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, a generalised character $R_{\mathbf{T},\theta}^{\mathbf{G}}$ of G.
(A generalised character of G is an element of $\mathbb{Z}[\mathrm{Irr}(G)]$.

Let $(\mathbf{T}, \theta)$ be a pair as above.

Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{T}$.
(In general $\mathbf{B}$ is **not** F-stable.)

Consider the Deligne-Lusztig variety associated to $\mathbf{B}$,

$$X_{\mathbf{B}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}.$$

This is an algebraic variety over $\bar{\mathbb{F}}_p$.

Deligne-Lusztig generalised characters

The finite groups $G = \mathbf{G}^F$ and $T = \mathbf{T}^F$ act on $X_{\mathbf{B}}$, and these actions commute.

Thus the étale cohomology group $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)$ is a $\bar{\mathbb{Q}}_r[G \times T]$-module,

and so its θ-isotypic component $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta$ is a $\bar{\mathbb{Q}}_r G$-module,

whose character is denoted by ch $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta$.

Only finitely many of the vector spaces $H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)$ are $\neq 0$.

Now put

$$R_{\mathbf{T},\theta}^{\mathbf{G}} = \sum_i (-1)^i \mathrm{ch}\, H_c^i(X_{\mathbf{B}}, \bar{\mathbb{Q}}_r)_\theta .$$

$R_{\mathbf{T},\theta}^{\mathbf{G}}$ is independent of the choice of $\mathbf{B}$ containing $\mathbf{T}$.

PROPERTIES OF DELIGNE-LUSZTIG CHARACTERS

The above construction and the following facts are due to Deligne and Lusztig (1976).

FACTS

Let $(\mathbf{T}, \theta)$ be a pair as above. Then

1. $R_{\mathbf{T},\theta}^{\mathbf{G}}(1) = \pm[G : T]_{p'}$.
2. *If $\mathbf{T}$ is contained in an F-stable Borel subgroup $\mathbf{B}$, then $R_{\mathbf{T},\theta}^{\mathbf{G}} = R_T^G(\theta)$ is the Harish-Chandra induced character.*
3. *If θ is in general position, i.e. $N_G(\mathbf{T}, \theta)/T = \{1\}$, then $\pm R_{\mathbf{T},\theta}^{\mathbf{G}}$ is an irreducible character.*

FACTS

4. *For $\chi \in \mathrm{Irr}(G)$, there is a pair $(\mathbf{T}, \theta)$ such that χ occurs in the (unique) expansion of $R_{\mathbf{T},\theta}^{\mathbf{G}}$ into $\mathrm{Irr}(G)$.*
   (Recall that $\mathrm{Irr}(G)$ is a basis of $\mathbb{Z}[\mathrm{Irr}(G)]$.)

UNIPOTENT CHARACTERS

DEFINITION (LUSZTIG)

An character χ of G is called unipotent, if χ is irreducible, and if χ occurs in $R^{\mathbf{G}}_{\mathbf{T},1}$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, where $\mathbf{1}$ denotes the trivial character of $T = \mathbf{T}^F$.

We write $\mathrm{Irr}^u(G)$ for the set of unipotent characters of G.

The above definition of unipotent characters uses étale cohomology groups.

So far, no elementary description known, except for $\mathrm{GL}_n(q)$; see below.

Lusztig classified $\mathrm{Irr}^u(G)$ in all cases, **independently** of q.

Harish-Chandra induction preserves unipotent characters, so it suffices to construct the cuspidal unipotent characters.

The unipotent characters of $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$.

Then $\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R_T^G(1)\}$.

Moreover, there is bijection

$$\mathcal{P}_n \leftrightarrow \mathrm{Irr}^u(G), \quad \lambda \leftrightarrow \chi_\lambda,$$

where $\mathcal{P}_n$ denotes the set of partitions of n.

The degrees of the unipotent characters are "polynomials in q":

$$\chi_\lambda(1) = q^{d(\lambda)} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{\prod_{h(\lambda)}(q^h - 1)},$$

with a certain $d(\lambda) \in \mathbb{N}$, and where $h(\lambda)$ runs through the hook lengths of λ.

THE DEGREES OF THE UNIPOTENT CHARACTERS OF $GL_5(q)$

λ	$\chi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$q^3(q^2+1)(q^2+q+1)$
$(2^2,1)$	$q^4(q^4+q^3+q^2+q+1)$
$(2,1^3)$	$q^6(q+1)(q^2+1)$
(1^5)	q^{10}

Jordan decomposition of elements

An important concept in the classification of elements of a finite reductive group is the Jordan decomposition of elements.

Since $\mathbf{G} \leq \mathrm{GL}_n(\bar{\mathbb{F}}_p)$, every $g \in \mathbf{G}$ has finite order.

Hence g has a unique decomposition as

$$g = su = us \tag{1}$$

with u a p-element and s a p'-element.

It follows from Linear Algebra that u is unipotent, i.e. all eigenvalues of u are equal to 1, and s is semisimple, i.e. diagonalisable.

(1) is called the Jordan decomposition of $g \in \mathbf{G}$.

If $g \in G = \mathbf{G}^F$, then so are u and s.

Jordan decomposition of conjugacy classes

This yields a model classification for Case 2 ($\ell = 0$) and, perhaps, Case 3 ($0 \neq \ell \neq p$).

