Clifford-Weil groups for finite group rings, some examples.

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Abstract. Finite group rings carry a natural involution that defines a form ring structure. We investigate the associated Clifford-Weil groups for the indecomposable representations of the groups of order $2, 3$ and the symmetric group Sym_3 over the fields with 2 and 3 elements as well as suitable symmetrizations. An analogue of Kneser's neighboring method is introduced, to classify all self-dual codes in a given representation.

1 Introduction.

Let G be a finite group and K be a finite field. Then the group algebra KG is a finite K-algebra with a natural K-linear involution

$$^{-} : \sum_{g\in G} a_g g \mapsto \sum_{g \in G} a_g g^{-1}.$$

This defines a form ring structure $\mathcal{R}^\epsilon(KG)$ on KG where $\epsilon = \pm 1$ (see Section 2).

A finite representation ρ of $\mathcal{R}^\epsilon(KG)$ consists of a finite KG-module V together with a G-invariant non-degenerate K-bilinear form $\beta : V \times V \to K$ which is symmetric, if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$. We do not deal with Hermitian forms here, since in our examples K will be a prime field.

In this language, a self-dual code C of length N for the representation ρ (for short, a code in ρ) is a KG-submodule of V^N that is self-dual with respect to

$$\beta^N : V^N \times V^N \to K, \beta^N((x_1,\dots,x_N),(y_1,\dots,y_N)) = \sum_{i=1}^N \beta(x_i,y_i).$$

The complete weight enumerator of a code $C \leq V^N$ is

$$\mathrm{cwe}(C) := \sum_{c\in C}\prod_{i=1}^N x_{c_i} \in \mathbb{C}[x_v \mid v \in V]$$

and a homogeneous polynomial of degree N in $|V|$ variables.

In Section 2 we will give explicit generators for a finite complex matrix group, the associated Clifford-Weil group $\mathcal{C}(\rho)$ such that $\mathrm{cwe}(C)$ is invariant under all variable substitutions defined by elements in $\mathcal{C}(\rho)$, hence $\mathrm{cwe}(C) \in \mathrm{Inv}(\mathcal{C}(\rho))$ lies in the invariant ring $\mathrm{Inv}(\mathcal{C}(\rho))$.

In fact the main results of [7] (Corollary 5.7.4 and 5.7.5) show that for a fairly general class of form rings $\mathrm{Inv}(\mathcal{C}(\rho))$ is generated as a vector space over $\mathbb{C}$ by the complete weight enumerators of self-dual codes in ρ. We conjecture that this is true for arbitrary finite form rings (cf. [7, Conjecture 5.7.2]) and in particular also for $\mathcal{R}^\epsilon(KG)$, but we do not know how to prove this for arbitrary finite group rings KG.

We denote the cyclic group of order n by Z_n and the symmetric group of degree n by Sym_n. Moreover we let $\mathbb{F}_p$ be the field with p elements.

2 Rings with involution.

Rings with involution define certain form rings as explained below. We will apply the theory developed in this section to group rings with the natural involution $^-$.

Let R be a ring with 1 and

$$^J : R \to R, x \mapsto x^J$$

an involution, i.e. a ring antiautomorphism of order 1 or 2. So $(ab)^J = b^J a^J$ and $(a^J)^J = a$ for all $a, b \in R$. Moreover let $\epsilon \in Z(R)$ be a central unit of R such that $\epsilon^J \epsilon = 1$. As explained in [7, Lemma 1.4.5] this setting defines a twisted ring $(R, \mathrm{id}, M = R)$ where the twist τ on $M = R$ is defined by

$$\tau : R \to R, a \mapsto a^J \epsilon.$$

The quadrupel

$$\mathcal{R}(R, J, \epsilon) := (R, \mathrm{id}, M = R, \Phi = R/\Lambda)$$

is a form ring (see [7, Definition 1.7.1]) where

$$\Lambda := \{m - \tau(m) \mid m \in M\}$$

with mappings

$$\{\!\{\quad\}\!\} : M \to \Phi, m \mapsto m + \Lambda \text{ and } \lambda : \Phi \to M, m + \Lambda \mapsto m + \tau(m).$$

The R-qmodule structure on Φ is given by

$$(m + \Lambda)[x] = x^J m x + \Lambda \text{ for all } m \in M, x \in R.$$

A representation of the form ring $\mathcal{R}(R, J, \epsilon)$ is given by a left R-module V together with a non-degenerate biadditive form $\beta : V \times V \to A$ into some abelian group A such that

$$\beta(v, rw) = \beta(r^J v, w) \text{ and } \beta(v, w) = \beta(w, \epsilon v) \text{ for all } v, w \in V, r \in R.$$

That β is non-degenerate means that it induces an isomorphism

$$\beta^* : V \to \mathrm{Hom}(V, A), v \mapsto (w \mapsto \beta(w, v))$$

which is then an isomorphism of R-left-modules, where $\mathrm{Hom}(V, A)$ is an R-left-module by

$$(rf)(v) := f(r^J v) \text{ for all } f \in \mathrm{Hom}(V, A), r \in R, v \in V.$$

The corresponding homomorphisms

$$\rho_M : R \to \mathrm{Bil}(V, A), \rho_\Phi : \Phi \to \mathrm{Quad}_0(V, A)$$

are given by

$$\rho_M(m) : (v, w) \mapsto \beta(v, mw), \;\; \rho_\Phi(m + \Lambda) : v \mapsto \beta(v, mv).$$

Note that for all $m \in M$ and all $v \in V$

$$\beta(v, mv) = \beta(m^J v, v) = \beta(v, \epsilon m^J v) = \beta(v, \tau(m) v)$$

so ρ_Φ is well-defined.

2.1 Symmetric idempotents

An idempotent $e^2 = e \in R$ is called symmetric, if $eR \cong e^J R$ as right R-modules. In this case there are $u_e \in eRe^J$ and $v_e \in e^J Re$ such that $u_e v_e = e$ and $v_e u_e = e^J$. A set of representatives of the R^*-conjugacy classes of symmetric idempotents in R will be denoted by $\mathrm{SymId}(R)$.

2.2 The associated Clifford-Weil group

In coding theory one is mainly interested in finite alphabets V. We now assume that R is a finite dimensional algebra over a finite field K such that the restriction of ^{J} is the identity on K. For any representation $\rho = (V, \beta)$ of the form ring $\mathcal{R}(R, J, \epsilon)$ we may take the abelian group A to be the field K and $\beta^* : V \to V^* := \mathrm{Hom}_K(V, K)$. Let p be the characteristic of K and $\mathrm{trace} : K \to \mathbb{F}_p$ denote the trace from K into its prime field $\mathbb{F}_p$. Identifying $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ with $\frac{1}{p}\mathbb{Z}/\mathbb{Z} \leq \mathbb{Q}/\mathbb{Z}$ the form $\beta : V \times V \to K$ defines a biadditive form

$$\tilde{\beta} : V \times V \to \mathbb{Q}/\mathbb{Z}, \tilde{\beta}(v, w) := \frac{1}{p}\mathrm{trace}(\beta(v, w))$$

which is again non-degenerate by the non-degeneracy of the trace form.

