On the Monge problem and multidimensional optimal control

Alban Quadrat and Daniel Robertz

***Abstract*— Using new results on the general Monge parametrization (see [25] and the references therein) recently obtained in [21], i.e., on the possibility to extend the concept of image representation to non-controllable multidimensional linear systems, we show that we can transform some quadratic variational problems (e.g., optimal control problems) with differential constraints into free variational ones directly solvable by means of the standard Euler-Lagrange equations. This result generalizes for non-controllable multidimensional linear systems the results obtained in [11], [19] for controllable ones. In particular, in the 1-D case, this result allows us to avoid the controllability condition commonly used in the behavioural approach literature for the study of optimal control problems with a finite horizon and replace it by the stabilizability condition for the ones with an infinite horizon.**

***Keywords*— Monge problem, parametrizability, multidimensional optimal control, variational problems, controllability, autonomous elements, stabilizability.**

I. Introduction

Let $D = A[\partial_1, \ldots, \partial_n]$ be a ring of differential operators with coefficients in the differential ring A (e.g., $A = \mathbb{R}$, $\mathbb{R}[x_1, \ldots, x_n]$, $\mathbb{R}(x_1, \ldots, x_n)$), $\partial_i = \partial/\partial x_i$. Moreover let $R \in D^{q\times p}$ be a matrix of differential operators and $\mathcal{F}$ a left D-module (e.g., $C^\infty(\mathbb{R}^n)$), namely, it satisfies:

$$\forall\, P_1,\ P_2 \in D,\ \forall\, y_1,\ y_2 \in \mathcal{F}:\ P_1\, y_1 + P_2\, y_2 \in \mathcal{F}.$$

A linear system of partial differential equations (PDEs) is then defined by:

$$\ker_{\mathcal{F}}(R.) \triangleq \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

$\ker_{\mathcal{F}}(R.)$ is called *behaviour* in the behavioural approach to multidimensional linear systems ([11], [13], [24]).

The classical Monge problem questions the existence of a matrix of differential operators $Q \in D^{p\times m}$ which satisfies:

$$\ker_{\mathcal{F}}(R.) = \mathrm{im}_{\mathcal{F}}(Q.) \triangleq Q\,\mathcal{F}^m.$$

See [25] for more historical details. If such a matrix Q exists, then we say that Q is a *parametrization* of the system $\ker_{\mathcal{F}}(R.)$. In the behavioural approach to multidimensional linear systems, we say that the behaviour $\ker_{\mathcal{F}}(R.)$ admits an *image representation* ([11], [13], [18], [24]). We refer to [3], [22] for an introduction to the Monge problem and new results. It was shown by Pillai and Shankar that the existence of a parametrization of a multidimensional linear system defined by PDEs with constant coefficients is equivalent to the C^∞-controllability of the system in terms of the possibility to patch two solutions ([11]). This last result extends for n-D linear systems with constant coefficients a result of J. C. Willems obtained for 1-D linear systems ([13]). See also [14], [16], [17].

Multidimensional optimal control theory has recently been developed in [11], [14], [19], [20]. Let us recall one of the main results. We refer to [11], [14], [19], [20] for more mathematical information.

Theorem 1 ([11], [14], [19], [20]): Let $R \in D^{q\times p}$, $\mathcal{F}$ a left D-module and $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ a linear system of PDEs of order r. Let us suppose that we have the following parametrization

$$\eta = Q\,\xi, \quad \forall\, \xi \in \mathcal{F}^m,$$

of the system $\ker_{\mathcal{F}}(R.)$, i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, where $Q \in D^{p\times m}$. Let us consider the problem of extremizing the quadratic cost

$$I = \int \frac{1}{2}\,\eta_r^T\,L\,\eta_r\,dx,$$

where:

$$\eta_r = (\partial^\alpha\,\eta = \partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}\,\eta,\ 0 \leq |\alpha| = \alpha_1+\ldots+\alpha_n \leq r)$$

and L is a symmetric matrix with entries in A, under the differential constraint $R\,\eta = 0$. The optimal system is then defined by

$$\begin{cases} \eta = Q\,\xi, \\ \mathcal{A}\,\xi = 0, \end{cases} \quad \forall\, \xi \in \mathcal{F}^m,$$

with $\eta = (\eta_l)_{1\leq l\leq p}$,

$$\pi_k^\alpha = \sum_{1\leq l\leq p, 0\leq|\beta|\leq r} L_{k,l}^{\alpha,\beta}\,\partial^\beta\,\eta_l, \quad \mathcal{A} = \widetilde{Q}\cdot\mathcal{B}\cdot Q,$$

