# 1. STLP - Short Theorems with Long proofs

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Using ideas from small cancellation theory and VR (Vor-Rückwärts) rewrites to investigate a presentation of a group.

# 2. “Traditional” methods

- How can we use a computer to investigate a finitely presented group?
  - Coset enumeration.
  - Knuth-Bendix and variations.
- Here is another idea that looks promising, hopefully by having a guide (“curvature”) as to which consequences of the relators are interesting.

# 3. Overview of idea.

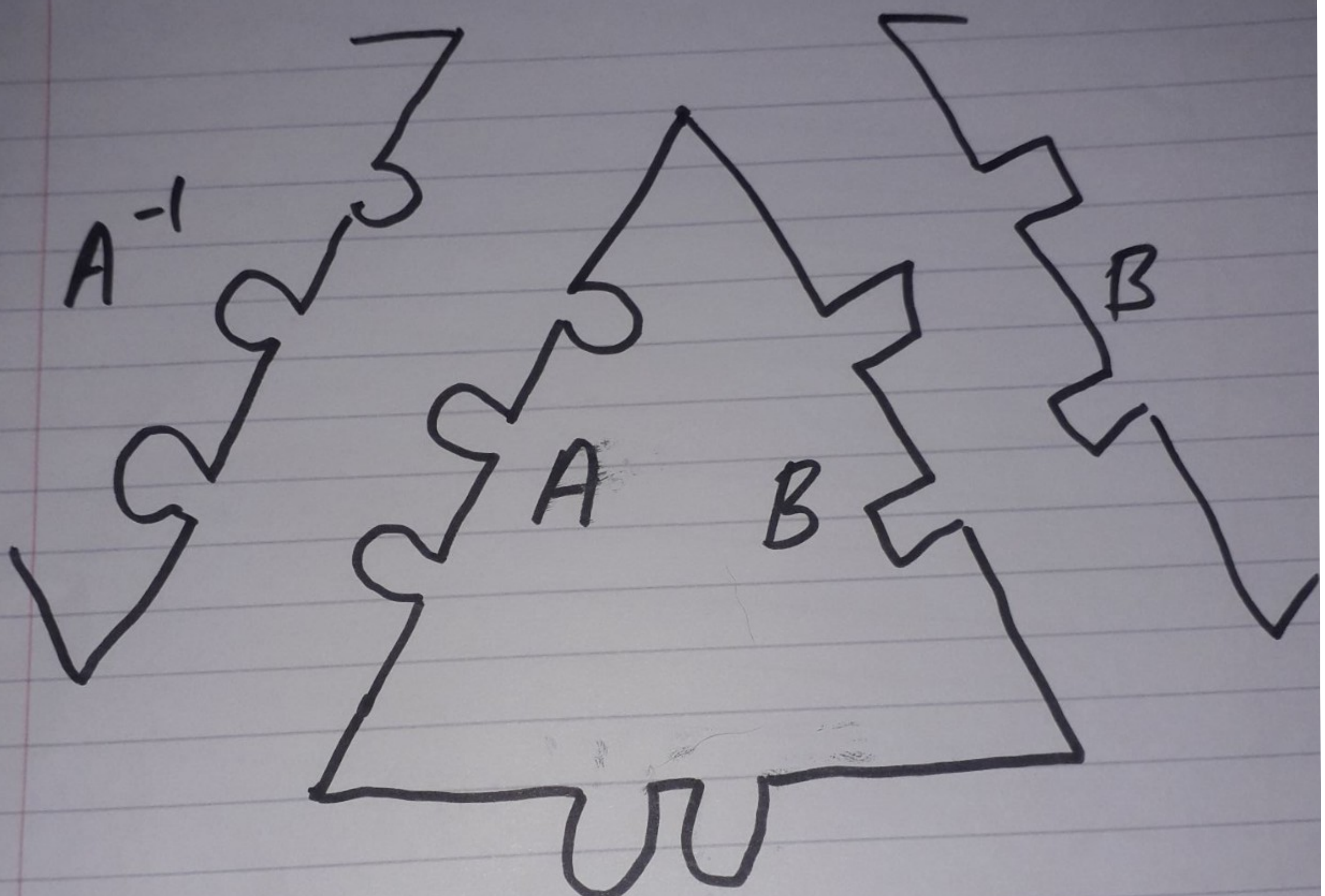
- Triangulate the presentation. All relators must be length three.
- Collect VR (Vor- and Rückwärds) rewrites W1 ↔ W2 that apply in either direction and do not increase length.
- Our word-problem-solver takes a (cyclic) word and applies the VR rewrites in all ways. If it makes the empty word, it was the identity, and otherwise it was not.
- Use curvature to show that every van Kampen diagram has some W1 on the boundary.