To define the associated Clifford-Weil group $\mathcal{C}(\rho)$ we index a basis $(e_v | v \in V)$ of $\mathbb{C}^{|V|}$ by the elements of V. Then $\mathcal{C}(\rho) \leq \mathrm{GL}_{|V|}(\mathbb{C})$ is the finite complex matrix group

$$\mathcal{C}(\rho) = \langle m_r, d_\phi, h_{e,u_e,v_e} : r \in R^*, \phi \in R, e = u_e v_e \in \mathrm{SymId}(R) \rangle$$

where

$$m_r : b_v \mapsto b_{rv}, \qquad d_\phi : b_v \mapsto \exp(2\pi i \tilde{\beta}(v, \phi v)) b_v$$

and

$$h_{e,u_e,v_e} : b_v \mapsto \frac{1}{|eV|^{1/2}} \sum_{w \in eV} \exp(2\pi i \tilde{\beta}(w, v_e v)) b_{w+(1-e)v}.$$

2.3 Symmetrized weight enumerators.

Very often certain elements of V share the same property (for instance they have the same Hamming weight). Then one might be interested in the symmetrized weight enumerators of the codes rather than the complete weight enumerators. One way to obtain the ring spanned by these symmetrized weight enumerators is of course to first calculate generators of the ring of complete weight enumerators and then apply the appropriate symmetrization. Since the ring spanned by the complete weight enumerators might be rather large, it is very helpful to have shortcuts to this procedure. This is only possible, if the action of the associated Clifford-Weil group commutes with the symmetrization.

Definition 2.1. *Let $G \leq \mathrm{Sym}(V)$ be a group permuting the elements of V and $X_0, \ldots, X_n$ denote the G-orbits on V. Then the G-*symmetrized weight enumerator *$\mathrm{swe}_G(C)$ of a code $C \leq V^N$ is the homogeneous polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ of degree N,*

$$\mathrm{swe}_G(C) := \sum_{c \in C} \prod_{i=0}^{n} x_i^{a_i(c)}$$

*where $a_i(c) := |\{j \in \{1, \ldots, N\} \mid c_j \in X_i\}|$ for $0 \leq i \leq n$. The V-*Hamming weight enumerator *of C is*

$$\mathrm{hwe}_V(C) := \sum_{c \in C} x^{N - w_V(c)} y^{w_V(c)} \in \mathbb{C}[x, y]$$

where the V-weight *of* $c = (c_1, \dots, c_N) \in V^N$ *is*

$$w_V(c) := |\{i \in \{1, \dots, N\} \mid c_i \neq 0\}|.$$

There are certain symmetrizations that commute with the action of the associated Clifford-Weil group, for instance if one takes G to be a subgroup of the central unitary group of R as defined and proven below. Usually the symmetization yielding the V-Hamming weight enumerators does not commute with $\mathcal{C}(\rho)$ and one may not expect that in general the V-Hamming weight enumerators of self-dual codes in a given representation generate the invariant ring of a finite group (see the end of Section 7.2 and [7, Section 5.8] for examples).

Definition 2.2. *Let* (R, J) *be a ring with involution. Then the* central unitary group

$$\mathrm{ZU}(R, J) := \{g \in Z(R) \mid gg^J = g^J g = 1\}.$$

Theorem 2.3. *Let* $\rho := (V, \beta)$ *be a finite representation of the form ring* $\mathcal{R}(R, J)$ *and* $U \leq \mathrm{ZU}(R, J)$*. Then*

$$\rho(U) := \langle m_u \mid u \in U \rangle$$

is in the center of $\mathcal{C}(\rho)$*.*

Proof. Clearly $\rho(U) \leq \mathcal{C}(\rho)$ commutes with the generators m_r for $r \in R^*$ since U is central in R^*. For $\phi \in \Phi$, $u \in U$ and $v \in V$ we have

$$\beta(uv, \phi uv) = \beta(uv, u\phi v) = \beta(u^J uv, \phi v) = \beta(v, \phi v)$$

so m_u commutes with d_ϕ. To see that m_u commutes with the last type h_{e,u_e,v_e} of generators of $\mathcal{C}(\rho)$ one has to note that $ueV = eV$ since u is a central unit and that $\beta(uw, v_e uv) = \beta(w, v_e v)$ for all $v, w \in V, u \in U$. $\square$

Remark 2.4. *The theorem uses that* $\{\!\{\ \ \}\!\}$ *is surjective in our situation. In general one has to replace* $\mathrm{ZU}(R, J)$ *by its subgroup*

$$U_\rho = \{g \in \mathrm{ZU}(R, J) \mid \rho(\phi)(gv) = \rho(\phi(v)) \text{ for all } v \in V, \phi \in \Phi\}$$

to obtain the same theorem as above.

Corollary 2.5. *Let* $\rho := (V, \beta)$ *be a finite representation of the form ring* $\mathcal{R}(R, J)$ *and* $U \leq \mathrm{ZU}(R, J)$*. Then* U *acts as permutations on the set* V *and the corresponding symmetrization commutes with the action of* $\mathcal{C}(\rho)$*.*

In this setup we can define the U-symmetrized Clifford-Weil group,

$$\mathcal{C}^{(U)}(\rho) \leq \mathrm{GL}_{n+1}(\mathbb{C}).$$

Generators for $g^{(U)}$ the symmetrized group may be obtained from the generators g of $\mathcal{C}(\rho)$ as follows. If

$$g \sum_{v \in X_i} e_v = \sum_{j=0}^{n} a_{ij} (\sum_{w \in X_j} e_w)$$

then

$$g^{(U)}(x_i) = \sum_{j=0}^{n} a_{ij} \frac{|X_j|}{|X_i|} x_j.$$

Of course $\rho(U)$ is in the kernel of this symmetrization $\mathcal{C}(\rho) \to \mathcal{C}^{(U)}(\rho)$.

Remark 2.6. *The invariant ring of $\mathcal{C}^{(U)}(\rho)$ consists of the U-symmetrized invariants of $\mathcal{C}(\rho)$. In particular, if the invariant ring of $\mathcal{C}(\rho)$ is spanned by the complete weight enumerators of self-dual codes in ρ, then the invariant ring of $\mathcal{C}^{(U)}(\rho)$ is spanned by the U-symmetrized weight-enumerators of self-dual codes in ρ.*

2.4 Form group rings

Let G be a finite group and K be a finite field. Then the group algebra KG is a finite K-algebra with a natural K-linear involution

$$\overline{} : \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g g^{-1}.$$

Since $\epsilon = 1$ and $\epsilon = -1$ are central units in KG, the construction of Section 2 defines a natural form ring structure $\mathcal{R}^\epsilon(KG)$ on KG where $\epsilon = \pm 1$.