where

$$\mathcal{B}\,\eta = \left(\sum_{0\leq|\alpha|\leq r}(-1)^{|\alpha|}\partial^\alpha\,\pi_k^\alpha\right)^T_{1\leq k\leq p}$$

and $\widetilde{Q}$ denotes the formal adjoint of Q obtained by contracting $Q\,\xi$ by a vector of test functions $\varphi \in \mathcal{D}^p$ and integrating by parts, i.e., $\int_{\mathbb{R}^n}(\varphi, Q\,\xi)\,dx = \int_{\mathbb{R}^n}(\widetilde{Q}\,\varphi, \xi)\,dx$, where $(\,\cdot\,,\,\cdot\,)$ denotes the standard inner product of $\mathbb{R}^p$.

A. Quadrat is with INRIA Sophia Antipolis, CAFE project, 2004 Route des Lucioles BP 93, 06902 Sophia Antipolis Cedex, France, Alban.Quadrat@sophia.inria.fr. D. Robertz is with Lehrstuhl B für Mathematik, RWTH - Aachen, Templergraben 64, 52056 Aachen, Germany, daniel@momo.math.rwth-aachen.de.

Let us illustrate this result on two explicit examples.

Example 1: Let us consider the quadratic optimal problem to minimize

$$\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\, dt \tag{1}$$

under the differential constraint defined by the Kalman system $\dot{x}(t) + x(t) - u(t) = 0$ and the initial condition $x(0) = x_0$.

We can easily check that the $\mathcal{F} = C^\infty(\mathbb{R})$-solutions of the system $\dot{x}(t) + x(t) - u(t) = 0$ are parametrized by:

$$\left\{ \begin{array}{l} x(t) = \xi(t), \\ u(t) = \dot{\xi}(t) + \xi(t), \end{array} \right. \quad \forall\, \xi \in \mathcal{F}. \tag{2}$$

Therefore, by substituting (2) into the cost functional (1), we are then led to minimize the free variational problem, i.e., the variational problem without differential constraint, defined by:

$$\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\, dt.$$

Therefore, the computation of the Euler-Lagrange equations then gives the following optimal system:

$$\left\{ \begin{array}{l} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0. \end{array} \right.$$

Integrating this last system and eliminating the initial condition x_0 from $u(t)$ and $x(t)$ finally gives the optimal controller:

$$u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\, e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\ e^{-\sqrt{2}(t-T)}}\, x(t).$$

We now illustrate Theorem 1 on a variational problem studied in mathematical physics.

Example 2: Let us extremize the electromagnetism Lagrangian defined by

$$\int \left(\frac{1}{2\,\mu_0}\ \parallel \vec{B} \parallel^2 - \frac{\epsilon_0}{2}\ \parallel \vec{E} \parallel^2 \right)\ dx_1\, dx_2\, dx_3\, dt, \tag{3}$$

where μ_0 (resp., ϵ_0) denotes the dielectric (magnetic) constant and the electromagnetism field $(\vec{B}, \vec{E})$ satisfies the following equations:

$$\left\{ \begin{array}{l} \vec{\nabla}.\vec{B} = 0, \\ \vec{\nabla} \wedge \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0. \end{array} \right. \tag{4}$$

If Ω is an open convex subset of $\mathbb{R}^4$, then the $C^\infty(\Omega)$-solutions of the first set of Maxwell equations (4) is known to be parametrizable by means of the quadri-potential $(\vec{A}, V)$, i.e.:

$$\left\{ \begin{array}{l} \dfrac{\partial \vec{B}}{\partial t} + \vec{\nabla} \wedge \vec{E} = \vec{0}, \\ \vec{\nabla}\,.\,\vec{B} = 0, \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla}\, V - \dfrac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}. \end{array} \right. \tag{5}$$

Hence, if we substitute (5) into (3), we then obtain a variational problem in $\vec{A}$ and V without differential constraint. Then, the Euler-Lagrange equations and the Lorentz gauge condition, namely,

$$\vec{\nabla}.\vec{A} + \frac{1}{c^2}\,\frac{\partial V}{\partial t} = 0,$$

give the following electromagnetic waves traveling at the speed of light $c = 1/\sqrt{(\epsilon_0\, \mu_0)}$ in the vacuum:

$$\left\{ \begin{array}{l} \dfrac{1}{c^2}\,\dfrac{\partial^2 \vec{A}}{\partial t^2} - \Delta\, \vec{A} = 0, \\ \dfrac{1}{c^2}\,\dfrac{\partial^2 V}{\partial t^2} - \Delta\, V = 0, \\ \vec{\nabla} \wedge \vec{A} = \vec{B}, \\ -\vec{\nabla} V - \dfrac{\partial \vec{A}}{\partial t} = \vec{E}. \end{array} \right.$$

II. Module-theoretic approach to linear systems

In this section, we recall the module-theoretic background ([23], [9]) for the study of multidimensional linear systems that follows. We refer to [3] for more details.

