



# Universität Stuttgart

Dynkin Diagrams of Simple Lie Algebras



3, C, 2, 3, 1  
1  
1

$A_1(1), A_1(3)$ $A_5$ 60 60	$A_1(2)$ $A_1(7)$ 300 300																					
$A_1(1), A_1(2)$ $A_6$ 360 360	$A_1(3)^3$ $A_1(8)$ 336 336																					
$A_7$ $A_1(11)$ 5040 5040	$A_1(2)$ $E_7(2)$ 5040 5040	$E_7(2)$ $E_7(2)$ 5040 5040	$E_8(2)$ $E_8(2)$ 5040 5040	$F_4(2)$ $F_4(2)$ 5040 5040	$G_2(3)$ $G_2(3)$ 5040 5040	${}^3D_4(2^3)$ ${}^3D_4(2^3)$ 5040 5040	${}^2E_6(2^2)$ ${}^2E_6(2^2)$ 5040 5040	${}^2B_2(2^3)$ ${}^2B_2(2^3)$ 5040 5040	${}^2F_4(2)^1$ ${}^2F_4(2)^1$ 5040 5040	${}^2G_2(3^3)$ ${}^2G_2(3^3)$ 5040 5040	$B_3(2)$ $B_3(2)$ 5040 5040	$C_4(3)$ $C_4(3)$ 5040 5040	$D_3(2)$ $D_3(2)$ 5040 5040	${}^2D_5(2^2)$ ${}^2D_5(2^2)$ 5040 5040								
$A_8$ $A_1(13)$ 16128 16128	$E_6(3)$ $E_7(3)$ 16128 16128	$E_8(3)$ $E_8(3)$ 16128 16128	$F_4(3)$ $F_4(3)$ 16128 16128	$G_2(4)$ $G_2(4)$ 16128 16128	${}^3D_4(3^3)$ ${}^3D_4(3^3)$ 16128 16128	${}^2E_6(3^2)$ ${}^2E_6(3^2)$ 16128 16128	${}^2B_2(2^5)$ ${}^2B_2(2^5)$ 16128 16128	${}^2F_4(2^2)$ ${}^2F_4(2^2)$ 16128 16128	${}^2G_2(3^5)$ ${}^2G_2(3^5)$ 16128 16128	$B_5(2)$ $B_5(2)$ 16128 16128												
$A_9$ $A_1(17)$ 181440 181440	$E_6(4)$ $E_7(4)$ 181440 181440	$E_8(4)$ $E_8(4)$ 181440 181440	$F_4(4)$ $F_4(4)$ 181440 181440	$G_2(5)$ $G_2(5)$ 181440 181440	${}^3D_4(4^3)$ ${}^3D_4(4^3)$ 181440 181440	${}^2E_6(4^2)$ ${}^2E_6(4^2)$ 181440 181440	${}^2B_2(2^7)$ ${}^2B_2(2^7)$ 181440 181440	${}^2F_4(2^4)$ ${}^2F_4(2^4)$ 181440 181440	${}^2G_2(3^7)$ ${}^2G_2(3^7)$ 181440 181440	$B_7(2)$ $B_7(2)$ 181440 181440												
$A_n$ $A_1(q)$ q	$E_6(q)$ $E_7(q)$ q	$E_8(q)$ $E_8(q)$ q	$F_4(q)$ $F_4(q)$ q	$G_2(q)$ $G_2(q)$ q	${}^3D_4(q^3)$ ${}^3D_4(q^3)$ q	${}^2E_6(q^2)$ ${}^2E_6(q^2)$ q	${}^2B_2(2^{n+1})$ ${}^2B_2(2^{n+1})$ q	${}^2F_4(2^{n+1})$ ${}^2F_4(2^{n+1})$ q	${}^2G_2(3^{n+1})$ ${}^2G_2(3^{n+1})$ q	$B_n(q)$ $B_n(q)$ q												