A representation of the form ring $\mathcal{R}^\epsilon(KG)$ is given by a finite KG-module V together with a G-invariant non-degenerate K-bilinear form $\beta : V \times V \to K$ which is symmetric, if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$.

3 A method to enumerate all self-dual codes.

There is a very nice and efficient method to enumerate all self-dual codes in a given length representation of a form ring. This is based on M. Kneser's ideas [4], described in [6] for codes over finite fields and in [5] for $\mathbb{Z}G$-lattices. We often apply it to find self-dual codes in representations of the finite form ring $\mathcal{R}(KG)$ and therefore we will describe it in a fairly general setting adopted to this situation.

Let $(V, \rho_M, \rho_\Phi, \beta)$ be a finite representation of a form ring (R, M, ψ, Φ) as defined in [7, Section 1]. In particular V is a finite left-module for the ring R and $\beta : V \times V \to \mathbb{Q}/\mathbb{Z}$ a non-degenerate form on V which induces an R-module isomorphism

$$\beta^* : V \to V^* := \mathrm{Hom}(V, \mathbb{Q}/\mathbb{Z}), w \mapsto (v \mapsto \beta(v, w)).$$

A self-dual code C in ρ is a R-submodule $C \leq V$ such that

$$C = C^\perp := \{v \in V \mid \beta(c, v) = 0 \text{ for all } c \in C\}.$$

Let

$$\mathcal{M}(V) := \{C \leq V \mid C = C^\perp\}$$

denote the set of all self-dual codes in V.

Lemma 3.1. *Let $C \in \mathcal{M}(V)$ and*

$$\star \quad \{0\} = V_0 < V_1 < \ldots < V_s = C < V_{s+1} < \ldots < V_t = V$$

be a composition series of V with simple R-left-module $S_i := V_i/V_{i-1}$ $(1 \leq i \leq t)$. Then $t = 2s$ and there is a bijection $\pi : \{1, \ldots, s\} \to \{s+1, \ldots, t\}$ such that $(S_i)^* = S_{\pi(i)}$.

Proof. The mapping $\beta^* : V \to \mathrm{Hom}(C, \mathbb{Q}/\mathbb{Z})$, $v \mapsto (c \mapsto \beta(v,c))$ is an epimorphism with kernel $C^\perp$. Hence $V/C = V/C^\perp \cong \mathrm{Hom}(C, \mathbb{Q}/\mathbb{Z}) = C^*$. Now the lemma follows since the composition factors of C^* are the dual $S^* = \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z})$ of the composition factors S of C.
Alternatively one may choose $V_{t-i} = V_i^\perp$ in the composition series $\star$. Then

$$V_{t-i}/V_{t-i-1} = V_i^\perp / V_{i+1}^\perp \cong (V_{i+1}/V_i)^*$$

and the lemma follows from the Jordan-Hölder theorem on the uniqueness of composition factors. □

Corollary 3.2. *If $\mathcal{M}(V) \neq \emptyset$ then each simple composition factor S of V that is isomorphic to its dual, $S \cong S^*$, occurs with even multiplicity in every composition series of V.*

Corollary 3.3. *Any two modules $C, D \in \mathcal{M}(V)$ have the same composition lengths: $\ell(C) = \ell(D) = s = \ell(V)/2$.*

Definition 3.4. *Two self-dual codes $C, D \in \mathcal{M}(V)$ are called* neighbors*, if the R-module $C/C \cap D$ is simple. The* neighbor-graph *is the graph Γ with vertex set $\mathcal{M}(V)$. Two vertices $C, D \in \mathcal{M}(V)$ are connected, if C and D are neighbors.*

Theorem 3.5. *The neighbor graph Γ is connected.*

Proof. We define a distance on the set $\mathcal{M}(V)$. For $C, D \in \mathcal{M}(V)$ let

$$d(C,D) := \ell(C/(C \cap D))$$

be the number of composition factors of the factor module $C/(C \cap D)$. Then clearly $d(C,D) = 0$ if and only if $C = D$ and $d(C,D) = d(D,C)$ by Corollary 3.3 and Jordan-Hölder. Also the triangle inequality follows easily from the fact that the number of composition factors is well defined. Clearly $d(C,D) \leq \ell(C) = s$ for all $C, D \in \mathcal{M}(V)$
We claim that this distance $d(C,D)$ is the number of edges in any shortest path in Γ connecting C and D, which shows that the diameter of Γ is bounded from above by s and in particular that Γ is connected.
To prove this claim we proceed by induction on $n := d(C,D)$. For $n = 0$ and $n = 1$ the claim is true by definition. Now assume that $n \geq 2$. Then we construct a code $C_1 \in \mathcal{M}(V)$ such that

$$d(C, C_1) = 1 \text{ and } d(C_1, D) = n - 1.$$

To this aim let $U := C \cap D$ and choose $D > U_1 > U$ such that $U_1/U \cong S$ is simple. This is possible since the composition length $n = \ell(D/U) \geq 2$. Then $U = U_1 \cap C$ and

$$S \cong U_1/(U_1 \cap C) \cong (U_1 + C)/C.$$

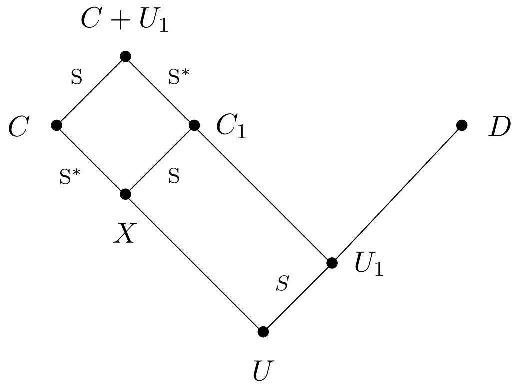
The module $X := (U_1 + C)^\perp < C = C^\perp$ is a submodule of C with $C/X \cong S^*$. Put

$$C_1 := X + U_1 = (U_1 + C)^\perp + U_1.$$

Then

$$C_1^\perp = (U_1 + C) \cap (U_1^\perp) \supseteq X + U_1 = C_1$$

since $U_1 \subseteq D = D^\perp \subseteq U_1^\perp$. Comparing the composition lengths we get $C_1^\perp = C_1 \in \mathcal{M}(V)$. Clearly $d(C, C_1) = 1$. Moreover $C_1 \cap D = U_1$ and hence $d(C_1, D) = n - 1$.



