An Atlas of Orthogonal Determinants

Gabriele Nebe

Lehrstuhl für Algebra und Zahlentheorie

joint work with Richard Parker and Thomas Breuer

$G_2(3)$ (Mod 2)

		;	@	@	@	@	@	@	@	@	@	@	@	@	;	;
				5	5											
		4245696	832	832	729	162	162	7	27	27	27	13	13			
		p power		A	A	A	A	A	A	C	C	C	A	A		
		p' part		A	A	A	A	A	A	A	A	A	A	A		
	OD	ind	1A	3A	3B	3C	3D	3E	7A	9A	9B	C**	13A	B*	fus	ind
X_1		+	1	1	1	1	1	1	1	1	1	1	1	1	:	+
X_2	O-	+	14	5	5	-4	2	-1	0	2	-1	-1	1	1	:	+
X_3		o	64	-8	-8	1	4	-2	1	1	b27	**	-1	-1	:	o
X_4		o	64	-8	-8	1	4	-2	1	1	**	b27	-1	-1	:	o
X_5	O-	+	78	-3	-3	-3	-3	6	1	0	0	0	0	0	:	+
X_6	O+	+	90	9	9	9	0	0	-1	0	0	0	-1	-1	:	+
X_7	O-	+	90	-9	18	0	3	-3	-1	-3	0	0	-1	-1	.	+
X_8	O-	+	90	18	-9	0	3	-3	-1	-3	0	0	-1	-1	|	
X_9	O-	+	378	-9	-9	9	-3	-6	0	3	0	0	1	1	:	+
X_10	O+	+	448	16	16	-11	-2	-2	0	1	1	1	b13	*	.	+
X_11	O+	+	448	16	16	-11	-2	-2	0	1	1	1	*	b13	|	
X_12	O+	+	832	-32	-32	-5	4	4	-1	1	1	1	0	0	:	+

Compute Orthogonal Determinants

A character χ is called orthogonal, if there is a representation ρ affording χ that fixes a non-degenerate quadratic form.

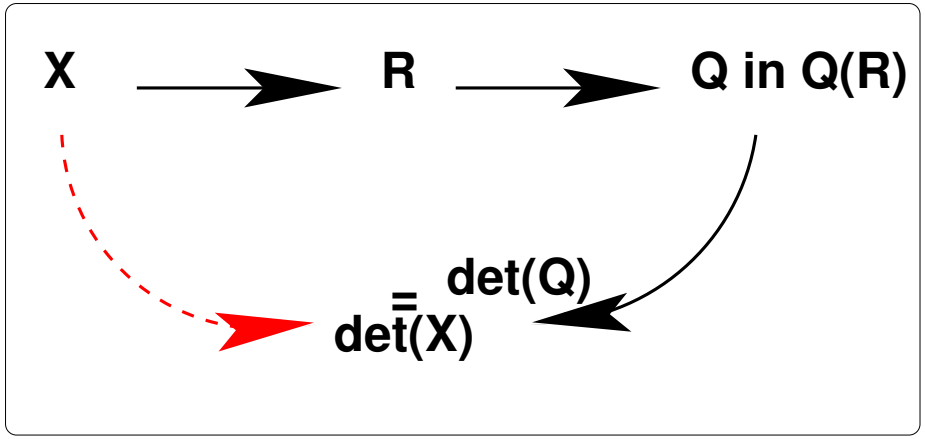
$$\mathfrak{Q}(\rho) := \{Q \text{ quadratic form } \mid Q(v\rho(g)) = Q(v) \text{ f. a. } g \in G\}$$

space of ρ-invariant quadratic forms.

- Joint project with Richard Parker and Thomas Breuer:
- For all but the largest few ATLAS groups compute all determinants of the invariant quadratic forms
- type of orthogonal group over finite fields

Orthogonal Determinants of characters

- Given an orthogonal character χ
- choose some orthogonal representation ρ affording χ
- construct some non-degenerate $Q \in \mathfrak{Q}(\rho)$
- and compute $\det(Q)$ (modulo squares).



Orthogonal stability

If $\det(Q)$ is independent of all these choices then χ is orthogonally stable and $\det(\chi) = \det(Q)$ (modulo squares) is the orthogonal determinant of χ.

Reduction to $\mathrm{Irr}^+(G)$

Theorem (GN 22)

χ is orthogonally stable, if and only if all its absolutely irreducible indicator + constituents have even degree.
Then $\det(\chi)$ is a square class of the character field of χ.