# 4. Assumptions

- In 20 minutes I must simplify. Inverses clutter up slides, so I've often omitted them. They are easily put back, or just imagine that all the generators have order two. I have deliberately glossed over some other issues, too.
- I must assume that all relators (except cancelling inverses) **have length 3**. This seems to be important, so to achieve this, we need to add generators and relators before we start.

# 5. Triangulation.

- All relators must be length 3. If there are longer relators e.g. ABCD... we introduce a new generator Z and replace AB by Z (a length 3 relator) so we now have ZCD...
- Some other fairly simple pre-processing of the presentation is needed to get it into a suitable form.




Jigsaw Triangles.

# 8. Main idea

- Either you can build a thing like that with your triangular jig-saw puzzle pieces, in which case the “base” is an STLP.
- Or you can’t, in which case you can solve the word problem with a VR rewrite system.

# 9. Heuristics.

- “Typical” problem
  - 100 letters (50 generators + inverses)
  - 400 triangles (65 relators, rotations, inverse)
- Each edge fits ~ 3 triangles.
- Big diagram has 3 edges for each 2 triangles . . .
- So you usually won’t be able to make it.

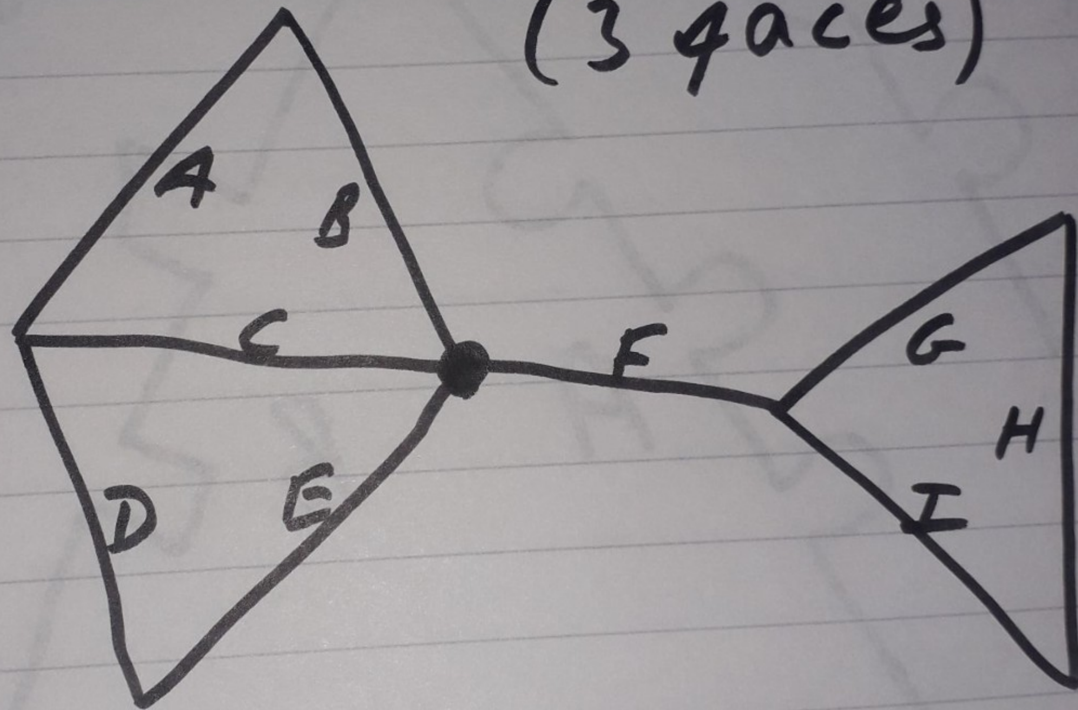
# 10. STLPs

- Traditional methods may struggle if there are Short Theorems with (only) a Long Proof.  Relations that hold, but not obviously so.