□

This provides an algorithm to enumerate all elements of $\mathcal{M}(V)$. Start with some self-dual code $C \in \mathcal{M}(V)$. For all composition factors S of V calculate all non-zero R-homomorphisms $\varphi : C \to S$. Their kernels $U := \ker(\varphi)$ provide all submodules $U \leq C$ such that $C/U \cong S$. The neighbors D of C such that $D \cap C = U$ can be obtained as full preimages of the self-dual submodules D/U of $U^\perp/U$ (not equal to C/U). Continue with all neighbors until all codes in $\mathcal{M}(V)$ have been found. Usually one is only interested in representatives of equivalence classes of codes in $\mathcal{M}(V)$, so there is a certain group G acting on $\mathcal{M}(V)$ that preserves submodules and duality. Then it is enough to work with representatives of the G-orbits. More details can be found in [3].

4 $\mathbb{F}_2 Z_2$

The Type of singly even self-dual codes over $\mathbb{F}_2 Z_2$ is one of the rare cases for which the invariant ring of the associated Clifford-Weil group is a polynomial ring. The Type of doubly even self-dual codes over $\mathbb{F}_2 Z_2$ is interesting because of the connection to Type IV codes over $\mathbb{Z}_4$. The Gray image of a Type IV code over $\mathbb{Z}_4$ is a doubly even $\mathbb{F}_2 Z_2$-linear self-dual code (see [1], [2]) However not all such codes are Gray images of a Type IV code over $\mathbb{Z}_4$.

Let $Z_2 = \langle a \rangle$. Then $\mathbb{F}_2 Z_2 \cong \mathbb{F}_2[x]/(x^2)$ via $(1 + a) \mapsto x$. In particular the unit group $(\mathbb{F}_2 Z_2)^* = \langle a \rangle \cong Z_2$ and $\mathbb{F}_2 Z_2$ has just two indecomposable modules, the simple module $S = \mathbb{F}_2$ and the projective module $P = \mathbb{F}_2 Z_2$. The representation ρ_S with underlying module S defines $\mathcal{C}(\rho_S) = \mathcal{C}(2_I)$ the Clifford-Weil group associated to the Type of singly even binary self-dual codes which is treated in detail in [7, Section 6.3].

Z_2 acts on the module $P \cong \mathbb{F}_2^2$ via

$$a \mapsto \rho_P(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the two non-degenerate a-invariant bilinear forms (with Gram matrices I_2 and $\rho_P(a)$) are in the same orbit under $(\mathbb{F}_2 Z_2)^*$ and hence define the same notion of duality. We choose β to be the standard form with Gram matrix I_2. Then with respect to the basis

$$e_{(0,0)}, e_{(1,0)}, e_{(0,1)}, e_{(1,1)}$$

of $\mathbb{C}[P]$ the associated Clifford-Weil group $\mathcal{C}(\mathbb{F}_2 Z_2)$ is generated by

$$m_a := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ d_\phi := \mathrm{diag}(1,-1,-1,1), \ h_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

($\phi = \{\!\{ \beta \}\!\}$) has order 16 and is isomorphic to $D_8 \times Z_2$, the direct product of the dihedral group of order 8 and the cyclic group of order 2. $\mathcal{C}(\mathbb{F}_2 Z_2)$ is a real reflection group and the invariant ring is the polynomial ring

$$\mathrm{Inv}(\mathcal{C}(\mathbb{F}_2 Z_2)) = \mathbb{C}[p_1, p_2, p_3, p_4]$$

with

$$p_1 = x+t, \ p_2 = x^2+y^2+z^2+t^2, \ p_3 = x^2+2yz+t^2, \ p_4 = x^4+y^4+z^4+t^4+8xyzt+2x^2t^2+2y^2z^2$$

where we put $x = x_{(0,0)}, y = x_{(1,0)}, z = x_{(0,1)}, t = x_{(1,1)}$ for simplicity. These polynomials are the complete weight enumerators of the codes $C_i \leq P^N$ with generator matrices

$$[(1,1)], \ \begin{bmatrix} (1,0) & (1,0) \\ (0,1) & (0,1) \end{bmatrix}, \ \begin{bmatrix} (1,1) & (1,1) \\ (1,0) & (0,1) \end{bmatrix},$$

and

$$\begin{bmatrix} (1,0) & (0,0) & (0,1) & (1,1) \\ (0,1) & (0,0) & (1,0) & (1,1) \\ (0,0) & (1,0) & (1,1) & (0,1) \\ (0,0) & (0,1) & (1,1) & (1,0) \end{bmatrix}.$$

For the module theoretic structure we get

$$C_1 \cong S, C_2 \cong C_3 \cong P, C_4 \cong P \oplus P$$

(as $\mathbb{F}_2 Z_2$-modules). As binary codes, C_2, C_3 and $C_1 \perp C_1$ are equivalent, and C_4 is equivalent to the extended Hamming code e_8 of length 8.

To obtain the type of doubly even binary codes in P^N, we may enlarge Φ and obtain one additional generator $d_\varphi := \mathrm{diag}(1, i, i, -1)$, with $i \in \mathbb{C}, i^2 = -1$. The group

$$\mathcal{C}_{\mathrm{II}}(\mathbb{F}_2 Z_2) = \langle \mathcal{C}(\mathbb{F}_2 Z_2), d_\varphi \rangle$$

has order 192 and Molien series

$$\frac{1 + \lambda^4 + 2\lambda^8}{(\lambda^4 - 1)^3 (\lambda^{12} - 1)}.$$

The invariant ring $\mathrm{Inv}(\mathcal{C}_{\mathrm{II}}(\mathbb{F}_2 Z_2))$ is a free module over the polynomial subring $R := \mathbb{C}[p_4, p_5, p_6, p_7]$,

$$\mathrm{Inv}(\mathcal{C}_{\mathrm{II}}(\mathbb{F}_2 Z_2)) = R \oplus Rq_1 \oplus Rq_2 \oplus Rq_3$$

where p_4 is as above, p_5, p_6, q_1 are complete weight enumerators of further Z_2-structures of e_8, $q_2 = \mathrm{cwe}(e_8 \otimes P)$, q_3 is the complete weight enumerators of a suitable Z_2-structure on d_{16}^+ and p_7 is the weight enumerator of any Z_2-structure of the Golay code.

To find the inequivalent doubly even codes in P^4 that are equivalent to e_8 as binary codes, we consider the automorphism group $G = \mathrm{Aut}(e_8)$. There are 2 conjugacy classes of elements of order 2 in G which are conjugate to $a = (1,2)(3,4)(5,6)(7,8)$ in Sym_8.