Let $D = A[\partial_1, \ldots, \partial_n]$ be a ring of differential operators[1], where A is a differential ring which is also an algebra over a field k containing $\mathbb{Q}$ as in Section I (e.g., $k = \mathbb{R}$). With a given linear system of partial differential equations $R\,\eta = 0$, $R \in D^{q\times p}$, for unknown functions $\eta_1, \ldots, \eta_p$ of independent variables $x_1, \ldots, x_n$ we associate the left D-module:

$$M = D^{1\times p}/(D^{1\times q}\, R).$$

If all coefficients of the linear system are real numbers, then the coefficient domain A of D can be chosen to be $\mathbb{R}$ so that D is a commutative polynomial ring and M is a D-module. More generally, if all coefficients of $R\,\eta = 0$ are polynomials in $x_1, \ldots, x_n$ (resp., rational functions in $x_1, \ldots, x_n$), then we choose $A = \mathbb{R}[x_1, \ldots, x_n]$ (resp., $A = \mathbb{R}(x_1, \ldots, x_n)$). Then, D is a Weyl algebra ([9]), which is a non-commutative ring, and $M = D^{1\times p}/(D^{1\times q}\, R)$ is a left D-module as a factor module of the left D-module $D^{1\times p}$.

The left D-module M which is associated with the given linear system is an intrinsic object, by which we

[1] All notions and statements of this section can be applied in the more general framework of Ore algebras D, but we shall consider only the case $D = A[\partial_1, \ldots, \partial_n]$ in what follows.

mean that two equivalent systems of equations $R_1\,\eta = 0$ and $R_2\,\eta = 0$ give rise to the same module M. This can be seen by viewing the row vectors $r \in D^{1\times p}$ as representatives of equations $r\,\eta = 0$. By construction of M, it is then clear that the equations $R\,\eta = 0$ and all their left D-linear combinations (i.e., consequences) are represented by zero in M. Therefore, the structural properties of the linear system can be studied with algebraic methods by considering the module M.

For the algebraic characterization of parametrizability of a linear system (and also controllability), the following submodule of M is of particular importance.

Definition 1: Let M be a left D-module. Then

$$t(M) \triangleq \{m \in M \mid \exists\, 0 \neq P \in D : P\,m = 0\}$$

is called the *torsion submodule* of M. Its elements are the *torsion elements* of M.

We introduce a few notations from homological algebra. We refer to [23] for more details.

Definition 2: 1) A family of left D-modules (resp., abelian groups) $(P_i)_{i\in\mathbb{Z}}$ together with a family of homomorphisms of left D-modules (resp. of abelian groups) $(d_i)_{i\in\mathbb{Z}}$, where $d_i : P_i \longrightarrow P_{i-1}$, is called a *complex* if $d_i \circ d_{i+1} = 0$, i.e., $\operatorname{im} d_{i+1} \subseteq \ker d_i$, for all $i \in \mathbb{Z}$.

2) A complex is said to be *exact at* P_r if $\operatorname{im} d_{r+1} = \ker d_r$. It is said to be *exact* if it is exact at P_i for all $i \in \mathbb{Z}$. Then, it is also called an *exact sequence* (of left D-modules, resp., of abelian groups).

3) If only three consecutive modules of an exact sequence are non-zero, then it is called a *short exact sequence*.

Example 3: Let M be a left D-module and N a submodule of M. Then

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

is a short exact sequence of left D-modules, where the first non-zero map is the canonical injection of N into M and the following morphism is the canonical projection of M onto M/N. In particular,

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0$$

is a short exact sequence of left D-modules.

The following notions will be of crucial importance in what follows. They form only a part of a more detailed classification of module properties.

Definition 3: [23] Let M be a finitely generated left D-module.

1) M is said to be *free* if there exists $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that $M \cong D^{1\times r}$, where $\cong$ denotes isomorphism of left D-modules.

2) M is said to be *projective* if there exist $r \in \mathbb{Z}_+$ and a left D-module P such that $M \oplus P \cong D^{1\times r}$.

3) M is said to be *torsion-free* if $t(M) = 0$.

4) M is said to be *torsion* if $M = t(M)$.

We have the following implications for the module-theoretic concepts introduced in the previous definition.

