Springer correspondance modulo ℓ

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December 8th, 2005

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Some perverse sheaves

Example : $G = SL_2$

Case $\ell \neq 2$

Case $\ell = 2$

Springer correspondance modulo ℓ : conjectures

Conjectures with the group algebra

Conjectures with the Schur algebra

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Background

G is a connected reductive group over k, of rank r, with Lie algebra $\mathfrak{g}$.

Two cases :

1. $k = \mathbb{C}$ complex topology
   $\mathbb{K} = \mathbb{Q}$ $\quad \mathcal{O} = \mathbb{Z}$ $\quad \mathbb{F} = \mathbb{F}_\ell$
2. $k = \overline{\mathbb{F}_p}$ étale topology
   $\mathbb{K} = \mathbb{Q}_\ell$ $\quad \mathcal{O} = \mathbb{Z}_\ell$ $\quad \mathbb{F} = \mathbb{F}_\ell$ $\quad \ell \neq p$

Λ will stand for $\mathbb{K}$, $\mathcal{O}$ or $\mathbb{F}$.

Notations

$\mathfrak{g}_{\mathrm{rs}}$ regular semisimple elements in $\mathfrak{g}$
$\mathcal{N}$ nilpotent elements in $\mathfrak{g}$
$\mathcal{B}$ variety of all Borel subalgebras of $\mathfrak{g}$ (the flag variety)
$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}_0) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}_0\}$
$\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ Grothendieck's simultaneous resolution

$\widetilde{\mathfrak{g}}_{\mathrm{rs}} = \pi^{-1}(\mathfrak{g}_{\mathrm{rs}})$
$\pi_{\mathrm{rs}} : \widetilde{\mathfrak{g}}_{\mathrm{rs}} \to \mathfrak{g}_{\mathrm{rs}}$

$\widetilde{\mathcal{N}} = \pi^{-1}(\mathcal{N})$
$\pi_{\mathcal{N}} : \widetilde{\mathcal{N}} \to \mathcal{N}$ Springer's resolution of the nilpotent variety

The fundamental diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[r, hook, "\widetilde{j_{\mathrm{rs}}}"] \arrow[d, "\pi_{\mathrm{rs}}"'] & \widetilde{\mathfrak{g}} \arrow[d, "\pi"'] & \widetilde{\mathcal{N}} \arrow[l, hook', "\widetilde{i_{\mathcal{N}}}"'] \arrow[d, "\pi_{\mathcal{N}}"] \\
\mathfrak{g}_{\mathrm{rs}} \arrow[r, hook, "j_{\mathrm{rs}}"'] & \mathfrak{g} & \mathcal{N} \arrow[l, hook', "i_{\mathcal{N}}"]
\end{tikzcd}$$

π_{rs}, π and $\pi_{\mathcal{N}}$ are proper.

π_{rs} is a finite étale Galois covering with group W

π is small and $\pi_{\mathcal{N}}$ is semi-small.

The fundamental diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{g}}_{\mathrm{rs}} \arrow[r, hook, "\widetilde{j_{\mathrm{rs}}}"] \arrow[d, "\pi_{\mathrm{rs}}"'] & \widetilde{\mathfrak{g}} \arrow[d, "\pi"'] & \widetilde{\mathcal{N}} \arrow[l, hook', "\widetilde{i_{\mathcal{N}}}"'] \arrow[d, "\pi_{\mathcal{N}}"] \\
\mathfrak{g}_{\mathrm{rs}} \arrow[r, hook, "j_{\mathrm{rs}}"'] & \mathfrak{g} & \mathcal{N} \arrow[l, hook', "i_{\mathcal{N}}"]
\end{tikzcd}$$

$$\mathcal{K}_{\mathrm{rs}} = \pi_{\mathrm{rs}*}\Lambda_{\widetilde{\mathfrak{g}}_{\mathrm{rs}}}[\dim G] \leftrightarrow \Lambda W \qquad \mathrm{End}(\mathcal{K}_{\mathrm{rs}}) \cong \Lambda W$$

$$\mathcal{K} = R\pi_!\Lambda_{\widetilde{\mathfrak{g}}}[\dim G] = j_{\mathrm{rs}!*}\mathcal{K}_{\mathrm{rs}} \qquad \mathrm{End}(\mathcal{K}) \cong \Lambda W$$

$$\mathcal{K}_{\mathcal{N}} = R\pi_{\mathcal{N}!}\Lambda_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = i_{\mathcal{N}}{}^*\mathcal{K}[-r]$$