The a-invariant codes C_k have generator matrices (I_4, J_k) with $k = 1, \ldots, 6$, where I_4 is the 4×4 unit matrix viewed as element of $P^{4\times 2}$ and

$$J_1 = \begin{bmatrix} (0,1) & (1,1) \\ (1,0) & (1,1) \\ (1,1) & (0,1) \\ (1,1) & (1,0) \end{bmatrix}, \; J_2 = \begin{bmatrix} (1,0) & (1,1) \\ (0,1) & (1,1) \\ (1,1) & (0,1) \\ (1,1) & (1,0) \end{bmatrix}, \; J_3 = \begin{bmatrix} (1,0) & (1,1) \\ (0,1) & (1,1) \\ (1,1) & (1,0) \\ (1,1) & (0,1) \end{bmatrix},$$

$$J_4 = \begin{bmatrix} (1,1) & (1,0) \\ (0,1) & (1,1) \\ (1,1) & (0,1) \\ (1,0) & (1,1) \end{bmatrix}, \; J_5 = \begin{bmatrix} (0,1) & (1,1) \\ (1,1) & (1,0) \\ (1,1) & (0,1) \\ (0,1) & (1,1) \end{bmatrix}, \; J_6 = \begin{bmatrix} (1,1) & (1,0) \\ (1,0) & (1,1) \\ (1,1) & (0,1) \\ (0,1) & (1,1) \end{bmatrix}.$$

with complete weight enumerators

$$\begin{array}{ll}
\mathrm{cwe}(C_1) = p_4 = & x^4 + 2x^2t^2 + 8xyzt + y^4 + 2y^2z^2 + z^4 + t^4 \\
\mathrm{cwe}(C_2) = p_5 = & x^4 + 2x^2t^2 + 2xy^2t + 4xyzt + 2xz^2t + 2y^3z + 2yz^3 + t^4 \\
\mathrm{cwe}(C_3) = & x^4 + 2x^2t^2 + 4xy^2t + 4xz^2t + 4y^2z^2 + t^4 \\
\mathrm{cwe}(C_4) = & x^4 + 3xy^2t + 6xyzt + 3xz^2t + y^3z + yz^3 + t^4 \\
\mathrm{cwe}(C_5) = p_6 = & x^4 + 12xyzt + y^4 + z^4 + t^4 \\
\mathrm{cwe}(C_6) = q_1 = & x^4 + 4xy^2t + 4xyzt + 4xz^2t + 2y^2z^2 + t^4
\end{array}$$

For the secondary invariants of degree 8 one may take

$$q_2 := \mathrm{cwe}(e_8 \otimes P) = x^8 + y^8 + z^8 + t^8 + 14(x^4y^4 + x^4z^4 + x^4t^4 + y^4z^4 + y^4t^4 + z^4t^4) + 168x^2y^2z^2t^2$$

(where Z_2 acts trivially on e_8) and the weight enumerator of a Z_2-structure of the indecomposable Type II code d_{16}^+ of length 16,

$$\begin{array}{l}
q_3 = \mathrm{cwe}(d_{16}^+) = x^8 + 4x^6t^2 + 2x^5y^2t + 8x^5yzt + 2x^5z^2t + 4x^4y^3z + 4x^4y^2z^2 + 4x^4yz^3 + \\
6x^4t^4 + 4x^3y^2t^3 + 32x^3yzt^3 + 4x^3z^2t^3 + 8x^2y^4t^2 + 16x^2y^3zt^2 + 24x^2y^2z^2t^2 + 16x^2yz^3t^2 + \\
8x^2z^4t^2 + 4x^2t^6 + 4xy^5zt + 24xy^4z^2t + 8xy^3z^3t + 24xy^2z^4t + 2xy^2t^5 + 4xyz^5t + \\
8xyzt^5 + 2xz^2t^5 + 2y^7z + 6y^5z^3 + 6y^3z^5 + 4y^3zt^4 + 4y^2z^2t^4 + 2yz^7 + 4yz^3t^4 + t^8.
\end{array}$$

A corresponding generator matrix is

$$\begin{bmatrix}
(1,0) & (0,0) & (0,1) & (0,0) & (0,0) & (0,1) & (0,0) & (0,1) \\
(0,1) & (0,0) & (1,0) & (0,0) & (0,0) & (0,1) & (0,0) & (0,1) \\
(0,0) & (1,0) & (1,1) & (0,1) & (0,0) & (0,0) & (1,1) & (1,1) \\
(0,0) & (0,1) & (1,1) & (0,1) & (0,0) & (0,1) & (1,1) & (1,0) \\
(0,0) & (0,0) & (0,0) & (1,1) & (0,0) & (0,1) & (0,0) & (0,1) \\
(0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,1) & (0,1) & (0,1) \\
(0,0) & (0,0) & (0,0) & (0,0) & (0,1) & (0,1) & (1,0) & (0,1) \\
(0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,1) & (0,0) & (1,1)
\end{bmatrix}$$

The automorphism group of the extended binary Golay code $\mathcal{G}_{24}$ has one conjugacy class of elements that are conjugate in Sym_{24} to

$$a = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24)$$

yielding an $\mathbb{F}_2 Z_2$-structure of $\mathcal{G}_{24}$ with generator matrix (I_{12}, J) where

$$J := \begin{bmatrix} (1,0) & (1,1) & (1,0) & (1,1) & (0,0) & (0,1) \\ (0,1) & (1,1) & (0,1) & (1,1) & (0,0) & (1,0) \\ (1,1) & (1,0) & (0,1) & (0,0) & (1,1) & (1,0) \\ (1,1) & (0,1) & (1,0) & (0,0) & (1,1) & (0,1) \\ (0,0) & (1,1) & (1,0) & (1,0) & (1,1) & (1,0) \\ (0,0) & (1,1) & (0,1) & (0,1) & (1,1) & (0,1) \\ (1,0) & (0,1) & (1,1) & (0,1) & (1,0) & (1,0) \\ (0,1) & (1,0) & (1,1) & (1,0) & (0,1) & (0,1) \\ (0,1) & (1,0) & (1,0) & (0,1) & (1,0) & (1,1) \\ (1,0) & (0,1) & (0,1) & (1,0) & (0,1) & (1,1) \\ (1,1) & (0,0) & (0,1) & (1,1) & (1,0) & (0,1) \\ (1,1) & (0,0) & (1,0) & (1,1) & (0,1) & (1,0) \end{bmatrix}$$