Proposition 1 ([23]): Let M be a finitely generated left D-module. If M is free, then M is projective. If M is projective, then M is torsion-free.

We are going to recall the characterization of the above module properties in the language of homological algebra. For more details, see [15], [18], [3].

Definition 4: Let M be a finitely generated left D-module. An exact sequence of left D-modules

$$\ldots \longrightarrow D^{1\times p_r} \xrightarrow{d_r} \ldots \xrightarrow{d_1} D^{1\times p_0} \xrightarrow{d_0} M \longrightarrow 0 \qquad (6)$$

is called a *free resolution* of M.

Remark 1: If a given left D-module M has a *finite presentation* $M = D^{1\times p}/(D^{1\times q}\,R)$, where $R \in D^{q\times p}$ has full row-rank, then

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow M \longrightarrow 0$$

is a free resolution of M. More generally, starting with a finite presentation of M, a free resolution of M can be constructed by iteratively computing generating sets of the kernels of d_i (syzygies), $i \geq 1$, starting with $d_1 = (.R)$. In the cases which are relevant here, i.e., $D = A[\partial_1, \ldots, \partial_n]$ is a commutative polynomial ring or a Weyl algebra over a field which contains $\mathbb{Q}$, every finitely generated left D-module has a free resolution in which at most d_0, d_1, $\ldots$, d_n are non-zero morphisms.

Definition 5: Let M be a finitely generated left D-module, $\mathcal{F}$ a left D-module and let (6) be a free resolution of M. Then

$$\ldots \longleftarrow \hom_D(D^{1\times p_r}, \mathcal{F}) \xleftarrow{d_r^*} \hom_D(D^{1\times p_{r-1}}, \mathcal{F})$$
$$\xleftarrow{d_{r-1}^*} \ldots \xleftarrow{d_1^*} \hom_D(D^{1\times p_0}, \mathcal{F}) \longleftarrow 0$$

is a complex of abelian groups, where d_i^* is defined by $d_i^*(f) = f \circ d_i$ for $f \in \hom_D(D^{1\times p_{i-1}}, \mathcal{F})$. The defects of exactness of this complex are denoted by:

$$\begin{aligned}\mathrm{ext}^0_D(M,\mathcal{F}) &= \ker(d_1^*),\\ \mathrm{ext}^i_D(M,\mathcal{F}) &= \ker(d_{i+1}^*)/\mathrm{im}(d_i^*), \quad i \geq 1.\end{aligned}$$

Proposition 2 ([23]): The abelian groups $\mathrm{ext}^i_D(M,\mathcal{F})$ only depend on M and not on the free resolution of M which is chosen to define $\mathrm{ext}^i_D(M,\mathcal{F})$.

Effective methods for computing these homological invariants were described in [3] and have been implemented in the Maple package OREMODULES [2]. We recall only two of the important characterizations of module properties in the language of homological algebra.

Theorem 2 ([3]): Let $D = A[\partial_1,\ldots,\partial_n]$ be a ring of differential operators which is either a commutative polynomial ring or a Weyl algebra, $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$ a finitely presented left D-module. We define the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R})$, where $\widetilde{R}$ denotes the formal adjoint of R. Then we have:

1) $t(M) \cong \mathrm{ext}^1_D(\widetilde{N},D)$. In particular, M is torsion-free if and only if $\mathrm{ext}^1_D(\widetilde{N},D)=0$.
2) M is projective if and only if $\mathrm{ext}^i_D(\widetilde{N},D)=0$ for all $i=1,\ldots,n$.

Using the module-theoretic approach to linear systems, structural properties of the behaviour $\ker_{\mathcal{F}}(R.)$, where $\mathcal{F}$ is a left D-module as in Section I, are deduced from the properties of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ which is associated with the linear system $R\,\eta = 0$. However, the relations between $\ker_{\mathcal{F}}(R.)$ and M also depend on the properties of the left D-module $\mathcal{F}$. A good duality between behaviours $\ker_{\mathcal{F}}(R.)$ and left D-modules M only holds for injective cogenerators $\mathcal{F}$ ([10]). We may think of an injective cogenerator $\mathcal{F}$ as a sufficiently rich space of functions. Before recalling the definition of an injective cogenerator, we state the following important remark by B. Malgrange ([8]).

Remark 2: Let $R \in D^{q\times p}$ and $M = D^{1\times p}/(D^{1\times q}\,R)$. Then, we have

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M,\mathcal{F})$$

as abelian groups (or k-vector spaces), i.e., the set of solutions of $R\,\eta = 0$ in $\mathcal{F}^p$ and the set of left D-morphisms from M to $\mathcal{F}$ are isomorphic abelian groups (resp. k-vector spaces).

