Some new decomposition numbers of finite classical groups

Emily Norton

December 12, 2020

An old story: Lie algebras acting on categories

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The idea: the generators e and f of $\mathfrak{sl}_2$ act on a module category $\mathcal{C}$ by a biadjoint pair of exact endofunctors E and F in such a way that the images of the functors in the Grothendieck group of $\mathcal{C}$ satisfy the $\mathfrak{sl}_2$-relations.

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Definition of $\mathfrak{g}$-categorification

$\mathfrak{g}$ a Lie algebra (finite or affine Dynkin type).

Example relevant for this talk: $\mathfrak{g} = \widehat{\mathfrak{sl}}_d$, $d \geq 2$. Dynkin diagram: d-gon (affine type A).

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A $\mathfrak{g}$-categorification on $\mathcal{C}$ is a collection of exact endofunctors $\{E_i, F_i\}$ of $\mathcal{C}$, where i ranges over the nodes of the Dynkin diagram of $\mathfrak{g}$, satisfying:

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- The classes $[S]$ in $[\mathcal{C}]$ of the simple objects $S \in \mathcal{C}$ are $\mathfrak{g}$-weight vectors;
- Strong: Set $E = \bigoplus_i E_i$, $F = \bigoplus_i F_i$. There are natural transformations $X \in \mathrm{End}(F)$ and $T \in \mathrm{End}(F^2)$ such that in $\mathrm{End}(F^n)$, $X_j := 1^{j-1}X1^{n-j}$ and $T_k := 1^{k-1}T1^{n-k-1}$ satisfy defining relations of an affine Hecke algebra.

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Consequence: $\mathrm{Soc}(E_i(S))$ and $\mathrm{Head}(F_i(S))$ are simple or 0 for all simple S
$\rightsquigarrow$ get a simply directed graph on the set of simple objects $\{S\}$ of $\mathcal{C}$.

Example: the symmetric groups (Chuang-Rouquier, LLT, Kleshchev)

$\mathrm{char}\ \Bbbk = p > 0$, $\mathcal{C} = \bigoplus_{n\geq 0} \Bbbk\mathfrak{S}_n\text{-mod}$.

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Theorem (Chuang-Rouquier, Lascoux-Leclerc-Thibon)

There is a $\widehat{\mathfrak{sl}}_p$-categorification on $\mathcal{C}$ with

$$\mathrm{Res} = E = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} E_i, \qquad \mathrm{Ind} = F = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} F_i$$

If Δ_λ, $\lambda \vdash n$, is a Specht module, then

$$[E_i(\Delta_\lambda)] = \sum_{\substack{\lambda \setminus b \\ \mathrm{ct}(b) \cong i \mod p}} [\Delta_{\lambda \setminus b}], \qquad [F_i(\Delta_\lambda)] = \sum_{\substack{\lambda \cup b \\ \mathrm{ct}(b) \cong i \mod p}} [\Delta_{\lambda \cup b}]$$

Illustration: $p = 3$, $\lambda = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3}\end{array}$.

Then

$$[E_1(\Delta_{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3}\end{array}})] = [\Delta_{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2}\end{array}}],$$

$$[F_1(\Delta_{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3}\end{array}})] = [\Delta_{\begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 0 & 1 \\ \cline{1-3}\end{array}}] + [\Delta_{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1}\end{array}}]$$

The symmetric groups, continued

$\lambda \vdash n$ a p-regular partition, S_λ a simple $\Bbbk\mathfrak{S}_n$-module. The head of $F_i(S_\lambda)$ is simple if $F_i(S_\lambda) \neq 0$, set it equal to $S_{\tilde{f}_i(\lambda)}$.

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There is a combinatorial rule for finding $\tilde{f}_i(\lambda)$: for $i \in \mathbb{Z}/p\mathbb{Z}$,

$$\tilde{f}_i(\lambda) = \lambda \cup \{\text{"good addable i-box"}\}$$

if a “good addable i-box” exists, otherwise $\tilde{f}_i(\lambda) = 0$.

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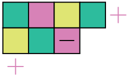

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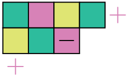

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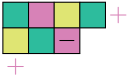

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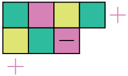

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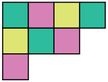
$\tilde{f}_1$ () = 

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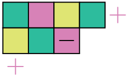

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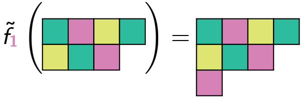

The graph with Vertices: $\{p\text{-regular partitions } \lambda\}$
Edges: $\{\lambda \to \mu \mid \mu = \tilde{f}_i(\lambda) \text{ for some } i \in \mathbb{Z}/p\mathbb{Z}\}$

is called the $\widehat{\mathfrak{sl}}_p$-crystal on the set of p-regular partitions.

