Root Graded Groups

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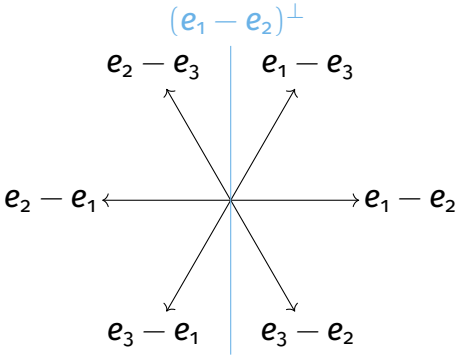
Example of a root system: A_n

$A_n := \{ e_i - e_j \mid i \neq j \in \{1, \dots, n+1\}\} \subseteq \mathbb{R}^{n+1}$ is a root system with $\dim_{\mathbb{R}} \langle A_n \rangle_{\mathbb{R}} = n$.
For $n = 2$:



For any $\alpha \in A_n$, we have a reflection $\sigma_\alpha \colon A_n \to A_n$ along

$$\alpha^\perp := \{ v \in \mathbb{R}^{n+1} \mid \alpha \cdot v = 0 \}.$$

Specifically, $\sigma_{e_i - e_j}$ acts as the transposition $\tau := (i\ j)$ on the indices: $\sigma_{e_i - e_j}(e_k) = e_{\tau(k)}$. Hence

$$\mathrm{Weyl}(A_n) := \langle \sigma_\alpha \mid \alpha \in A_n \rangle \cong \mathrm{Sym}(n+1).$$

Notation. $e_i - e_j \leftrightarrow (i, j) \leftrightarrow ij$.

Example of an A_n-graded group

Let $(R, +, *)$ be a (unital, not necessarily commutative) ring and $n \geqslant 2$.
Define for all $(i, j) \in A_n = \{ e_i - e_j \mid i \neq j \in \{1, \dots, n+1\}\}$:

$$x_{ij} = x_{e_i - e_j} \colon (R, +) \to \mathrm{GL}_{n+1}(R), a \mapsto \mathrm{id}_{n+1} + a e_{ij}, \quad U_{ij} := U_{e_i - e_j} := x_{ij}(R).$$

Put $\mathrm{E}_{n+1}(R) := \langle U_\alpha \mid \alpha \in A_n \rangle$ ($= \mathrm{SL}_{n+1}(R)$ if R is a field). Then we have:

1. *Commutator relations:* $[U_{ij}, U_{jk}] \subseteq U_{ik}$ for all pairwise distinct i, j, k. In other words, $[U_\alpha, U_\beta] \subseteq U_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$. Even more,

$$[x_{ij}(a), x_{jk}(b)] = x_{ik}(a * b) \quad \text{for all } a, b \in R.$$

2. *Existence of Weyl elements:* Let $(i, j) \in A_n$ and put $\tau := (i\ j)$. Then the element

$$w_{ij} := x_{ji}(-1_R) x_{ij}(1_R) x_{ji}(-1_R) \approx \text{“permutation matrix for } \tau\text{”}$$

satisfies $U_{k,l}^{w_{ij}} = U_{\tau(k), \tau(l)}$ for all $(k, l) \in A_n$. That is, $U_\beta^{w_\alpha} = U_{\sigma_\alpha(\beta)}$ for all $\beta \in A_n$.

Definition (Root graded group)

Let Φ be a root system and let G be a group. A *Φ-grading of G* is a family $(U_\alpha)_{\alpha\in\Phi}$ of non-trivial subgroups of G with the following properties:

1. $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.
2. *Commutator relations:* For all $\alpha, \beta \in \Phi$ such that $\alpha \neq \beta \neq -\alpha$, we have
$$[U_\alpha, U_\beta] \subseteq \langle U_{i\alpha+j\beta} \mid i, j \in \mathbb{N}_{\geqslant 1}, i\alpha + j\beta \in \Phi \rangle.$$
3. *Existence of Weyl elements:* For all $\alpha \in \Phi$, we have
$$U_\alpha^\sharp := \left\{ b_\alpha \in U_\alpha \;\middle|\; \begin{array}{c} \text{There exist } a_{-\alpha}, c_{-\alpha} \in U_{-\alpha} \text{ such that} \\ U_\beta^{a_{-\alpha} b_\alpha c_{-\alpha}} = U_{\sigma_\alpha(\beta)} \text{ for all } \beta \in \Phi \end{array} \right\} \neq \varnothing.$$
4. A non-degeneracy axiom.

If even $U_\alpha^\sharp = U_\alpha \setminus \{1_G\}$ for all $\alpha \in \Phi$, then $(G, (U_\alpha)_{\alpha\in\Phi})$ is called an *RGD-system*.

Remark. In our example, $x_\alpha(r) \in U_\alpha^\sharp$ if and only if r is invertible.

Example. Chevalley groups yield more examples of root graded (matrix) groups which are "coordinatised by a ring R", i.e. we have isomorphisms $x_\alpha\colon (R,+) \to U_\alpha$ and the *Chevalley commutator formula:*

$$[x_\alpha(a), x_\beta(b)] = \prod_{\substack{i,j\geqslant 1 \text{ s.t.} \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(\underbrace{c_{\alpha,\beta,i,j}}_{\in\mathbb{Z}} a^i b^j).$$

Goal. Show that every root grading of rank at least 3 is *coordinatised by some algebraic structure* (which depends on the root system). For RGD-systems, this is known to be true by Tits' classification of thick spherical buildings, and all involved algebraic structures are "division structures" (e.g. fields in place of rings) in this case.

Summary of results

Root system	Algebraic structure
A_2	Nonassociative rings (Moufang 1933, Faulkner 1977)
A_n $(n \geqslant 3)$	Associative rings (Shi 1991)
B_n $(n \geqslant 3)$	Quadratic modules over associative commutative rings (W.)
C_3	Alternative (W.) rings with involution (Zhang 2014)
C_n $(n \geqslant 4)$	Associative rings with involution (Zhang 2014)
D_n $(n \geqslant 4)$	Associative commutative rings (Shi 1991)
E_6, E_7, E_8	Associative commutative rings (Shi 1991)
F_4	"Composition algebras" over associative commutative rings (W.)
BC_n $(n \geqslant 3)$	Pseudo-quadratic modules over alternative rings with involution (W.)
H_3, H_4	$R \times R$ where R is an associative commutative ring (Berg, W.)

Root graded groups are defined

in a *purely combinatorial* way,

but they are always

of *algebraic origin*

(for rank at least 3)