Definition 6: 1) [23] A left D-module $\mathcal{F}$ is called *injective* if, for every left D-module M, and, for all $i \geq 1$, we have $\mathrm{ext}^i_D(M,\mathcal{F})=0$.

2) [23] A left D-module $\mathcal{F}$ is called *cogenerator* if, for every left D-module M, we have:

$$\hom_D(M,\mathcal{F})=0 \quad\Longrightarrow\quad M=0.$$

We note the following (non-constructive) existence theorem.

Theorem 3 ([23]): An injective cogenerator left D-module $\mathcal{F}$ exists for every ring D.

Lemma 1: 1) If $\mathcal{F}$ is an injective left D-module, then $\hom_D(\,\cdot\,,\mathcal{F})$ transforms exact sequences of left D-modules into exact sequences of abelian groups.

2) If the left D-module $\mathcal{F}$ is an injective cogenerator, then the exactness of the complex of abelian groups obtained by applying the functor $\hom_D(\,\cdot\,,\mathcal{F})$ to a complex of left D-modules implies the exactness of this latter complex.

We give a few examples of modules over the commutative polynomial ring $\mathbb{R}[\partial_1,\ldots,\partial_n]$ and the localized Weyl algebra $\mathbb{R}(t)\left[\frac{d}{dt}\right]$ which are injective cogenerators.

Example 4: 1) If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the ring $\mathbb{R}[\partial_1,\ldots,\partial_n]$ of differential operators with coefficients in $\mathbb{R}$ [8].

2) [26] If $\mathcal{F}$ denotes the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathcal{F}$ is an injective cogenerator left $\mathbb{R}(t)\left[\frac{d}{dt}\right]$-module.

We finish this section by recalling the characterization of parametrizability of a behaviour in terms of the associated module.

Proposition 3 ([3]): Let $R \in D^{q\times p}$ and $\mathcal{F}$ an injective cogenerator left D-module. The behaviour $\ker_{\mathcal{F}}(R.)$ has a parametrization (or image representation) $Q \in D^{p\times m}$, i.e., $\ker_{\mathcal{F}}(R.) = Q\,\mathcal{F}^m$, if and only if the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ which is associated with the linear system $R\,\eta = 0$ is torsion-free. By Theorem 2, M is torsion-free if and only if $\mathrm{ext}^1_D(\widetilde{N},D)=0$ for the left D-module:

$$\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R}).$$

III. MAIN RESULTS ON THE GENERAL MONGE PROBLEM

The first main purpose of this paper is to prove the following new theorem.

Theorem 4: Let $R \in D^{q \times p}$, $M = D^{1 \times p}/(D^{1 \times q} R)$, $\mathcal{F}$ an injective cogenerator left D-module ([3], [8], [11], [18], [24]) and consider the linear system

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

Then, we obtain a parametrization of $\ker_{\mathcal{F}}(R.)$ by applying the following algorithm:

1) Following the constructive algorithms developed in [2], [3], [16], compute $R' \in D^{q' \times p}$ and $R'' \in D^{q \times q'}$ such that:

$$\begin{cases} R = R''\,R', \\ t(M) = (D^{1 \times q'}\,R')/(D^{1 \times q}\,R), \\ M/t(M) = D^{1 \times p}/(D^{1 \times q'}\,R'). \end{cases}$$

2) Compute a matrix $Q \in D^{p \times m}$ such that:

$$\ker_D(.Q) = (D^{1 \times q'}\,R').$$

This is always possible and general algorithms are given in [3], [16] and implemented in OREMODULES ([2]). Then, using the fact that $\mathcal{F}$ is an injective left D-module, we obtain

$$\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m,$$

i.e., Q is a parametrization of the system $\ker_{\mathcal{F}}(R'.)$ ([3], [11], [18]).

3) Compute a matrix $T \in D^{r' \times q'}$ such that:

$$\ker_D(.R') = D^{1 \times r'}\,T,$$

i.e., compute the first syzygy module of $(D^{1 \times q'}\,R')$ [2], [3]. Using the fact that $\mathcal{F}$ is an injective left D-module, we then have:

$$\ker_{\mathcal{F}}(T.) = R'\,\mathcal{F}^p.$$

4) Find a fundamental solution $\overline{\tau} \in \mathcal{F}^{q'}$ of the autonomous linear system:

$$\begin{cases} R''\,\tau = 0, \\ T\,\tau = 0. \end{cases} \qquad (7)$$

Such a fundamental solution always exists as $\mathcal{F}$ is a cogenerator left D-module.