$\widehat{\mathfrak{sl}}_d$-action on module categories of finite classical groups (DVV, GHJ)

$B_n(q)$, $C_n(q)$ finite reductive groups, Weyl group $W = B_n$
$\mathrm{char}\ \mathbb{k} = \ell > 0$, $|q| = d \mod \ell$, $d \geq 2$ even.

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$B_n(q)$, $C_n(q)$ finite reductive groups, Weyl group $W = B_n$
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That means:

$$\mathrm{Res} \cong \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} E_i, \qquad \mathrm{Ind} \cong \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} F_i$$

and on $[\mathcal{C}]$ there is a $\widehat{\mathfrak{sl}}_d$-crystal graph with vertices $\{[S] \mid S \text{ simple}\}$ and arrows $[S] \to [T]$ if T is the head of $F_i(S)$ for some $i \in \mathbb{Z}/d\mathbb{Z}$.

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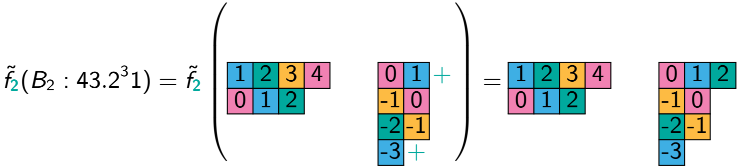

Dec matrix of $B_{2n}(q)$, $C_{2n}(q)$ when $\ell \mid \Phi_{2n}(q)$

$(\mathbb{K}, \mathbb{O}, \mathbb{k})$ an ℓ-modular system large enough for the group, $d = |q| \mod \ell$ even
$\Delta_{B_{t^2+t}:\lambda}$: the ℓ-modular reduction of an $\mathbb{O}$-lattice of the irreducible ordinary unipotent representation $\rho_{B_{t^2+t}} : \lambda$,
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Labels of simple modules in the principal block:

- bipartitions of $2n$ (principal series),
- bipartitions of $2n-2$ (B_2 series),
- bipartitions of $2n-6$ (B_6 series),

which have empty n-co-core.

Dec matrix of $B_6(q)$ and $C_6(q)$, $\ell \mid \Phi_6(q)$

First found by Dudas-Malle [DM '20]. It is also given by our theorem.

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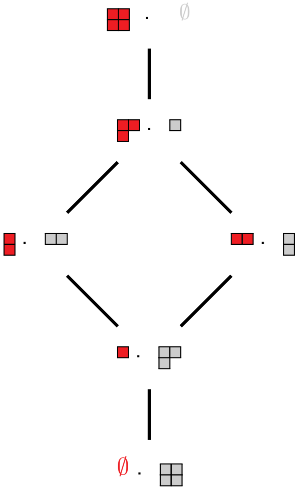
$6.$	1																										
$.6$		1																									
$51.$	1		1																								
2.4	1	1		1																							
$B_2 : 2^2.$					1																						
$41^2.$			1			1																					
$.51$		1					1																				
21.3	1		1	1				1																			
1.41		1		1			1		1																		
$B_2 : 21.1$					1					1																	
B_6											1																
$21^2.2$			1			1		1				1															
$1^2.31$				1				1	1				1														
$B_2 : 1^2.2$										1				1													
$31^3.$						1									1												
$.41^2$							1		1							1											
$B_2 : 2.1^2$										1							1										
$21^3.1$						1						1			1			1									
$1^3.21$								1				1	1						1								
$B_2 : 1.21$					1					1				1			1			1							
$B_2 : .2^2$					1															1	1						
$1^4.1^2$												1						1	1			1					
$21^4.$															1			1					1				
$.31^3$									1				1			1								1			
$1^6.$											2							1				1	1		1		
$.21^4$									1										1					1		1	
$.1^6$																			1		2	1				1	1

Any patterns in this matrix?

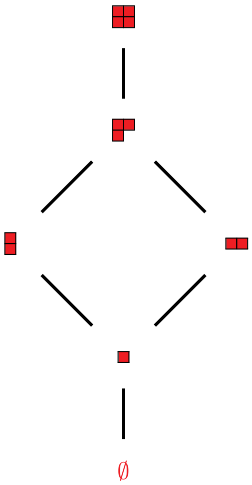
1. Consider submatrix labeled $B_2 : \lambda^1.\lambda^2$:

$B_2 : 2^2.$	1					
$B_2 : 21.1$	1	1				
$B_2 : 1^2.2$		1	1			
$B_2 : 2.1^2$		1		1		
$B_2 : 1.21$	1	1	1	1	1	
$B_2 : .2^2$	1				1	1

2. Draw poset determined by nonzero dec numbers:



Project $\lambda^1.\lambda^2$ onto λ^1:



Question: where have we seen this poset before?

