Universität Stuttgart

An Elementary Characterisation of Special Nilpotent Orbits

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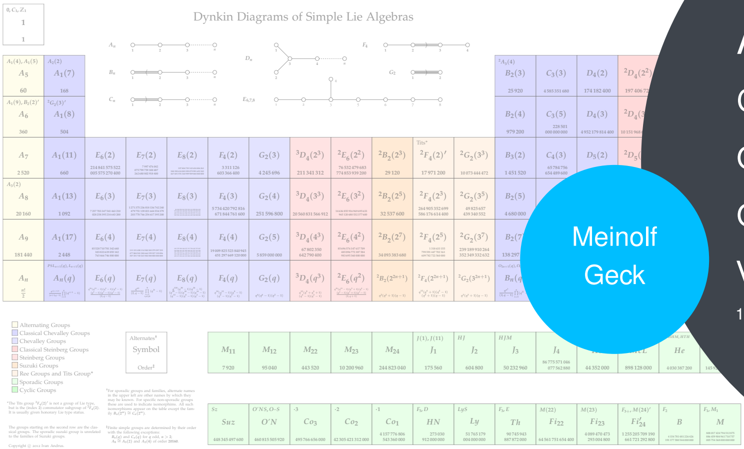
Virtual Nikolaus Conference

11-12 December 2020



Exercise Linear Algebra II. Compute the following determinants:

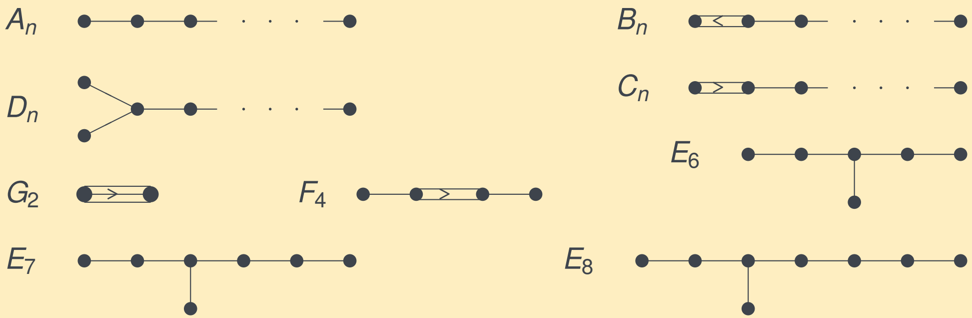
$$\det\begin{pmatrix} 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & -x_1 & 0 & -x_2 & -x_3 & -x_5 \\ 0 & 0 & 0 & 0 & -x_2 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & 0 & -x_3 & 2x_4 & 0 & 0 \\ x_1 & 0 & x_2 & x_3 & 0 & x_5 & 0 & 0 \\ 0 & x_2 & 0 & -2x_4 & -x_5 & 0 & 0 & 0 \\ 0 & x_3 & -x_4 & 0 & 0 & 0 & 0 & 0 \\ x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 9(x_1x_4^2x_5 - x_2x_3x_4^2)^2.$$

$$\det\begin{pmatrix} 0 & 0 & 0 & 0 & -2x_1 & -2x_2 & -2x_3 & -x_4 \\ 0 & 0 & 2x_1 & 2x_2 & 0 & 0 & -x_4 & -2x_5 \\ 0 & -2x_1 & 0 & 2x_3 & 0 & x_4 & 0 & -2x_6 \\ 0 & -2x_2 & -2x_3 & 0 & -x_4 & 0 & 0 & -2x_7 \\ 2x_1 & 0 & 0 & x_4 & 0 & 2x_5 & 2x_6 & 0 \\ 2x_2 & 0 & -x_4 & 0 & -2x_5 & 0 & 2x_7 & 0 \\ 2x_3 & x_4 & 0 & 0 & -2x_6 & -2x_7 & 0 & 0 \\ x_4 & 2x_5 & 2x_6 & 2x_7 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$(-16x_1^2x_7^2+32x_1x_2x_6x_7-32x_1x_3x_5x_7+8x_1x_4^2x_7-16x_2^2x_6^2+32x_2x_3x_5x_6-8x_2x_4^2x_6-16x_3^2x_5^2+8x_3x_4^2x_5-x_4^4)^2.$$

Note: The matrices are skew-symmetric, so $\det$ = square of the pfaffian (as noted by U. Thiel).