whose complete weight enumerator yields the last generator

$$p_7 = x^{12} + 15x^8t^4 + 14x^6y^4t^2 + 64x^6y^3zt^2 + 84x^6y^2z^2t^2 + 64x^6yz^3t^2 + 14x^6z^4t^2 + 32x^6t^6 + 4x^5y^6t+ 40x^5y^5zt + 92x^5y^4z^2t + 112x^5y^3z^3t + 92x^5y^2z^4t + 40x^5yz^5t + 4x^5z^6t + x^4y^8 + 4x^4y^7z + 10x^4y^6z^2+ 28x^4y^5z^3 + 34x^4y^4z^4 + 28x^4y^4t^4 + 28x^4y^3z^5 + 128x^4y^3zt^4 + 10x^4y^2z^6 + 168x^4y^2z^2t^4 + 4x^4yz^7+ 128x^4yz^3t^4 + x^4z^8 + 28x^4z^4t^4 + 15x^4t^8 + 24x^3y^6t^3 + 112x^3y^5zt^3 + 296x^3y^4z^2t^3 + 416x^3y^3z^3t^3+ 296x^3y^2z^4t^3 + 112x^3yz^5t^3 + 24x^3z^6t^3 + 2x^2y^8t^2 + 24x^2y^7zt^2 + 76x^2y^6z^2t^2 + 168x^2y^5z^3t^2+ 180x^2y^4z^4t^2 + 14x^2y^4t^6 + 168x^2y^3z^5t^2 + 64x^2y^3zt^6 + 76x^2y^2z^6t^2 + 84x^2y^2z^2t^6+ 24x^2yz^7t^2 + 64x^2yz^3t^6 + 2x^2z^8t^2 + 14x^2z^4t^6 + 4xy^6t^5 + 40xy^5zt^5 + 92xy^4z^2t^5 + 112xy^3z^3t^5+ 92xy^2z^4t^5 + 40xyz^5t^5 + 4xz^6t^5 + 2y^10z^2 + 16y^8z^4 + y^8t^4 + 4y^7zt^4 + 28y^6z^6 + 10y^6z^2t^4 + 28y^5z^3t^4+ 16y^4z^8 + 34y^4z^4t^4 + 28y^3z^5t^4 + 2y^2z^10 + 10y^2z^6t^4 + 4yz^7t^4 + z^8t^4 + t^{12}.$$

5 $\mathbb{F}_2 \operatorname{Sym}_3$

The group ring $\mathbb{F}_2 \operatorname{Sym}_3 = \mathbb{F}_2 Z_2 \oplus \mathbb{F}_2^{2\times 2}$ is the direct product of two blocks that are invariant under the canonical involution. The first block is already dealt with in Section 4. For the second block, we should note that the left modules of the matrix ring $R = \mathbb{F}_2^{2\times 2}$ are of the form $M = \mathbb{F}_2^{2\times 1} \otimes V$ for some $\mathbb{F}_2$-vector space V. The self-dual R-submodules of M are of the form $\mathbb{F}_2^{2\times 1} \otimes C = C(2)$ for a self-dual binary code $C \leq V$. The associated Clifford-Weil group is the real Clifford group $\mathcal{C}_2(2_I)$ of genus 2 (see [7, Section 6.3]) of which the invariant ring is spanned by the genus 2 complete weight enumerators of the self-dual binary codes.

6 $\mathbb{F}_3 \operatorname{Sym}_3$

$\mathbb{F}_3 \operatorname{Sym}_3$ has 6 indecomposable modules:

$$S_+, S_-, V_+, V_- = V_+ \otimes S_-, P_+, P_- = P_+ \otimes S_-$$

where S_+ and S_- are the two simple modules (with trivial character, respectively the signum character), P_+ and P_- the two corresponding projective indecomposable modules, P_+ is just the natural permutation module of the symmetric group Sym_3, and $V_+ = P_+/\operatorname{soc}(P_+)$, $V_- = P_-/\operatorname{soc}(P_-)$ are the two indecomposables with composition length 2. Since $V_- \cong \operatorname{Hom}_{\mathbb{F}_3}(V_+, \mathbb{F}_3)$, both modules V_+ and V_- do not carry a Sym_3-invariant non-degenerate bilinear form. $\mathbb{F}_3 \operatorname{Sym}_3$ acts on the simple modules S_+ and S_- just as $\mathbb{F}_3$, so the self-dual codes in S_+^N and S_-^N are the self-dual ternary codes of length N. The corresponding Clifford-Weil group is described in [7, Section

7.4.1]. The self-dual codes in P_+^N are the same as the ones in P_-^N, so it is enough to consider the representation ρ_{P_+}. The projective indecomposable Sym_3-module P_+ is uniserial,

$$P_+ > J(P_+) > \mathrm{soc}(P_+) > 0$$

with composition factors (S_+, S_-, S_+). The Clifford-Weil group $\mathcal{C}(P_+) \leq \mathrm{GL}_{27}(\mathbb{C})$ has order $2^8 3^9$. Its invariant ring is far from being a polynomial ring. The Molien series starts with

$$1 + 5\lambda^4 + 40\lambda^8 + 2321\lambda^{12} + 140997\lambda^{16} + \ldots = f(\lambda)/N(\lambda)$$

with

$$N(\lambda) = (1-\lambda^4)^5(1-\lambda^8)^4(1-\lambda^{12})^{12}(1-\lambda^{36})^6$$

and a positive polynomial f of degree 376 with $f(1) > 10^{22}$. So it is hopeless to calculate the full invariant ring here. The 5 invariants of degree 4 are provided by the complete weight enumerators $p_1, \ldots, p_5$ of the codes $C_1, \ldots, C_5$ with generator matrices

$$\begin{bmatrix} (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,1,1) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) \\ (0,1,2) & (0,0,0) & (0,1,2) & (0,1,2) \\ (0,0,0) & (0,1,2) & (0,1,2) & (0,2,1) \end{bmatrix}, \quad \begin{bmatrix} (1,1,1) & (1,1,1) & (1,1,1) & (1,1,1) \\ (1,1,1) & (0,0,0) & (0,0,0) & (2,2,2) \\ (0,0,0) & (1,1,1) & (0,0,0) & (2,2,2) \\ (1,0,0) & (1,0,0) & (0,0,0) & (1,0,0) \\ (0,1,2) & (0,0,0) & (0,1,2) & (0,2,1) \\ (0,0,0) & (0,1,2) & (0,2,1) & (0,2,1) \end{bmatrix},$$

$$\begin{bmatrix} (1,0,0) & (2,0,0) & (0,0,0) & (2,0,0) \\ (0,1,0) & (0,2,0) & (0,0,0) & (0,2,0) \\ (0,0,1) & (0,0,2) & (0,0,0) & (0,0,2) \\ (0,0,0) & (1,0,0) & (2,0,0) & (2,0,0) \\ (0,0,0) & (0,1,0) & (0,2,0) & (0,2,0) \\ (0,0,0) & (0,0,1) & (0,0,2) & (0,0,2) \end{bmatrix}, \quad \begin{bmatrix} (1,0,0) & (2,0,0) & (0,0,0) & (2,0,0) \\ (0,1,0) & (0,2,0) & (0,0,0) & (0,2,0) \\ (0,0,1) & (0,0,2) & (0,0,0) & (0,0,2) \\ (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) \\ (1,2,0) & (0,0,0) & (0,2,1) & (1,2,0) \\ (0,1,2) & (0,0,0) & (0,2,1) & (0,1,2) \end{bmatrix},$$

$$\begin{bmatrix} (1,0,0) & (1,0,0) & (1,1,1) & (1,0,0) \\ (0,1,0) & (0,1,0) & (1,1,1) & (0,1,0) \\ (0,0,1) & (0,0,1) & (1,1,1) & (0,0,1) \\ (1,0,0) & (2,0,0) & (2,0,0) & (1,1,1) \\ (0,1,0) & (0,2,0) & (0,2,0) & (1,1,1) \\ (0,0,1) & (0,0,2) & (0,0,2) & (1,1,1) \end{bmatrix}.$$