5) Find the general solution of the inhomogeneous linear system:

$$R'\,\eta = \overline{\tau}, \quad \eta \in \mathcal{F}^p.$$

It is well-known that this problem can be decomposed into the following two subproblems:

a) Find a particular solution $\overline{\eta} \in \mathcal{F}^p$ of the inhomogeneous linear system $R'\,\overline{\eta} = \overline{\tau}$.

b) Find the general solution of $R'\,\eta = 0$. However, we already know that we have:

$$\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m.$$

6) Finally, the general solution of $R\,\eta = 0$ in $\mathcal{F}^p$ is of the form:

$$\eta = \overline{\eta} + Q\,\xi, \quad \forall\,\xi \in \mathcal{F}^m. \qquad (8)$$

Proof: We are going to verify the assertions stated in the given algorithm.

1) An algorithm to compute $R' \in D^{q' \times p}$ such that $M/t(M) = D^{1 \times p}/(D^{1 \times q'}\,R')$ was described in [3]. The existence and the possibility to compute $R'' \in D^{q \times q'}$ satisfying $R''\,R' = R$ was explained in [21]. Using OREMODULES [2], R'' can be computed using the command `Factorize`.

2) We have the exact sequence of left D-modules

$$D^{1 \times q'} \xrightarrow{.R'} D^{1 \times p} \xrightarrow{.Q} D^{1 \times m}.$$

Since $\mathcal{F}$ is injective, by Lemma 1 1) the complex

$$\mathcal{F}^{q'} \xleftarrow{R'.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$$

is also exact, i.e., $\ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m$.

3) Similarly to 2), the injectivity of $\mathcal{F}$ implies that $\hom_D(\,\cdot\,, \mathcal{F})$ transforms the exact sequence of left D-modules

$$D^{1 \times r'} \xrightarrow{.T} D^{1 \times q'} \xrightarrow{.R'} D^{1 \times p}$$

into the exact sequence

$$\mathcal{F}^{r'} \xleftarrow{T.} \mathcal{F}^{q'} \xleftarrow{R'.} \mathcal{F}^p,$$

which means $\ker_{\mathcal{F}}(T.) = R'\,\mathcal{F}^p$.

4) By combining Remark 2 and Definition 6 2) we conclude that a fundamental solution $\overline{\tau}$ of (7) always exists in $\mathcal{F}^{q'}$.

5) is clear.

6) Finally, we show that (8) is the general solution of $R\,\eta = 0$ in $\mathcal{F}^p$. Due to step 1), $R\,\eta = 0$ is equivalent to $R''\,R'\,\eta = 0$, and therefore equivalent to

$$\begin{cases} R''\,\tau = 0, \\ \tau = R'\,\eta. \end{cases} \qquad (9)$$

Because of step 3) we know that $R'\,\mathcal{F}^p = \ker_{\mathcal{F}}(T.)$. Hence, the component τ of every solution (τ, η) of (9) satisfies (7), and conversely, every solution τ of (7) yields a solution $(\tau, R'\,\eta)$ of (9). Having computed a fundamental solution of (7) in step 4), the second equation in (9) is solved for $\eta \in \mathcal{F}^p$ in step 5). Therefore, the algorithm described in this theorem determines the general solution of $R\,\eta = 0$ in $\mathcal{F}^p$.

$\blacksquare$

Remark 3: If $t(M) = 0$ in Theorem 4, then we can choose $R' = R$ and R'' as the identity matrix. Then the algorithm can already be stopped in step 2) because

$$\ker_{\mathcal{F}}(R.) = \ker_{\mathcal{F}}(R'.) = Q\,\mathcal{F}^m.$$

In this case all solutions of $R\,\eta = 0$ in $\mathcal{F}^p$ are parametrized in terms of arbitrary functions $\xi_1, \ldots, \xi_m$ of the independent variables $x_1, \ldots, x_n$. When $t(M) \neq 0$, the algorithm described in Theorem 4 constructs a parametrization of $\ker_{\mathcal{F}}(R.)$ in terms of such arbitrary functions, (integration) constants and arbitrary functions depending only on certain of the independent variables.

Let us illustrate Theorem 4 first on a simple ordinary differential example.