${}^5A_1(4)$ $B_2(3)$ 25200 25200	$C_3(3)$ $C_3(3)$ 420000000 420000000	$D_4(2)$ $D_4(2)$ 176400000 176400000	${}^2D_4(2^2)$ ${}^2D_4(2^2)$ 397400000 397400000	$B_2(4)$ $B_2(4)$ 30000 30000	$C_3(5)$ $C_3(5)$ 300000000 300000000	$D_4(3)$ $D_4(3)$ 876000000 876000000	${}^2D_4(3^2)$ ${}^2D_4(3^2)$ 18107100000 18107100000
$B_3(5)$ $B_3(5)$ 460800 460800	$C_4(3)$ $C_4(3)$ 504000000 504000000	$D_5(2)$ $D_5(2)$ 14537280 14537280	${}^2D_5(2^2)$ ${}^2D_5(2^2)$ 213127600 213127600	$B_5(5)$ $B_5(5)$ 72000 72000	$C_4(5)$ $C_4(5)$ 600000000 600000000	$D_5(3)$ $D_5(3)$ 2592000000 2592000000	${}^2D_5(3^2)$ ${}^2D_5(3^2)$ 5237760000 5237760000

Meinolf Geck

# AN ELEMENTARY CHARACTERISATION OF SPECIAL NILPOTENT ORBITS

11-12 December 2020

- Alternating Groups
- Classical Chevalley Groups
- Classical Groups
- Classical Steinberg Groups
- Suzuki Groups
- Twisted Groups
- Unitary Groups and This Group\*
- Sporadic Groups
- Symplectic Groups
- Simple Groups

Alphabet? Symbol? Order?

$M_{11}$	$M_{12}$	$M_{22}$	$M_{23}$	$M_{24}$	$F(1), F(1)$	$H_1$	$H_2$	$H_{3M}$	$J_3$	$J_4$	$He$
7920	93000	443520	10200960	244825000	378768	4034900	301232640	66770376000	643320000	600128000	100000000

\*The groups  ${}^3D_4(q^3)$  and  ${}^2E_6(q^2)$  are defined as the fixed points of the Frobenius map  $x \mapsto x^q$  of the natural embedding of  ${}^3D_4(q^3)$  in  ${}^3D_4(q^3)$  and  ${}^2E_6(q^2)$  in  ${}^2E_6(q^2)$ , respectively. For more details see [21].

## Exercise Linear Algebra II. Compute the following determinants:

$$\det \begin{pmatrix} 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & -x_1 & 0 & -x_2 & -x_3 & -x_5 \\ 0 & 0 & 0 & 0 & -x_2 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & 0 & -x_3 & 2x_4 & 0 & 0 \\ x_1 & 0 & x_2 & x_3 & 0 & x_5 & 0 & 0 \\ 0 & x_2 & 0 & -2x_4 & -x_5 & 0 & 0 & 0 \\ 0 & x_3 & -x_4 & 0 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 9(x_1 x_4^2 x_5 - x_2 x_3 x_4^2)^2.$$

$$\det \begin{pmatrix} 0 & 0 & 0 & 0 & -2x_1 & -2x_2 & -2x_3 & -x_4 \\ 0 & 0 & 2x_1 & 2x_2 & 0 & 0 & -x_4 & -2x_5 \\ 0 & -2x_1 & 0 & 2x_3 & 0 & x_4 & 0 & -2x_6 \\ 0 & -2x_2 & -2x_3 & 0 & -x_4 & 0 & 0 & -2x_7 \\ 2x_1 & 0 & 0 & x_4 & 0 & 2x_5 & 2x_6 & 0 \\ 2x_2 & 0 & -x_4 & 0 & -2x_5 & 0 & 2x_7 & 0 \\ 2x_3 & x_4 & 0 & 0 & -2x_6 & -2x_7 & 0 & 0 \\ x_4 & 2x_5 & 2x_6 & 2x_7 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$(-16x_1^2 x_7^2 + 32x_1 x_2 x_6 x_7 - 32x_1 x_3 x_5 x_7 + 8x_1 x_4^2 x_7 - 16x_2^2 x_6^2 + 32x_2 x_3 x_5 x_6 - 8x_2 x_4^2 x_6 - 16x_3^2 x_5^2 + 8x_3 x_4^2 x_5 - x_4^4)^2.$$

Note: The matrices are skew-symmetric, so  $\det = \text{square of the pfaffian}$  (as noted by U. Thiel).