Cartan–Killing: The finite-dimensional simple Lie algebras over $\mathbb{C}$ are classified by the following "Dynkin diagrams".



Infinite families: Lie algebras of matrices

$$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C}), \quad B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C}), \quad C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C}), \quad D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C}).$$

Exceptional algebras:

$$\dim \mathfrak{g}_2 = 14, \quad \dim \mathfrak{f}_4 = 52, \quad \dim \mathfrak{e}_6 = 78, \quad \dim \mathfrak{e}_7 = 133, \quad \dim \mathfrak{e}_8 = 248.$$

Let $\mathfrak{g}$ be a Lie algebra with a given Dynkin diagram. Then $\mathfrak{g}$ has a "Chevalley basis"

$$B = \{h_i \mid i \in I\} \cup \{e_\alpha \mid e_\alpha \in \Phi\}$$

- where $I =$ indexing set for the nodes of the diagram,
- $\mathfrak{h} = \langle h_i \mid i \in I \rangle_{\mathbb{C}} \subseteq \mathfrak{g}$ Cartan subalgebra,
- $\Phi \subseteq \mathfrak{h}^*$ root system such that $[h, e_\alpha] = \alpha(h) e_\alpha$ for all $h \in \mathfrak{h}$ and $\alpha \in \Phi$.

The e_α (eigenvectors for $\mathfrak{h}$) are uniquely determined up to non-zero scalars.
"Canonical choice", hence, canonical matrix realisation of $\mathfrak{g}$ (G., Proc. AMS 2017).

```
#####################################################################
##     Welcome to  version 1.1  of the Julia module  ‘ChevLie‘:     ##
##        CONSTRUCTING  LIE  ALGEBRAS  AND  CHEVALLEY  GROUPS       ##
##     https://pnp.mathematik.uni-stuttgart.de/iaz/iaz2/geckmf/     ##
##     Type   ?LieAlg     for first help; all comments welcome!     ##
#####################################################################

julia> canchevbasis_adj(LieAlg(:f,4))                  # dim lie = 52 = 4 + 48
[...]        # list of 48 matrices of size 52x52 representing the e_\alpha’s
```

An element $e \in \mathfrak{g}$ is called "nilpotent" if $\mathrm{ad}(e)\colon \mathfrak{g} \to \mathfrak{g}$ is a nilpotent linear map.

All e_α ($\alpha \in \Phi$) are nilpotent, so can form $x_\alpha(t) := \exp\big(t{\cdot}\mathrm{ad}(e_\alpha)\big) \in \mathrm{GL}(\mathfrak{g})$ for $t \in \mathbb{C}$.

Obtain algebraic group $G = \langle x_\alpha(t) \mid \alpha \in \Phi, t \in \mathbb{C}\rangle \leqslant \mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$.

Dynkin–Kostant theory (See, e.g., Carter's book on finite groups of Lie type).

The nilpotent G-orbits of $\mathfrak{g}$ are classified by "weighted Dynkin diagrams", i.e., maps

$$d\colon I \to \{0,1,2\}, \qquad \text{where } I = \text{vertices of Dynkin diagram.}$$

Can extend d linearly to function $d\colon \Phi \to \mathbb{Z}$. Obtain grading

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_d(n) \qquad \text{with} \qquad [\mathfrak{g}_d(n), \mathfrak{g}_d(m)] \subseteq \mathfrak{g}_d(n+m),$$

where $\mathfrak{g}_d(0) := \mathfrak{h} \oplus \langle e_\alpha \mid d(\alpha) = 0\rangle_{\mathbb{C}}$ and $\mathfrak{g}_d(n) := \langle e_\alpha \mid d(\alpha) = n\rangle_{\mathbb{C}}$ for $n \neq 0$.

The nilpotent orbit defined by d intersects $\mathfrak{g}_d(2)$ in a dense open set.

Lusztig (1979): Using Springer correspondence + new definition of "special" characters of Weyl group W, single out nilpotent orbits called "special".