$\mathrm{Irr}^+(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = +, \chi(1) \text{ even } \}$.

Suffices to find $\det(\chi)$ for all $\chi \in \mathrm{Irr}^+(G)$ and all simple groups G.

The determinant of the adjoint involution

- B non-degenerate symmetric bilinear form on V
- adjoint involution ι_B on $\mathrm{End}(V)$

$$B(\alpha(v), w) = B(v, \iota_B(\alpha)(w)) \text{ for all } v, w \in V.$$

$$E_-(B) := \{\alpha \in \mathrm{End}_K(V) \mid \iota_B(\alpha) = -\alpha\}$$

- basis $(v_1, \ldots, v_n)$, $\mathrm{End}(V) \cong K^{n \times n}$
- $\iota_B(A) = BA^{tr}B^{-1}$ and $E_-(B) = \{BX \mid X = -X^{tr}\}$ as
- $\iota_B(BX) = B(BX)^{tr}B^{-1} = BX^{tr}$.
- $X = -X^{tr}$ then $\det(X)$ is a square.

Proposition (Knus, Merkurjev, Rost, Tignol, 1998)

$\dim(V)$ even $\Rightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^\times)^2$ for any invertible $\alpha \in E_-(B)$.

Computing the determinant

Proposition (Knus, Merkurjev, Rost, Tignol, 1998)

$\dim(V)$ even $\Rightarrow E_-(B) \cap \mathrm{GL}(V) \neq \{\}$.
Then $\det(B) = \det(\alpha)(K^\times)^2$ for any invertible $\alpha \in E_-(B)$.

- $Q \in \mathfrak{Q}(\rho)$ non-degenerate, $\rho(G) \leq O(Q)$, $n := \dim(\rho)$ even
- $\iota_{B_Q}(g) = g^{-1}$ for all $g \in \rho(G)$
- Take three random elements g, h, k in $\rho(G)$
- compute $X = X(g,h,k) = g - g^{-1} + h - h^{-1} + k - k^{-1}$
- If $\det(X) \neq 0$ then
  (a) χ_ρ is orthogonally stable and
  (b) $\det(\chi_\rho) = \det(X)(F(\chi)^\times)^2$.

$\Sigma^-(G) := \langle g - g^{-1} \mid g \in G \rangle \leq \mathbb{Z}G$.

χ orthogonally stable $\Leftrightarrow$ there is $X \in \Sigma^-(G)$ such that $\det(\rho(X)) \neq 0$ for any representation ρ affording χ. Then