- What is a “Short Theorem”?
- A word with few letters that is the identity in the group.

# 11. What is a Long Proof?

- By definition of normal subgroup, every word that is the identity in the group is the cancelled product of conjugates of relators.
- I define the length of the proof as the number of relators needed.
- This is the number of “faces” in a van Kampen diagram.

# A van Kampen diagram (3 faces)




EDC. CAB. FGHIF

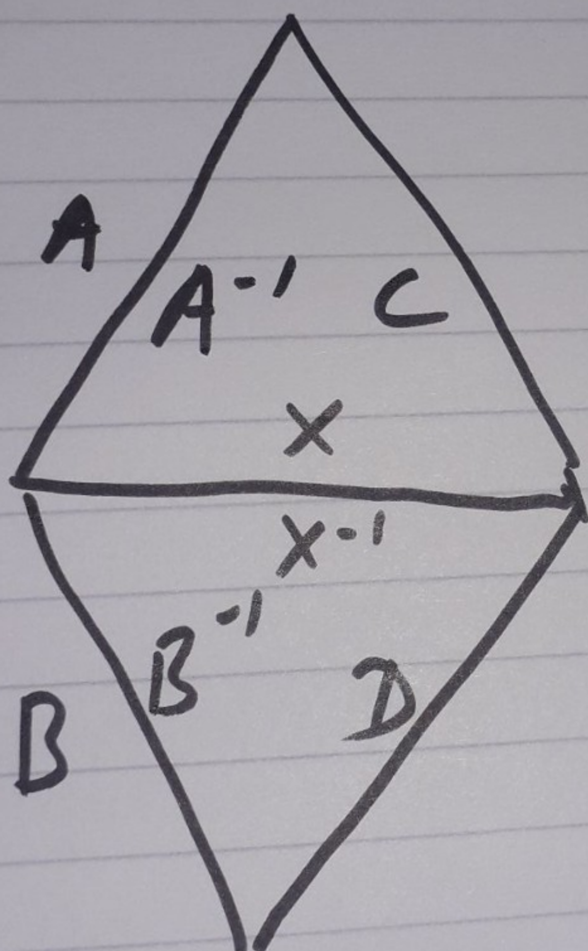
# 13. Dehn rewrites.

- If AA=1 we can delete AA and make our word shorter.
- If ABC is a relator, we can replace AB by C and make our word shorter.
- These are the “Dehn” rewrites.  They are not usually powerful enough to do much.

# 14. Diamond VR

- From two relators ACX and DBX we can conclude that AB = CD.  This is the “diamond” relation.
- Diamond VR takes a word and does diamond rewrites and rotations in all possible ways hoping to reduce to the empty word.
- If the length can ever goes down (by a Dehn rewrite) we start again with that new, shorter word.

# Diamonds




$$A^{-1} C X . X^{-1} D B^{-1} = 1$$

$$A^{-1} C D B^{-1} = 1$$

$$CD = AB$$

# 16. Diamond VR is powerful.

- There are actually plenty of (triangulated) presentations where Diamond VR solves the word-problem.
- Even if it doesn't always work, we may consider words where it *does* work as boring - the opposite of interesting.
- This definition of interesting is usable!