These play a key role in several problems in representation theory, but definition . . . "un-natural".

```
julia> lie=LieAlg(:f,4); wdd=weighted_dynkin_diagrams(lie)
```

$[0,0,0,0], [1,0,0,0], [0,0,0,1], [0,1,0,0], [2,0,0,0], [0,0,0,2], [0,0,1,0], [2,0,0,1],$

$[0,1,0,1], [1,0,1,0], [0,2,0,0], [2,2,0,0], [1,0,1,2], [0,2,0,2], [2,2,0,2], [2,2,2,2]$

(16 nilpotent orbits in total, 11 of which are special.)

Recall: the nilpotent orbit defined by $d\colon I \to \{0,1,2\}$ intersects $\mathfrak{g}_d(2)$ in a dense open set. Dually: there is a dense open set of linear maps $\lambda\colon \mathfrak{g}_d(2) \to \mathbb{C}$ such that

$$\sigma_\lambda\colon \mathfrak{g}_d(1) \times \mathfrak{g}_d(1) \to \mathbb{C}, \qquad (y,z) \mapsto \lambda([y,z]),$$

is a non-degenerate, symplectic form. Consider Gram matrix $\mathcal{G}_\lambda$ of σ_λ.

- Let $\beta_1, \ldots, \beta_n \in \Phi$ be those $\alpha \in \Phi$ with $d(\alpha) = 1$ ($\rightsquigarrow$ basis of $\mathfrak{g}_d(1)$) and $\gamma_1, \ldots, \gamma_m \in \Phi$ be those $\alpha \in \Phi$ with $d(\alpha) = 2$ ($\rightsquigarrow$ basis of $\mathfrak{g}_d(2)$).
- An arbitrary linear $\lambda\colon \mathfrak{g}_d(2) \to \mathbb{C}$ is specified by $x_l := \lambda(e_{\gamma_l})$ for $1 \leqslant l \leqslant m$; then the (i,j)-entry of $\mathcal{G}_\lambda$ is given by $\lambda([e_{\beta_i}, e_{\beta_j}]) \in \mathbb{Z}[x_1, \ldots, x_m]$.

Matrices on first slide: $\mathfrak{g} = \mathfrak{f}_4$ with $d = [0,1,0,1]$ and $d = [0,0,0,1]$.
1st matrix: $\det = 9(\ldots)^2$; 2nd: $\det = (-16x_1^2x_7^2 \pm \ldots - x_4^4)^2$. Guess a pattern ?

Main observation: Everything above works over $\mathbb{Z}$!

- Chevalley basis $B = \{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ spans Lie algebra $\mathfrak{g}_\mathbb{Z}$ over $\mathbb{Z}$.
- Given $d\colon I \to \{0, 1, 2\}$, let $\mathfrak{g}_{\mathbb{Z},d}(n) = \langle e_\alpha \mid d(\alpha) = n\rangle_\mathbb{Z}$ for $n = 1, 2$.

Integrality condition, see G. (Transf. Groups, online May 2020, and JSAG 2020).
We say that d is $\mathbb{Z}$-special if there is some $\lambda \in \mathrm{Hom}(\mathfrak{g}_{\mathbb{Z},d}(2), \mathbb{Z})$ such that

$$\sigma_\lambda\colon \mathfrak{g}_{\mathbb{Z},d}(1) \times \mathfrak{g}_{\mathbb{Z},d}(1) \to \mathbb{Z}, \qquad (y, z) \mapsto \lambda([y, z]),$$

is non-degenerate over $\mathbb{Z}$. (If $\mathfrak{g}_{\mathbb{Z},d}(1) = \{0\}$, then d is declared $\mathbb{Z}$-special.)

Conjecture (now Theorem): d is Lusztig-special if and only if d is $\mathbb{Z}$-special.

- Using our computations of Gram matrices, true for all exceptional types. (For large matrices, relies on Groebner basis techniques as suggested by U. Thiel and A. Steel.)
- For classical type, see Dong and Yang, Advances in Math. (online Nov. 2020).
- Relevance for real Lie groups: see Vogan's MIT Virtual Lie Group Seminar talk "What's special about special?" at `http://www-math.mit.edu/~dav/LG/`.