$$\det(\chi) = \det(\rho(X))(F(\chi)^\times)^2.$$

The ordinary character table of J_2

	;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@	@	@	@	@	@	
			1		1																															
	604800	920	240	080	36	96	300	300	50	50	24	12	7	8	20	20	10	10	12	15	15				336	48	12	6	48	16	6	6	7	12	12	
p power		A	A	A	A	A	A	A	A	A	AA	BB	A	A	BB	BB	DA	CA	AA	BA	AA				A	A	B	BC	A	A	AB	BC	AC	AB	AB	
p' part		A	A	A	A	A	A	A	A	A	AA	BB	A	A	AB	BB	CA	DA	AA	AA	BA				A	A	A	BC	A	A	AB	BC	AC	AB	AB	
ind	1A	2A	2B	3A	3B	4A	5A	B*	5C	D*	6A	6B	7A	8A	10A	B*	10C	D*	12A	15A	B*	fus	ind		2C	4B	4C	6C	8B	8C	12B	12C	14A	24A	B*	
χ_1	+ 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++		1	1	1	1	1	1	1	1	1	1	1	χ_1
χ_2	+ 14	−2	2	5	−1	2	−3b5	*	b5+2	*	1	−1	0	0	b5	*	−b5	*	−1	0	0		+		0	0	0	0	0	0	0	0	0	0	0	χ_2
χ_3	+ 14	−2	2	5	−1	2	*	−3b5	*	b5+2	1	−1	0	0	*	b5	*	−b5	−1	0	0															χ_3
χ_4	+ 21	5	−3	3	0	1	b5+4	*	−2b5	*	−1	0	0	−1	b5	*	0	0	1	*	−b5		+		0	0	0	0	0	0	0	0	0	0	0	χ_4
χ_5	+ 21	5	−3	3	0	1	*	b5+4	*	−2b5	−1	0	0	−1	*	b5	0	0	1	−b5	*															χ_5
χ_6	+ 36	4	0	9	0	4	−4	−4	1	1	1	0	1	0	0	0	−1	−1	1	−1	−1	:	++		6	−2	0	0	2	2	1	0	−1	−1	−1	χ_6
χ_7	+ 63	15	−1	0	3	3	3	3	−2	−2	0	−1	0	1	−1	−1	0	0	0	0	0	:	++		7	3	−1	1	−3	1	0	−1	0	0	0	χ_7
χ_8	+ 70	−10	−2	7	1	2	−5b5	*	0	0	−1	1	0	0	−b5	*	0	0	−1	b5	*		+		0	0	0	0	0	0	0	0	0	0	0	χ_8
χ_9	+ 70	−10	−2	7	1	2	*	−5b5	0	0	−1	1	0	0	*	−b5	0	0	−1	*	b5															χ_9
χ_{10}	+ 90	10	6	9	0	−2	5	5	0	0	1	0	−1	0	1	1	0	0	1	−1	−1	:	++		6	2	0	0	4	0	−1	0	−1	1	1	χ_{10}
χ_{11}	+ 126	14	6	−9	0	2	1	1	1	1	−1	0	0	0	1	1	−1	−1	−1	1	1	:	++		0	4	0	0	2	−2	1	0	0	−1	−1	χ_{11}
χ_{12}	+ 160	0	4	16	1	0	−5	−5	0	0	0	1	−1	0	−1	−1	0	0	0	1	1	:	++		8	0	2	−1	0	0	0	−1	1	0	0	χ_{12}
χ_{13}	+ 175	15	−5	−5	1	−1	0	0	0	0	3	1	0	−1	0	0	0	0	−1	0	0	:	++		7	−1	1	1	−1	−1	−1	1	0	−1	−1	χ_{13}
χ_{14}	+ 189	−3	−3	0	0	−3	−3b5	*	b5+2	*	0	0	0	1	b5	*	b5	*	0	0	0		+		0	0	0	0	0	0	0	0	0	0	0	χ_{14}
χ_{15}	+ 189	−3	−3	0	0	−3	*	−3b5	*	b5+2	0	0	0	1	*	b5	*	b5	0	0	0															χ_{15}
χ_{16}	+ 224	0	−4	8	−1	0	2r5−1	−2r5−1	2b5	*	0	−1	0	0	1	1	0	0	0	*	−b5		+		0	0	0	0	0	0	0	0	0	0	0	χ_{16}
χ_{17}	+ 224	0	−4	8	−1	0	−2r5−1	2r5−1	*	2b5	0	−1	0	0	1	1	0	0	0	−b5	*															χ_{17}
χ_{18}	+ 225	−15	5	0	3	−3	0	0	0	0	0	−1	1	−1	0	0	0	0	0	0	0	:	++		1	−3	−1	1	3	−1	0	−1	1	0	0	χ_{18}
χ_{19}	+ 288	0	4	0	−3	0	3	3	−2	−2	0	1	1	0	−1	−1	0	0	0	0	0	:	++		8	0	−2	−1	0	0	0	1	1	0	0	χ_{19}
χ_{20}	+ 300	−20	0	−15	0	4	0	0	0	0	1	0	−1	0	0	0	0	0	1	0	0	:	++		6	−2	0	0	−2	−2	1	0	−1	1	1	χ_{20}
χ_{21}	+ 336	16	0	−6	0	0	−4	−4	1	1	−2	0	0	0	0	0	1	1	0	−1	−1	:	++		0	0	0	0	0	0	0	0	0	r6	−r6	χ_{21}
$a_{2,13}$																						fus	ind		4	4	8	12	8	8	12	24	28	24	24	
ind	1	2	4	3	3	4	5	5	5	5	6	12	7	8	20	20	10	10	12	15	15						8			8		12	24	28	24	
	2	2		6	6	4	10	10	10	10	6		14	8			10	10	12	30	30															
χ_{22}																							−		0	0	0	0	0	0	0	0	0	0	0	χ_{22}
χ_{23}	6	2	0	−3	0	2	−2b5	*	*	b5−1	1	0	−1	0	0	0	−b5	*	−1	b5	*															χ_{23}