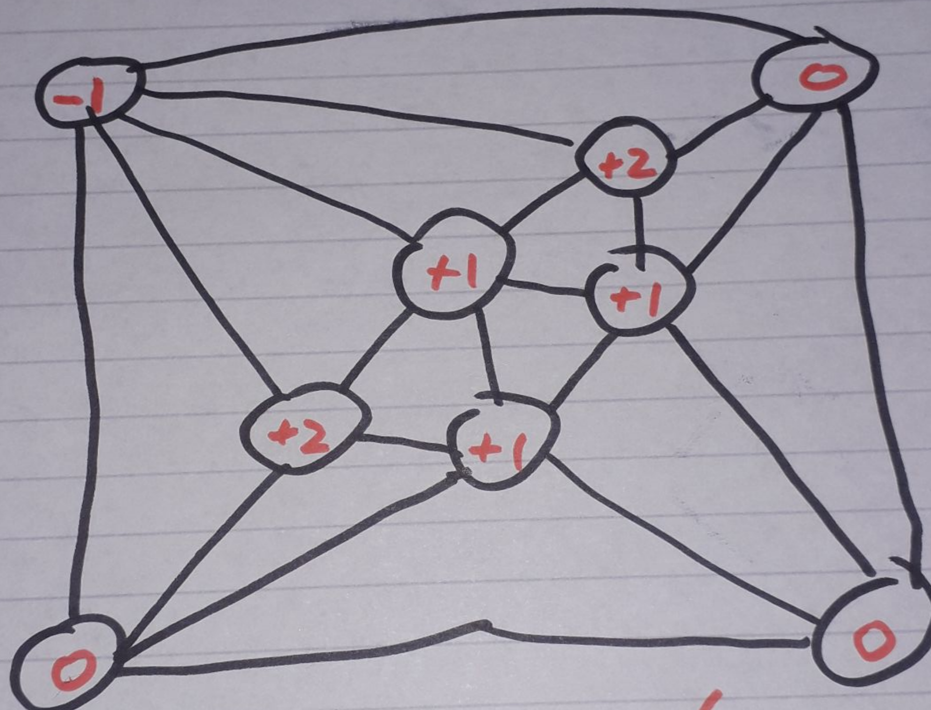
# 17. “Curvature”

- In the case of triangles, I define the “curvature” of a vertex where T triangles and E edges meet as

  Curvature = 6 + 2.T − 3.E

- A simple application of Euler’s formula (faces+vertices=edges +2) then tells us that the sum of the curvature over a whole diagram is always 6.

A van Kampen diagram
with curvature marked in red.




The total curvature is 6.

# 19. The driving force.

- If Diamond VR cannot reduce the number of triangles in a diagram, the curvature of every boundary vertex is zero or negative, so we conclude that . . .
- There must be quite a few internal vertices with positive curvature . . . at most five triangles meet there.

# 20. So what can we actually do?

- The first idea is to jig-saw-fit at most five triangles together at an internal vertex in all possible ways . . .
- This is a finite process and, under reasonable conditions, is entirely practical.
- If there are none, we have proved that Diamond VR solves the word problem.

# 21. If we find a bad diagram?

- We could triangulate these too.
- We could pass them on to Knuth-Bendix
- We could pass them on to Todd-Coxeter
- **We could add more VR rules for them**.

- I think it fair to say that this area is ripe for further research.

# 22. Add More VR rules

- VR rules can go either way, so we potentially have a large search problem.
- We an associate a **price** to each rule to limit the search to a given total price.
- Here the price is the number of triangles that the rule removes – diamonds, for example, have price 2 (2 triangles).

# 23. An algorithm?

- Given a word that is the identity, is there a van Kampen diagram to prove it with T triangles.  For definite.
- We collect some rules W1 → W2 in t triangles.
- And prove (using curvature) that every van Kampen diagram has some W1 on the boundary.