Imposing the additional condition that the codes contain the all-ones vector $\mathbf{1}$, one gets a Clifford-Weil group of order $2^8 3^{11}$ with Molien series

$$1 + 2\lambda^4 + 10\lambda^8 + 403\lambda^{12} + 16200\lambda^{16} + \ldots = g(\lambda)/N_1(\lambda)$$

with

$$N_1(\lambda) = (1-\lambda^4)^2(1-\lambda^8)^7(1-\lambda^{12})^{12}(1-\lambda^{36})^6$$

and a positive polynomial g of degree 388 with $g(1) > 10^{22}$. The two invariants of degree 4 are p_1 and p_2.

7 $\mathbb{F}_3 Z_3$

$\mathbb{F}_3 Z_3$ has 3 indecomposable modules: the simple module $S \cong \mathbb{F}_3$, the projective module $P \cong \mathbb{F}_3 Z_3$ and $P/\operatorname{soc}(P) = V$ of composition length 2.

7.1 The 3-dimensional module P

The module P is just the restriction of the $\mathbb{F}_3 \operatorname{Sym}_3$-module P_+ to Z_3. The associated Clifford-Weil group $\mathcal{C}(P)$ has order $2^5 3^5$ and Molien series starting with

$$1 + 37\lambda^4 + 9294\lambda^8 + \ldots .$$

The additional condition that the codes contain the all-ones vector yields a Clifford-Weil group of order $2^5 3^7$ whose Molien series starts with

$$1 + 6\lambda^4 + 911\lambda^8 + 148842\lambda^{12} + \ldots .$$

A system of representatives for the Sym_4-equivalence classes of self-dual $\mathbb{F}_3 Z_3$-codes in P^4 may be calculated as follows.

An $\mathbb{F}_3 Z_3$-code C in P^4 is a self-dual code in $\mathbb{F}_3^{12}$, with the additional property that $a := (1,2,3)(4,5,6)(7,8,9)(10,11,12)$ is contained in the permutation group $P(C)$ of C.

Up to monomial equivalence, there exist three self-dual codes in $\mathbb{F}_3^{12}$. Hence for each of these three codes D we have to determine the set $\mathcal{G}_D := \{\pi \in \mathcal{M}on \mid a \in P(D\pi)\}$, where $\mathcal{M}on$ is the group of monomial permutations on twelve points. Since the condition $a \in P(D\pi)$ is equivalent with $\pi a \pi^{-1} \in \operatorname{Aut}(D)$, the set $\mathcal{G}_D$ can be determined with elementary calculations. Now $\mathcal{G}_D$ consists of right cosets of the subgroup

$$< (1,4)(2,5)(3,6), (1,4,7,10)(2,5,8,11)(3,6,9,12) > \cong \operatorname{Sym}_4$$

in $\mathcal{M}on$, hence may be reduced to a set of coset representatives. The union of the reduced sets $\mathcal{G}_D$ then yields a system of representatives for the Sym_4-equivalence classes of self-dual $\mathbb{F}_3 Z_3$-codes in P^4, consisting of 48 codes.

Since it is hopeless to calculate generators for the invariant ring here, it is useful to apply the strategy described in Section 2.3 to obtain generators for the ring spanned by the U-symmetrized weight enumerators of the codes, where $U \cong Z_6$ is the full central unitary group of $(\mathbb{F}_3 Z_3, ^{-})$. U preserves the composition series

$$P > V > S > 0$$

and has 3 orbits X_3, X_4, X_5 of length 6 on $P - V$ (distinguished by their Hamming weight) one orbit X_2 of length 6 on $V - S$, one orbit X_1 on $S - \{0\}$ and the orbit $X_0 = \{0\}$. The symmetrized Clifford-Weil group $\mathcal{C}^{(U)}(P)$ has order $2^4 3^4$ and Molien series starting with

$$1 + 3\lambda^4 + 9\lambda^8 + 34\lambda^{12} + \ldots = \frac{f}{g}$$

with

$$g(\lambda) = (1-\lambda^{36})(1-\lambda^{12})^2(1-\lambda^4)^3$$

and

$$f(\lambda) = \lambda^{60} + 5\lambda^{56} + 17\lambda^{52} + 18\lambda^{48} + 25\lambda^{44} + 25\lambda^{40} + 32\lambda^{36} + 26\lambda^{32} + 27\lambda^{28} + 31\lambda^{24} + 21\lambda^{20} + 11\lambda^{16} + 13\lambda^{12} + 3\lambda^8 + 1.$$

The 48 codes of length 4 yield four different symmetrized weight enumerators which generate the 3-dimensional space of invariants of degree 4.

$$\begin{array}{l}
x_0^4 + 72x_4x_3x_5(x_0+x_1) + 24(x_0x_2^3 + x_1(2x_2^3 + x_3^3 + x_4^3 + x_5^3)) + 144x_2(x_3x_4^2 + x_3^2x_5 + x_4x_5^2) + 8x_0x_1^3, \\
x_0^4 + 24x_0(x_4^3 + x_3^3 + x_2^3 + x_5^3) + 144(x_2x_3x_4^2 + x_1x_3x_4x_5 + x_2x_4x_5^2 + x_2x_3^2x_5) + 8x_0x_1^3 + 48x_1x_2^3, \\
x_0^4 + 8x_0^3x_1 + 24x_0^2x_1^2 + 216x_0x_2^3 + 32x_0x_1^3 + 432x_2^3x_1 + 16x_1^4, \\
x_0^4 + 2x_0^3x_1 + 6x_0(x_0x_1^2 + x_4^3 + x_3^3 + x_5^3) + +36(x_0+2x_1)(x_3x_4x_5 + 2x_2^3) + \\
12x_1(x_3^3 + x_4^3 + x_5^3) + 108x_2(x_3x_4^2 + x_4x_5^2 + x_3^2x_5) + 14x_0x_1^3 + 4x_1^4.
\end{array}$$

7.2 The 2-dimensional module V

The 2-dimensional indecomposable $\mathbb{F}_3Z_3$-module V has an $\mathbb{F}_3$-basis with respect to which a acts as