Example 5: Let us parametrize all $\mathcal{F} = C^\infty(\mathbb{R})$-solutions of the time-varying linear system:

$$\ddot{y}(t) - t\,\dot{u}(t) - u(t) = 0.$$

We consider $D = A_1(\mathbb{R}) = \mathbb{R}[t]\left[\frac{d}{dt}\right]$ and:

$$R = \begin{pmatrix} \dfrac{d^2}{dt^2} & -t\,\dfrac{d}{dt} - 1 \end{pmatrix} \in D^{1\times 2}.$$

We can check that $R'' = \frac{d}{dt}$, $R' = \begin{pmatrix} \frac{d}{dt} & -t \end{pmatrix}$ and $T = 0$. Then, we need to find all $\mathcal{F}$-solutions of:

$$\dot{y}(t) - t\,u(t) = C, \quad C \in \mathbb{R}. \qquad (10)$$

We easily check that $(y_\star \quad u_\star)^T = (C\,t \quad 0)^T$ is a particular solution of (10). Hence, we only need to find a parametrization of all $\mathcal{F}$-solutions of the homogeneous linear system $\dot{y}(t) - t\,u(t) = 0$. But, it is well-known that $\mathcal{F}$ is not an injective cogenerator left D-module ([5]). However, we can prove that we have the following *split exact sequence*

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.Q} D^{1\times 2} \xrightarrow{.P} D \longrightarrow 0,$$

where

$$Q = \begin{pmatrix} t^2 & t\,\frac{d}{dt} - 1 \\ t\,\frac{d}{dt} + 2 & \frac{d^2}{dt^2} \end{pmatrix} \in D^{2\times 2}$$

and

$$P = \begin{pmatrix} \frac{d}{dt} \\ -t \end{pmatrix} \in D^2.$$

See [3], [15], [16] for more details. Therefore, if we apply the functor $\hom_D(\cdot, \mathcal{F})$ to the previous split exact sequence, we then obtain the exact sequence

$$0 \longleftarrow \mathcal{F} \xleftarrow{R.} \mathcal{F}^2 \xleftarrow{Q.} \mathcal{F}^2 \xleftarrow{P.} \mathcal{F} \longleftarrow 0,$$

which shows that we have the following parametrization of $\ker_{\mathcal{F}}(R'.)$:

$$\begin{cases} y(t) = t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ u(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) + \ddot{\xi}_2(t), \end{cases} \quad \forall\, \xi_1,\ \xi_2 \in \mathcal{F}.$$

Using the previous parametrization of all $\mathcal{F}$-solutions of the homogeneous part of (10), we obtain the following parametrization of $\ker_{\mathcal{F}}(R.)$:

$$\begin{cases} y(t) = C\,t + t^2\,\xi_1(t) + t\,\dot{\xi}_2(t) - \xi_2(t), \\ u(t) = t\,\dot{\xi}_1(t) + 2\,\xi_1(t) + \ddot{\xi}_2(t), \end{cases} \quad \forall\, \xi_1,\ \xi_2 \in \mathcal{F}.$$

Finally, we can easily prove that $C = \dot{y}(0)$.

Let us now give an example of a multidimensional linear system defined by PDEs.

Example 6: We consider the system $\mathrm{grad}\,(\mathrm{div}\,\vec{B}) = \vec{0}$, i.e.:

$$\begin{cases} \partial_1\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0, \\ \partial_2\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0, \\ \partial_3\,(\partial_1\,B_1 + \partial_2\,B_2 + \partial_3\,B_3) = 0. \end{cases} \qquad (11)$$

This system commonly appears in mathematical physics ([6], [7]). Let us parametrize all the $\mathcal{F} = C^\infty(\mathbb{R}^3)$-solutions of (11). Using the algorithms developed in [3], [15], [16], we obtain the following matrices:

$$R = \mathrm{grad}\,(\mathrm{div}), \quad R' = \mathrm{div}, \quad R'' = \mathrm{grad}, \quad T = 0.$$

See [2] for explicit computations. Therefore, there is one autonomous element in (11) defined by:

$$\begin{cases} \tau = \mathrm{div}\,\vec{B}, \\ \partial_1\,\tau = 0, \\ \partial_2\,\tau = 0, \\ \partial_3\,\tau = 0. \end{cases}$$

Hence, we need to parametrize all $\mathcal{F}$-solutions of the following PDE:

$$\mathrm{div}\,\vec{B} = C, \quad C \in \mathbb{R}. \qquad (12)$$

We then easily check that a particular solution of (12) is given by:

$$\vec{B}_\star = (C\,x_1 \quad 0 \quad 0)^T.$$

Therefore, all $\mathcal{F}$-solutions of (11) are finally given by

$$\vec{B} = \vec{B}_\star + \mathrm{curl}\,\vec{\Psi}, \quad \forall\,\vec{\Psi} \in \mathcal{F}^3,$$

where curl denotes the standard curl operator in $\mathbb{R}^3$.