For $g \in G$ with Jordan decomposition $g = us = su$, we write $C^G_{u,s}$ for the G-conjugacy class containing g.

This gives a labelling

$$\{\text{conjugacy classes of } G\} \\ \updownarrow \\ \{C^G_{s,u} \mid s \text{ semisimple}, u \in C_G(s) \text{ unipotent}\}.$$

(In the above, the labels s and u have to be taken modulo conjugacy in G and $C_G(s)$, respectively.)

Moreover, $|C^G_{s,u}| = |G : C_G(s)||C^{C_G(s)}_{1,u}|$.

This is the Jordan decomposition of conjugacy classes.

Example: The general linear group once more

$G = \mathrm{GL}_n(q)$, $s \in G$ semisimple. Then

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$$

with $\sum_{i=1}^m n_i d_i = n$. (This gives finitely many class types.)

Thus it suffices to classify the set of unipotent conjugacy classes $\mathcal{U}$ of G.

By Linear Algebra we have

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$C_{1,u}^G \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of q.

In general, i.e. for other groups, it depends slightly on q.

JORDAN DECOMPOSITION OF CHARACTERS

Let $(\mathbf{G}, F)$ be a connected reductive group.

Let $(\mathbf{G}^*, F)$ denote the dual reductive group.

If $\mathbf{G}$ is determined by the root datum $(X, \Phi, Y, \Phi^\vee)$, then $\mathbf{G}^*$ is defined by the root datum $(Y, \Phi^\vee, X, \Phi)$.

EXAMPLES

(1) If $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_p)$*, then* $\mathbf{G}^* = \mathbf{G}$*.*

(2) If $\mathbf{G} = \mathrm{SO}_{2m+1}(\bar{\mathbb{F}}_p)$*, then* $\mathbf{G}^* = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$*.*

MAIN THEOREM (LUSZTIG; JORDAN DEC. OF CHAR'S, 1984)

Suppose that $Z(\mathbf{G})$ *is connected. Then there is a bijection*

$$\mathrm{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \textit{ semisimple}, \lambda \in \mathrm{Irr}^u(C_{G^*}(s))\}$$

Moreover, $\chi_{s,\lambda}(1) = |G^* : C_{G^*}(s)|_{p'}\, \lambda(1)$*.*

The irreducible characters of $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then

$$\mathrm{Irr}(G) = \{\chi_{s,\lambda} \mid s \in G \text{ semisimple}, \lambda \in \mathrm{Irr}^u(C_G(s))\}.$$

We have $C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$ with $\sum_{i=1}^m n_i d_i = n$.

Thus $\lambda = \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_m$ with $\lambda_i \in \mathrm{Irr}^u(\mathrm{GL}_{n_i}(q^{d_i})) \longleftrightarrow \mathscr{P}_{n_i}$.

Moreover,

$$\chi_{s,\lambda}(1) = \frac{(q^n-1)\cdots(q-1)}{\prod_{i=1}^m [(q^{d_i n_i}-1)\cdots(q^{d_i}-1)]} \prod_{i=1}^m \lambda_i(1).$$

THE DEGREES OF THE IRREDUCIBLE CHARACTERS OF $\mathrm{GL}_3(q)$

$C_G(s)$	λ	$\chi_{s,\lambda}(1)$
$\mathrm{GL}_1(q^3)$	(1)	$(q-1)^2(q+1)$
$\mathrm{GL}_1(q^2)\times\mathrm{GL}_1(q)$	$(1)\boxtimes(1)$	$(q-1)(q^2+q+1)$
$\mathrm{GL}_1(q)^3$	$(1)\boxtimes(1)\boxtimes(1)$	$(q+1)(q^2+q+1)$
$\mathrm{GL}_2(q)\times\mathrm{GL}_1(q)$	$(2)\boxtimes(1)$ $(1,1)\boxtimes(1)$	q^2+q+1 $q(q^2+q+1)$
$\mathrm{GL}_3(q)$	(3) $(2,1)$ $(1,1,1)$	1 $q(q+1)$ q^3

Concluding remarks

1. There are also results by Lusztig (1988) in case $Z(\mathbf{G})$ is not connected, e.g. if $\mathbf{G} = \mathrm{SL}_n(\bar{\mathbb{F}}_p)$ or $\mathbf{G} = \mathrm{Sp}_{2m}(\bar{\mathbb{F}}_p)$ with p odd.

   For such groups, $C_{\mathbf{G}^*}(s)$ is not always connected, and the problem then is to define unipotent characters for $C_{\mathbf{G}^*}(s)^F$.

2. The Jordan decomposition of conjugacy classes and characters allow for the construction of generic character tables in all cases.

3. Let $\{G(q) \mid q$ a prime power$\}$ be a series of finite groups of Lie type, e.g. $\{\mathrm{GU}_n(q)\}$ or $\{\mathrm{SL}_n(q)\}$ (n fixed).

   Then there exists a **finite** set $\mathcal{D}$ of polynomials in $\mathbb{Q}[x]$ s.t.:

   If $\chi \in \mathrm{Irr}(G(q))$, then there is $f \in \mathcal{D}$ with $\chi(1) = f(q)$.

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Thank you for your listening!