**Cartan–Killing:** The finite-dimensional simple Lie algebras over  $\mathbb{C}$  are classified by the following “Dynkin diagrams”.



**Infinite families:** Lie algebras of matrices

$$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C}), \quad B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C}), \quad C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C}), \quad D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C}).$$

**Exceptional algebras:**

$$\dim \mathfrak{g}_2 = 14, \quad \dim \mathfrak{f}_4 = 52, \quad \dim \mathfrak{e}_6 = 78, \quad \dim \mathfrak{e}_7 = 133, \quad \dim \mathfrak{e}_8 = 248.$$

Let  $\mathfrak{g}$  be a Lie algebra with a given Dynkin diagram. Then  $\mathfrak{g}$  has a “Chevalley basis”

$$B = \{h_i \mid i \in I\} \cup \{e_\alpha \mid e_\alpha \in \Phi\}$$

- where  $I =$  indexing set for the nodes of the diagram,
- $\mathfrak{h} = \langle h_i \mid i \in I \rangle_{\mathbb{C}} \subseteq \mathfrak{g}$  Cartan subalgebra,
- $\Phi \subseteq \mathfrak{h}^*$  root system such that  $[h, e_\alpha] = \alpha(h)e_\alpha$  for all  $h \in \mathfrak{h}$  and  $\alpha \in \Phi$ .

The  $e_\alpha$  (eigenvectors for  $\mathfrak{h}$ ) are uniquely determined up to non-zero scalars.

“Canonical choice”, hence, canonical matrix realisation of  $\mathfrak{g}$  (G., Proc. AMS 2017).

```
#####  
##      Welcome to version 1.1 of the Julia module 'ChevLie':      ##  
##      CONSTRUCTING LIE ALGEBRAS AND CHEVALLEY GROUPS          ##  
##      https://pnp.mathematik.uni-stuttgart.de/iaz/iaz2/geckmf/    ##  
##      Type ?LieAlg      for first help; all comments welcome!    ##  
#####
```

```
julia> canchevbasis_adj(LieAlg(:f,4))                                # dim lie = 52 = 4 + 48  
[...]                      # list of 48 matrices of size 52x52 representing the e_\alpha's
```

An element  $e \in \mathfrak{g}$  is called “nilpotent” if  $\text{ad}(e): \mathfrak{g} \rightarrow \mathfrak{g}$  is a nilpotent linear map. All  $e_\alpha$  ( $\alpha \in \Phi$ ) are nilpotent, so can form  $x_\alpha(t) := \exp(t \cdot \text{ad}(e_\alpha)) \in \text{GL}(\mathfrak{g})$  for  $t \in \mathbb{C}$ . Obtain algebraic group  $G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{C} \rangle \leq \text{GL}(\mathfrak{g})$  with Lie algebra  $\mathfrak{g}$ .

**Dynkin–Kostant theory** (See, e.g., Carter’s book on finite groups of Lie type).

The nilpotent  $G$ -orbits of  $\mathfrak{g}$  are classified by “weighted Dynkin diagrams”, i.e., maps

$$d: I \rightarrow \{0, 1, 2\}, \quad \text{where } I = \text{vertices of Dynkin diagram.}$$

Can extend  $d$  linearly to function  $d: \Phi \rightarrow \mathbb{Z}$ . Obtain grading

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_d(n) \quad \text{with} \quad [\mathfrak{g}_d(n), \mathfrak{g}_d(m)] \subseteq \mathfrak{g}_d(n+m),$$

where  $\mathfrak{g}_d(0) := \mathfrak{h} \oplus \langle e_\alpha \mid d(\alpha) = 0 \rangle_{\mathbb{C}}$  and  $\mathfrak{g}_d(n) := \langle e_\alpha \mid d(\alpha) = n \rangle_{\mathbb{C}}$  for  $n \neq 0$ .

The nilpotent orbit defined by  $d$  intersects  $\mathfrak{g}_d(2)$  in a dense open set.

**Lusztig (1979)**: Using Springer correspondence + new definition of “special” characters of Weyl group  $W$ , single out nilpotent orbits called “special”.