Fiber at x of the cohomology sheaves : $H^i(\mathcal{B}_x)$,
where $\mathcal{B}_x = \pi^{-1}(x)$ is the Springer fiber

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Perverse sheaf $\mathcal{K} = C \oplus C_\varepsilon$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	$-1-\varepsilon$	$\Lambda \oplus \Lambda_\varepsilon$	.	.	.
C^{reg}	2	-1	Λ	.	.	.
C^{triv}	0	-2	Λ	.	Λ	.

Simple perverse sheaves

- $C := IC(\mathfrak{g}, \Lambda_{\mathfrak{g}_{\mathrm{rs}}})[3] = \Lambda_{\mathfrak{g}}[3]$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	-1	Λ	.	.	.
C^{reg}	2	-1	Λ	.	.	.
C^{triv}	0	-1	Λ	.	.	.

- $C_\varepsilon := IC(\mathfrak{g}, \Lambda_\varepsilon)[3]$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	$-\varepsilon$	Λ_ε	.	.	.
C^{reg}	2	0	.	.	.	.
C^{triv}	0	-1	.	.	Λ	.

Simple perverse sheaves

- $B := IC(\overline{C^{reg}}, \Lambda)[2] = \Lambda_{\mathcal{N}}[2]$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	0	.	.	.	.
C^{reg}	2	1	.	Λ	.	.
C^{triv}	0	1	.	Λ	.	.

- $A := \Lambda_{C^{triv}}$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	0	.	.	.	.
C^{reg}	2	0	.	.	.	.
C^{triv}	0	1	.	.	.	Λ

Restriction to nilpotents ${i_{\mathcal{N}}}^*[-1]$

$$
\begin{aligned}
C &\mapsto B\\
C_\varepsilon &\mapsto A\\
\mathcal{K} = C \oplus C_\varepsilon &\mapsto \mathcal{K}_{\mathcal{N}} = A \oplus B
\end{aligned}
$$

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Perverse sheaf $\mathcal{K}$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	-2	ΛW	.	.	.
C^{reg}	2	-1	Λ	.	.	.
C^{triv}	0	-2	Λ	.	Λ	.

Simple perverse sheaves

- $c = C$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	-1	Λ	.	.	.
C^{reg}	2	-1	Λ	.	.	.
C^{triv}	0	-1	Λ	.	.	.

Simple perverse sheaves

- b

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	0	.	.	.	.
C^{reg}	2	1	.	Λ	.	.
C^{triv}	0	0	.	Λ	Λ	.

- $a = A$

Stratum	Dim	χ	-3	-2	-1	0
$\mathfrak{g}_{\mathrm{rs}}$	3	0	.	.	.	.
C^{reg}	2	0	.	.	.	.
C^{triv}	0	1	.	.	.	Λ

Loewy structures

Reduction of simples :

$$A = a \qquad B = \begin{matrix} b \\ a \end{matrix}$$

$$C = c \qquad C_\varepsilon = \begin{matrix} c \\ b \end{matrix}$$

Reduction of $\mathcal{K}$:

$$0 \to C \to \mathcal{K} \to C_\varepsilon \to 0$$

$$\mathcal{K} = \begin{matrix} c \\ b \\ c \end{matrix}$$

Restriction to nilpotents

$$C = \begin{array}{c} c \end{array} \mapsto B = \begin{array}{c} b \\ a \end{array}$$

$$C_\varepsilon = \begin{array}{c} c \\ b \end{array} \mapsto A = \begin{array}{c} a \end{array}$$

Reduction of $\mathcal{K}_\mathcal{N}$:

$$0 \to B \left(= \begin{array}{c} b \\ a \end{array} \right) \to \mathcal{K}_\mathcal{N} \to a \to 0$$

$$\mathcal{K} = \begin{array}{c} c \\ b \\ c \end{array} \mapsto \mathcal{K}_\mathcal{N} = \begin{array}{c} a \\ b \\ a \end{array}$$

A Springer correspondance ?