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

and an A-invariant bilinear form with Gram matrix

$$F = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

There are no symmetric non-degenerated invariant forms on V, so here we need to work with $\mathcal{R}^-(\mathbb{F}_3Z_3)$ and $\epsilon = -1$. The associated Clifford-Weil group is isomorphic to $Z_2 \times Z_3 \times Z_3 \times \mathrm{Sym}_3$ of order 108. The Molien series is

$$d(\lambda)/n(\lambda) = 1 + \lambda + \lambda^2 + 7\lambda^3 + 11\lambda^4 + 11\lambda^5 + 49\lambda^6 + 91\lambda^7 + \ldots$$

with denominator

$$n(\lambda) = (1-\lambda)(1-\lambda^3)^4(1-\lambda^6)^4$$

and numerator

$$\begin{array}{l}
d(\lambda) = 2\lambda^{25} + 4\lambda^{24} + 18\lambda^{22} + 22\lambda^{21} + 16\lambda^{20} + 43\lambda^{19} + 65\lambda^{18} + 89\lambda^{17} + \\
83\lambda^{16} + 91\lambda^{15} + 123\lambda^{14} + 89\lambda^{13} + 78\lambda^{12} + 71\lambda^{11} + 59\lambda^{10} + \\
45\lambda^9 + 25\lambda^8 + 26\lambda^7 + 16\lambda^6 + 4\lambda^4 + 2\lambda^3 + 1
\end{array}$$

The invariant of degree 1 is of course the weight enumerator p of the code $C_1 := C = \langle (1,1) \rangle \leq V$. There are 13 self-dual codes in V^3, one of which is C^3. The other yield 6 different weight enumerators providing in total seven invariants of degree 3, that are linearly independent. Generator matrices $(I_3|J_i)$ of 6 such codes with distinct weight enumerators are as follows:

$$J_1 = \begin{bmatrix} 2) & (0,2) \\ 1) & (2,0) \\ 1) & (2,2) \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1) & (0,1) \\ 2) & (1,0) \\ 1) & (2,2) \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1) & (2,0) \\ 2) & (2,1) \\ 1) & (2,2 \end{bmatrix},$$

$$J_4 = \begin{bmatrix} 2) & (1,2) \\ 1) & (0,2) \\ 1) & (1,1) \end{bmatrix}, \quad J_5 = \begin{bmatrix} 2) & (2,1) \\ 1) & (0,1) \\ 1) & (2,2) \end{bmatrix}, \quad J_6 = \begin{bmatrix} 2) & (1,0) \\ 1) & (1,2) \\ 1) & (2,2) \end{bmatrix}.$$

The submodule structure of V is $V > S > 0$ with $S = \langle (1,1) \rangle$. So $V = X_0 \cup X_1 \cup X_2$ with $X_0 = \{(0,0)\}$, $X_1 = V - S = \{(1,0),(2,0),(0,1),(0,2),(1,2),(2,1)\}$, $X_2 = S - \{(0,0)\} = \{(1,1),(2,2)\}$. This partition of V is the set of orbits of the central unitary group of the group ring and hence the corresponding symmetrization commutes with the action of the Clifford-Weil group. The resulting symmetrized Clifford-Weil group is generated by

$$d := \mathrm{diag}(1, \zeta_3, 1) \text{ and } h = \begin{pmatrix} 1/3 & 2 & 2/3 \\ 1/3 & 0 & -1/3 \\ 1/3 & -1 & 2/3 \end{pmatrix}$$

has order 18 and is isomorphic to the complex reflection group $G = Z_3 \times \mathrm{Sym}_3$. All 12 codes of length 3 that are $\neq C^3$ have the same symmetrized weight enumerator

$$p_3 := x_0^3 + 6x_0x_2^2 + 18x_1^3 + 2x_2^3.$$

The invariant ring of G is the polynomial ring $\mathbb{C}[p_1, p_3, p_6]$, where $p_1 := x_0 + 2x_2$ is the symmetrized weight enumerator of C and

$$p_6 = x_0^6 + 30x_0^4x_2^2 + 40x_0^3x_2^3 + 90x_0^2x_2^4 + 60x_0x_2^5 + 486x_1^6 + 22x_2^6$$

the symmetrized weight enumerator of a suitable code of length 6, for instance C_6 with generator matrix

$$\begin{bmatrix} (1,0) & (0,2) & (0,2) & (0,2) & (0,2) & (1,0) \\ (0,1) & (0,1) & (0,1) & (0,1) & (0,1) & (1,2) \\ (0,0) & (1,1) & (0,0) & (0,0) & (0,0) & (2,2) \\ (0,0) & (0,0) & (1,1) & (0,0) & (0,0) & (2,2) \\ (0,0) & (0,0) & (0,0) & (1,1) & (0,0) & (2,2) \\ (0,0) & (0,0) & (0,0) & (0,0) & (1,1) & (2,2) \end{bmatrix}.$$

Continuing to symmetrize to obtain V-Hamming weight enumerators $q_i := p_i(x, y, y) = \mathrm{hwe}_V(C_i)$ we will not obtain an invariant ring of a group. The subgroup of $\mathrm{GL}_2(\mathbb{Q})$ that stabilizes q_1 and q_3 is of order 2 and its invariant ring is

$$\mathbb{C}[x + 2y, x^2 + 8y^2]$$

which properly contains the ring spanned by q_1, q_3 and

$$q_6 = (\frac{1}{3}q_1^9q_3 - \frac{1}{2}q_1^6q_3^2 + q_1^3q_3^3 + \frac{1}{6}q_3^4)/(q_1^3q_3).$$

This shows that the assumption that the symmetrization commutes with the action of the Clifford-Weil group is necessary.

References

[1] Stefka Bouyuklieva, Some results on Type IV codes over $\mathbb{Z}_4$. IEEE Trans. Inf. Theory 48 (March 2002) 768-773.

[2] S. Dougherty, P. Gaborit, M. Harada, A. Munemasa, P. Solé, Type IV self-dual codes over rings. IEEE Trans. Inf. Theory 45 (Nov. 1999) 2345-2360.

[3] A. Günther, Self-dual group ring codes, PhD Thesis, RWTH Aachen University (in preparation).

[4] M. Kneser, Klassenzahlen definiter quadratischer Formen, Archiv der Math. 8 (1957) 241–250.

[5] J. Morales, Maximal hermitian forms over $\mathbb{Z}G$. Comment. Math. Helvetici 63 (1988) 209–225.

[6] G. Nebe, Kneser-Hecke-operators in coding theory. Abh. Math. Sem. Univ. Hamburg 76 (2006) 79–90.

[7] G. Nebe, E. Rains, N. Sloane, *Self-dual codes and invariant theory.* Springer (2006)

[8] *The Magma Computational Algebra System for Algebra, Number Theory and Geometry.* available via the magma home page http://www.maths.usyd.edu.au:8000/u/magma/