Answer:

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When $d > |\lambda|$ we can expect similar behavior to $d = \infty$.

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Idea of proof: show that when $d > n - t^2 - t$, the dec numbers are controlled by combinatorics of $\widehat{\mathfrak{sl}}_d$-crystal. Proof by induction, not too difficult.

Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

By the *B_{t^2+t}-submatrix* we mean the submatrix of the unipotent decomposition matrix of $\Bbbk B_n(q)$ or $\Bbbk C_n(q)$ whose rows and columns are labeled by $B_{t^2+t} : \lambda^1.\lambda^2$.

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Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

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- Explicit, closed formulas for the entries of the B_{t^2+t}-submatrix.

Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

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Idea of proof: show that when $d > n - t^2 - t$, the dec numbers are controlled by combinatorics of $\widehat{\mathfrak{sl}}_d$-crystal. Proof by induction, not too difficult. Tiefhängende Früchte.

Consequences:

- Explicit, closed formulas for the entries of the B_{t^2+t}-submatrix.
- The B_{t^2+t}-submatrix depends only on the order of q mod ℓ, not on ℓ.

Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

By the B_{t^2+t}*-submatrix* we mean the submatrix of the unipotent decomposition matrix of $\Bbbk B_n(q)$ or $\Bbbk C_n(q)$ whose rows and columns are labeled by $B_{t^2+t} : \lambda^1.\lambda^2$.

Theorem

(Dudas-N., '20, work in progress) Let $d > n - t^2 - t$ be even and let $\ell = \mathrm{char}\, \Bbbk$ be any prime such that $|q| = d \mod \ell$. Then the decomposition numbers in the B_{t^2+t}-submatrix of the unipotent dec matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$ are given by Brundan-Stroppel's algorithm.

Idea of proof: show that when $d > n - t^2 - t$, the dec numbers are controlled by combinatorics of $\widehat{\mathfrak{sl}}_d$-crystal. Proof by induction, not too difficult. Tiefhängende Früchte.

Consequences:

- Explicit, closed formulas for the entries of the B_{t^2+t}-submatrix.
- The B_{t^2+t}-submatrix depends only on the order of q mod ℓ, not on ℓ.
- The B_{t^2+t}-submatrix is the same as the decomposition matrix of Category $\mathcal{O}$ of the rational Cherednik algebra $H_{d,\mathbf{s}_t}(n - t^2 - t)$.

Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

By the B_{t^2+t}*-submatrix* we mean the submatrix of the unipotent decomposition matrix of $\Bbbk B_n(q)$ or $\Bbbk C_n(q)$ whose rows and columns are labeled by $B_{t^2+t} : \lambda^1.\lambda^2$.

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Consequences:

- Explicit, closed formulas for the entries of the B_{t^2+t}-submatrix.
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- The B_{t^2+t}-submatrix is the same as the decomposition matrix of Category $\mathcal{O}$ of the rational Cherednik algebra $H_{d,\mathbf{s}_t}(n - t^2 - t)$.
- All dec numbers in the B_{t^2+t}-submatrix are 0 or 1.

Generic submatrices of the decomposition matrix of $\Bbbk B_n(q)$ and $\Bbbk C_n(q)$

By the *B_{t^2+t}-submatrix* we mean the submatrix of the unipotent decomposition matrix of $\Bbbk B_n(q)$ or $\Bbbk C_n(q)$ whose rows and columns are labeled by $B_{t^2+t} : \lambda^1.\lambda^2$.

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Idea of proof: show that when $d > n - t^2 - t$, the dec numbers are controlled by combinatorics of $\widehat{\mathfrak{sl}}_d$-crystal. Proof by induction, not too difficult. Tiefhängende Früchte.

Consequences:

- Explicit, closed formulas for the entries of the B_{t^2+t}-submatrix.
- The B_{t^2+t}-submatrix depends only on the order of q mod ℓ, not on ℓ.
- The B_{t^2+t}-submatrix is the same as the decomposition matrix of Category $\mathcal{O}$ of the rational Cherednik algebra $H_{d,\mathbf{s}_t}(n - t^2 - t)$.
- All dec numbers in the B_{t^2+t}-submatrix are 0 or 1.
- This is the first result identifying large submatrices of the unip dec matrix in blocks of arbitrary complexity since the 1990s.

THANK YOU