$\mathcal{K}_{\mathrm{rs}}$	$\mathcal{K}$	$\mathcal{K}_{\mathcal{N}}$
	c	a
c_{rs}	b	b
c_{rs}	c	a

A Springer correspondance ?

$\mathcal{K}_{\mathrm{rs}}$	$\mathcal{K}$	$\mathcal{K}_{\mathcal{N}}$
$\begin{matrix} c_{\mathrm{rs}} \\ c_{\mathrm{rs}} \end{matrix}$	$\begin{matrix} c \\ b \\ c \end{matrix}$	$\begin{matrix} a \\ b \\ a \end{matrix}$

$\mathcal{K}'_{\mathrm{rs}}$	$\mathcal{K}'$	$\mathcal{K}'_{\mathcal{N}}$
$c_{\mathrm{rs}} \oplus \begin{matrix} c_{\mathrm{rs}} \\ c_{\mathrm{rs}} \end{matrix}$	$c \oplus \begin{matrix} c \\ b \\ c \end{matrix}$	$\begin{matrix} b \\ a \end{matrix} \oplus \begin{matrix} a \\ b \\ a \end{matrix}$

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Conjecture on the endomorphism algebra of $\mathcal{K}_{\mathcal{N}}$

The natural morphism $\mathrm{End}(\mathcal{K}) \to \mathrm{End}(\mathcal{K}_{\mathcal{N}})$ should be an isomorphism.

Remarks :

- $\Lambda W \cong \mathrm{End}(\mathcal{K}_{\mathrm{rs}}) \cong \mathrm{End}(\mathcal{K})$ because $j_{\mathrm{rs}!*}$ is fully faithful.
- In characteristic 0, injectivity follows from $End(\mathcal{K}_0) \cong \Lambda W$, and then an argument of dimension gives the isomorphism.
- Other approach (Brylinski) : Fourier transform (for $\mathfrak{g}$ only)

Conjecture on the Springer correspondance for $G = GL_n$

There should be natural bijections

$$
\begin{aligned}
&\{\text{column } \ell\text{-regular partitions of } n\}\\
&\cong \{\text{simples of the head of } \mathcal{K}_{\mathcal{N}}\}\\
&\cong \{\text{indecomposable summands of } \mathcal{K}_{\mathcal{N}}\}\\
&\cong \{\text{projective indecomposable modules of } \mathrm{End}(\mathcal{K}_{\mathcal{N}}) \cong \Lambda\mathfrak{S}_n\}\\
&\cong \{\text{simple modules of } \Lambda\mathfrak{S}_n\}\\
&\cong \{\ell\text{-regular partitions of } n\}
\end{aligned}
$$

whose composition would be the conjugation of partitions.

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Introducing the Schur algebra

In order for every partition of n to appear, one should consider the local system on $\mathfrak{g}_{\mathrm{rs}}$ corresponding to the following representation of $\mathfrak{S}_n$

$$\mathcal{K}'_{\mathrm{rs}} \leftrightarrow \bigoplus_{\lambda \in P(n)} \mathrm{Ind}_{\Lambda\mathfrak{S}_\lambda}^{\Lambda\mathfrak{S}_n} \Lambda$$

whose endomorphism algebra is the Schur algebra $S(n)$.
Then put $\mathcal{K}' = j_{\mathrm{rs}!*}(\mathcal{K}'_{\mathrm{rs}})[\dim G]$, and $\mathcal{K}'_{\mathcal{N}} = i^*_{\mathcal{N}}(\mathcal{K})[-r]$.