These play a key role in several problems in representation theory, but definition . . . “un-natural”.

```
julia> lie=LieAlg(:f,4); wdd=weighted_dynkin_diagrams(lie)
[0,0,0,0],[1,0,0,0],[0,0,0,1],[0,1,0,0],[2,0,0,0],[0,0,0,2],[0,0,1,0],[2,0,0,1],
[0,1,0,1],[1,0,1,0],[0,2,0,0],[2,2,0,0],[1,0,1,2],[0,2,0,2],[2,2,0,2],[2,2,2,2]
(16 nilpotent orbits in total, 11 of which are special.)
```

Recall: the nilpotent orbit defined by  $d: l \rightarrow \{0, 1, 2\}$  intersects  $\mathfrak{g}_d(2)$  in a dense open set. Dually: there is a dense open set of linear maps  $\lambda: \mathfrak{g}_d(2) \rightarrow \mathbb{C}$  such that

$$\sigma_\lambda: \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \rightarrow \mathbb{C}, \quad (y, z) \mapsto \lambda([y, z]),$$

is a non-degenerate, symplectic form. Consider Gram matrix  $\mathcal{G}_\lambda$  of  $\sigma_\lambda$ .

- Let  $\beta_1, \dots, \beta_n \in \Phi$  be those  $\alpha \in \Phi$  with  $d(\alpha) = 1$  ( $\leadsto$  basis of  $\mathfrak{g}_d(1)$ ) and  $\gamma_1, \dots, \gamma_m \in \Phi$  be those  $\alpha \in \Phi$  with  $d(\alpha) = 2$  ( $\leadsto$  basis of  $\mathfrak{g}_d(2)$ ).
- An arbitrary linear  $\lambda: \mathfrak{g}_d(2) \rightarrow \mathbb{C}$  is specified by  $x_l := \lambda(e_{\gamma_l})$  for  $1 \leq l \leq m$ ; then the  $(i, j)$ -entry of  $\mathcal{G}_\lambda$  is given by  $\lambda([e_{\beta_i}, e_{\beta_j}]) \in \mathbb{Z}[x_1, \dots, x_m]$ .

Matrices on first slide:  $\mathfrak{g} = \mathfrak{f}_4$  with  $d = [0, 1, 0, 1]$  and  $d = [0, 0, 0, 1]$ .

1st matrix:  $\det = 9(\dots)^2$ ; 2nd:  $\det = (-16x_1^2 x_7^2 \pm \dots - x_4^4)^2$ . **Guess a pattern ?**

Main observation: Everything above works over  $\mathbb{Z}$  !

- Chevalley basis  $B = \{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$  spans Lie algebra  $\mathfrak{g}_{\mathbb{Z}}$  over  $\mathbb{Z}$ .
- Given  $d: I \rightarrow \{0, 1, 2\}$ , let  $\mathfrak{g}_{\mathbb{Z},d}(n) = \langle e_\alpha \mid d(\alpha) = n \rangle_{\mathbb{Z}}$  for  $n = 1, 2$ .

**Integrality condition**, see G. (Transf. Groups, online May 2020, and JSAG 2020).

We say that  $d$  is  $\mathbb{Z}$ -special if there is some  $\lambda \in \text{Hom}(\mathfrak{g}_{\mathbb{Z},d}(2), \mathbb{Z})$  such that

$$\sigma_\lambda: \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \rightarrow \mathbb{Z}, \quad (y, z) \mapsto \lambda([y, z]),$$

is non-degenerate over  $\mathbb{Z}$ . (If  $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$ , then  $d$  is declared  $\mathbb{Z}$ -special.)

**Conjecture (now Theorem):**  $d$  is Lusztig-special if and only if  $d$  is  $\mathbb{Z}$ -special.

- Using our computations of Gram matrices, true for all exceptional types. (For large matrices, relies on Groebner basis techniques as suggested by U. Thiel and A. Steel.)
- For classical type, see Dong and Yang, Advances in Math. (online Nov. 2020).
- Relevance for real Lie groups: see Vogan's MIT Virtual Lie Group Seminar talk "What's special about special?" at <http://www-math.mit.edu/~dav/LG/>.