Finite Simple 3'-Groups are Cyclic or Suzuki Groups

December 6th, 2013

Imke Toborg

Martin-Luther-Universität
Halle-Wittenberg

Nikolaus Conference 2013

This is joint work with Rebecca Waldecker

This is joint work with Rebecca Waldecker

Theorem (Thompson, unpublished)

Theorem (Thompson, unpublished)

Let G be a finite simple $3'$-group.
Then G is cyclic or a Suzuki group.

Definition

Let G be a group and S a Sylow 2-subgroup of G.
A subgroup A of S is **strongly closed** in G
(with respect to S) if and only if
for all $g \in G$ we have $A^g \cap S \leq A$.

Theorem (Glauberman)

Factorizations in Local Subgroups of Finite Groups; Regional Conference Series in Mathematics, No. 33. AMS, Prov., R.I., 1977.

Let G be a finite that does not involve a S_4.
Suppose that, whenever G^* is a 2-constrained section of G, then every non-abelian composition factor of G^* has the form $Sz(2^{2n+1})$ or $PSL_2(3^n)$.

Then G has an elementary abelian strongly closed 2-subgroup.

Theorem (Glauberman)

Factorizations in Local Subgroups of Finite Groups; Regional Conference Series in Mathematics, No. 33. AMS, Prov., R.I., 1977.

Let G be a finite that does not involve a S_4.
Suppose that, whenever G^* is a 2-constrained section of G, then every non-abelian composition factor of G^* has the form $Sz(2^{2n+1})$ or $PSL_2(3^n)$.

Then G has an elementary abelian strongly closed 2-subgroup.

There is a theorem of Stellmacher which implies the same result.

A characteristic subgroup of Σ_4-free groups; Israel J. Math. 94 (1996).

Theorem (Goldschmidt)

2-Fusion in finite groups; Ann. of Math. (2) 99 (1974), 70 - 117.

Let G be a finite group and suppose that A is an abelian strongly closed 2-subgroup of G.
Let $K := \langle A^G \rangle$ and let $\bar{K} := K/O(K)$. Then:

- $\bar{K}$ is a central product of an abelian 2-group and quasi-simple groups L of a list[1] which includes Suzuki groups,
- $\bar{A} = O_2(\bar{K}) \cdot \Omega_1(\bar{T})$ for some Sylow 2-subgroup T of K containing A.

Theorem (Goldschmidt)

2-Fusion in finite groups; Ann. of Math. (2) 99 (1974), 70 - 117.

Let G be a finite group and suppose that A is an abelian strongly closed 2-subgroup of G.
Let $K := \langle A^G \rangle$ and let $\bar{K} := K/O(K)$. Then:

- $\bar{K}$ is a central product of an abelian 2-group and quasi-simple groups L of a list[1] which includes Suzuki groups,
- $\bar{A} = O_2(\bar{K}) \cdot \Omega_1(\bar{T})$ for some Sylow 2-subgroup T of K containing A.

[1] $L/Z(L)$ is a group of Ree type, $PSL_2(2^n)$ with $n \geq 3$, $Sz(2^{2n+1})$ with $n \geq 1$, $PSU_3(2^n)$ with $n \geq 2$, $PSL_2(q)$ with $q \equiv 3, 5 (mod 8)$ and $q \geq 5$ or J_1.

Corollary (Aschbacher)

S_3-free 2-fusion systems; Proc. of the Edingburgh Math. Soc. 56 (2013), 27-48.

Assume that G is a finite S_3-free non-abelian simple group. Then $G \cong Sz(2^{2n+1})$ for some $n \geq 1$.

Corollary (Aschbacher)

S_3-free 2-fusion systems; Proc. of the Edingburgh Math. Soc. 56 (2013), 27-48.

Assume that G is a finite S_3-free non-abelian simple group. Then $G \cong Sz(2^{2n+1})$ for some $n \geq 1$.

Its proof also uses Goldschmidt's theorem.

How did Goldschmidt prove his theorem?

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.

- He investigates a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- He shows that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then he chooses wisely a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two sections he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two and a half pages he investigates the group H to conclude that $O(F(H))$ has no elementary abelian subgroup of order p^3 for any prime p or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two and a half pages we show that $O(H) = 1$ or that $|A| \leq 4$.
- In the first case he shows that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two and a half pages we show that $O(H) = 1$ or that $|A| \leq 4$.
- In the first case we show that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. Then he shows that G is one of his list.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two and a half pages we show that $O(H) = 1$ or that $|A| \leq 4$.
- In the first case we show that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. This is a contradiction.
- The second case he analyses separately.

- We investigate a minimal counterexample G with a minimal abelian strongly closed 2-subgroup A.
- We show that G is simple and $G = \langle C_G(a) \mid a \in A^{\#} \rangle$.
- Then we just take a maximal subgroup H of G that contains $C_G(a)$ for some $a \in A^{\#}$.
- In the next two and a half pages we show that $O(H) = 1$ or that $|A| \leq 4$.
- In the first case we show that $|A| = 8$ and $N_G(A)/C_G(A)$ is non-abelian of order 21. This is a contradiction.
- The second case does not occur, because of the Z^*-Theorem of Glauberman.

Thank you for your attention.

Wenn es raschelt im Haus heute Nacht
still und heimlich 'dass niemand erwacht,
sind es ganz bestimmt keine Mäuse,
sondern viele liebe kleine Nikoläuse

Denn dem Alten fehlt längst schon die Kraft
und weil er es allein nicht mehr schafft,
setzte er sich im Himmel zur Ruhe
und schaut nur noch von oben auf die Schuhe.

Seine Augen sind müde und krank
'drum putzt die Stiefel blitzeblank.

Rolf Zuckowski