72 ordinaries 20 Dec 2021

	11	19	29	31	41	59		13	17
14	0−	0+	0−	0+	0−	0−	−3	×	×
14	0−	0+	0−	0+	0−	0−	−3	×	×
36	0+	0+	0+	0+	0+	0+	~~1~~ 5	0−	0−
70	0−	0+	0−	0+	0−	0−	−3	×	×
70	0−	0+	0−	0+	0−	0−	−3	×	×
90	0+	0−	0+	0−	0−	0−	−7	0−	0−
126	0−	0−	0+	0−	0+	0−	~~−1~~ −5	0−	0−
160	0+	0+	0+	0+	0+	0+	1	0+	0+
224	0+	0+	0+	0+	0+	0+	1	×	×
224	0+	0+	0+	0+	0+	0+	1	×	×
288	0−	0−	0−	0−	0+	0+	~~21~~ 105	0+	0−
300	0−	0−	0−	0−	0+	0+	21	0−	0+
336	0+	0+	0+	0+	0+	0+	1	0+	0+

lab 63 175 189_{ab} 225

1 2 4 8 3 6
12 11 9 5 10 7 13

$\langle prime \rangle = 1 + k63 + k36$ | $e2\ k36 = 224_{ab} + 70_{ab} + 21_{ab}$

Further applications

Split extensions $G : 2$ (GN 22)

Groups with non-trivial center (ind. o, Spinor norm)

Reduction to simple groups

Enough to compute $\det(\chi)$ for absolutely irreducible even degree orthogonal characters of simple groups (with 4 exceptions).

p-groups (GN 22)

- Let G be a p-group, χ orthogonally stable
- $\det(\chi) =$ formula (degree, character field)
- $\mathbb{Q}(\chi) = \mathbb{Q}$ then $\det(\chi) = p^{\chi(1)/(p-1)}$.

Orthogonal Condensation

- V a permutation module, p odd prime
- $H \in \mathrm{Syl}_p(G)$, $e := \frac{1}{|H|}\sum_{h\in H} h$, $V = Ve \perp V(1-e)$
- $V(1-e)_{|H}$ orthogonally stable of known orthogonal determinant
- Ve spanned by the H-orbit sums
- $A = e\mathbb{Q}Ge$ algebra with involution
- Composition factors of Ve modulo primes $\neq p$
- Determinant of skew elements of A on composition factors
- Enough primes $\Rightarrow$ determinant of this composition factor.

Harada Norton group

- $G = HN$, $\dim(V) = 108,345,600$, $p = 5$.
- $\dim(Ve) = 7008$.
- $\det(\chi) = 4\sqrt{5} + 17$ for $\chi(1) = 5,103,000$.

Parker's conjecture

Determinants of rational orthogonally stable characters are odd.
If $\det(\chi) = d(K^\times)^2$ then $\nu(d)$ is even for all dyadic valuations ν of K.

Parker's conjecture holds

- for the ATLAS groups up to HN (RP, TB, TB, GN)
- for orthogonally simple characters of the form $\psi + \overline{\psi}$ (Navarro, Tiep, Isaacs, Liebeck)
- for solvable groups (GN)
- $\mathrm{SL}_2(q)$, $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ (all q, OB,LH,GN),
- for all Coxeter groups (Linda Hoyer)
- for all groups $\mathrm{GL}_n(q)$, $\mathrm{G}_2(q)$ with q odd (Linda Hoyer)

No counterexamples to Parker's conjecture so far.

Conclusion

- χ is orthogonally stable, if and only if all its indicator $+$ constituents have even degree.
- Computed the orthogonal determinants of the characters in

$$\mathrm{Irr}^+(G) := \{\chi \in \mathrm{Irr}(G) \mid \mathrm{ind}(\chi) = +, \chi(1) \text{ even } \}.$$

for the ATLAS groups up to HN.

$\Sigma^-(G) := \langle g - g^{-1} \mid g \in G \rangle \leq \mathbb{Z}G.$

χ orthogonally stable $\Leftrightarrow$ there is $X \in \Sigma^-(G)$, $\det(\rho(X)) \neq 0$ for any representation ρ affording χ. Then

$$\det(\chi) = \det(\rho(X))(F(\chi)^\times)^2.$$

Parker's conjecture

$X \in \Sigma^-(G) \Rightarrow \nu(\det(\rho(X)))$ even for all dyadic valuations ν.