Conjecture for the Schur algebra $S(n)$

There should be natural bijections

$$\begin{aligned}
&\{\text{partitions of } n\}\\
&\cong \{\text{simples of the head of } \mathcal{K}'_{\mathcal{N}}\}\\
&\cong \{\text{indecomposable summands of } \mathcal{K}'_{\mathcal{N}}\}\\
&\cong \{\text{projective indecomposable modules of } \mathrm{End}(\mathcal{K}'_{\mathcal{N}}) \cong S(n)\}\\
&\cong \{\text{simple modules of } S(n)\}\\
&\cong \{\text{partitions of } n\}
\end{aligned}$$

whose composition would be the conjugation of partitions.

Conjecture on decomposition matrices

Up to conjugation of partitions, the decomposition matrices for GL_n-equivariant perverse sheaves on $\mathcal{N}$ and for the Schur algebra $S(n)$ coincide.

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Definition : equivalent singularities

$$Sing(X, x) = Sing(Y, y)$$

if and only if

there exist (Z,z) and two morphims smooth at z as follows

$$\begin{tikzcd} Z \arrow[rr, "\phi"] \arrow[dd, "\psi"'] & & X \\ & z \arrow[r, mapsto] \arrow[d, mapsto] & x \\ Y & y & \end{tikzcd}$$

Singularity of an orbit

If an algebraic group G acts on X, $Sing(X, x)$ only depends on the orbit O of x, and then one writes

$$Sing(X, O) := Sing(X, x)$$

where x is any element of O.

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Theorem of Brieskorn, Slodowy

If G is almost simple of type A, D or E, then

$$Sing(\overline{C_{reg}}, C_{subreg})$$

is a kleinian singularity of the same type.

This is a surface singularity of the form $(\mathbb{A}^2/\Gamma, 0)$ where Γ is a finite group of SL_2.

In the type A_{n-1}, Γ is a cyclic group of order n.

Corollary

For $G = GL_n$, let a_λ be the perverse extension of the constant sheaf $\mathbb{F}$ on the regular nilpotent class, shifted by $\dim \mathcal{N}$. Here is part of the table of cohomology stalks of $a_{(n)}$:

1. if $\ell \nmid n$

Stratum	Dim	χ	$-2\nu_G$	$-2\nu_G + 1$	$-2\nu_G + 2$
(n)	$2\nu_G$	1	Λ	.	.
$(n-1,1)$	$2\nu_G - 2$	1	Λ	.	.

2. if $\ell \mid n$

Stratum	Dim	χ	$-2\nu_G$	$-2\nu_G + 1$	$-2\nu_G + 2$
(n)	$2\nu_G$	1	Λ	.	.
$(n-1,1)$	$2\nu_G - 2$	0	Λ	Λ	.

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Here is the table of cohomology stalks of $a_{(2,1^{n-2})}$:

1. if $\ell \nmid n$

Str	χ	$-2n+2$	$-2n+3$	$-2n+4$	$\dots$	-2	-1	0
C_{min}	1	Λ	.	.	.	.	.	.
C_{triv}	$n-1$	Λ	.	Λ	$\dots$	Λ	.	.

2. if $\ell \mid n$

Str	χ	$-2n+2$	$-2n+3$	$-2n+4$	$\dots$	-2	-1	0
C_{min}	1	Λ	.	.	.	.	.	.
C_{triv}	$n-2$	Λ	.	Λ	$\dots$	Λ	Λ	.

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Deleting rows and columns

Theroem (Kraft and Procesi, 1981) :

Let λ_1 and μ_1 be obtained from λ and μ by suppressing the first common rows and columns. Then

$$Sing(\overline{C_\lambda}, C_\mu) = Sing(\overline{C_{\lambda_1}}, C_{\mu_1}) \text{ and } codim_{\overline{C_\lambda}}(C_\mu) = codim_{\overline{C_{\lambda_1}}}(C_{\mu_1})$$

A fact on adjacent partitions

Let $\lambda > \mu$ be adjacent partitions of n
(that is, there is no ν such that $\lambda > \nu > \mu$).

Then, after deleting common rows and columns, one obtains either $((k),(k-1))$ or $((2,1^{k-2}),(1^k))$ for some $k \leq n$.

Corollary

The decomposition numbers for perverse sheaves and for the Schur algebra coincide for adjacent partitions.