We can wonder when it is possible to obtain a particular solution of the inhomogeneous system $R'\,\eta = \overline{\tau}$, where $\overline{\tau} \in \mathcal{F}^{q'}$ is fixed, by means of purely algebraic techniques, i.e., by means of a kind of *variation of constants* which does not use any integration. This problem was solved in [21].

Theorem 5 ([21]): Let $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ presented by R. Then, we have

$$M \cong t(M) \oplus (M/t(M)) \qquad (13)$$

if and only if there exist $S \in D^{p \times q'}$ and $V \in D^{q' \times q}$ such that:

$$R' - R'\, S\, R' = V\, R. \tag{14}$$

We note that (14) always holds if D is a left hereditary ring as, for instance, $D = K\left[\frac{d}{dt}\right]$, where K is a *differential field* [16] (e.g., $K = \mathbb{R}$, $\mathbb{R}(t)$), or if D is the first Weyl algebra $A_1(k) = k[t]\left[\frac{d}{dt}\right]$. Moreover, (14) also holds if $M/t(M)$ is a projective left D-module, a fact that can be constructively checked ([3], [15]). Finally, constructive algorithms have been developed in [21] for computing the matrices S and V appearing in (14). See [2] for the implementations of all these algorithms in OREMODULES.

We then have the following interesting corollary of Theorem 5.

Corollary 1 ([21]): We assume the same notations and hypotheses as in Theorem 4. Let us consider a fundamental solution $\overline{\tau} \in \mathcal{F}^{q'}$ of the following system:

$$\begin{cases} R''\, \tau = 0, \\ T\, \tau = 0. \end{cases}$$

Then, $S\, \overline{\tau}$ is a particular solution of the inhomogeneous linear system $R'\, \eta = \overline{\tau}$ and the general solution of $\ker_{\mathcal{F}}(R.)$ is exactly of the form:

$$\eta = S\, \overline{\tau} + Q\, \xi, \quad \forall\, \xi \in \mathcal{F}^m.$$

Let us illustrate Theorem 5 and Corollary 1 on two explicit examples.

Example 7: Let us consider $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$, the divergence operator $\operatorname{div} = (\partial_1 \quad \partial_2 \quad \partial_3)$ and the left D-module $\mathcal{F} = C^\infty(\mathbb{R}^3)$. We parametrize the linear system of PDEs

$$P(\partial)\,(\partial_1\, B_1 + \partial_2 B_2 + \partial_3 B_3) = 0, \quad 0 \neq P(\partial) \in D, \tag{15}$$

namely, $\ker_{\mathcal{F}}((P\, \operatorname{div}).)$. Using the constructive algorithms developed in [3], [15], [16], we obtain:

$$\begin{cases} R' = \operatorname{div}, \\ R'' = P(\partial). \end{cases}$$

Therefore, $\ker_{\mathcal{F}}((P\, \operatorname{div}).)$ admits the following autonomous element:

$$\begin{cases} \tau = \partial_1 B_1 + \partial_2\, B_2 + \partial_3\, B_3, \\ P(\partial)\, \tau = 0. \end{cases}$$

Let $\overline{\tau} \in \mathcal{F}$ be a fundamental solution of $P(\partial)\, \tau = 0$ (it always exists because $\mathcal{F} = C^\infty(\mathbb{R}^3)$ was recalled in Example 4 1) to be an injective cogenerator). We then have to solve the inhomogeneous system:

$$\operatorname{div} \vec{B} = \overline{\tau}. \tag{16}$$

Therefore, we need to:

1) Find a particular solution $\vec{B}_\star$ of the inhomogeneous linear system (16). Using Theorem 5, we can try to find $S = (S_1 \quad S_2 \quad S_3)^T \in D^3$ and $V \in D$ satisfying (14), namely:

$$R' - R'\, S\, R' = V\, R \;\Leftrightarrow\; \sum_{i=1}^{3} \partial_i S_i - V\, P(\partial) = 1$$
$$\Leftrightarrow\; (\partial_1, \partial_2, \partial_3, P(\partial)) = D \;\Leftrightarrow\; P(0) \neq 0.$$

Hence, if $P(0) \neq 0$, then we obtain (13), where $M = D^{1\times 3}/(D\,(P(d)\, \operatorname{div}))$ and:

$$M/t(M) = D^{1\times 3}/(D\, \operatorname{div}).$$

Then, by Corollary 1, we get that

$$\vec{B}_\star = (S_1 \quad S_2 \quad S_3)^T\, \overline{\tau}$$

is a particular solution of the inhomogeneous linear system (16) as we have:

$$(\partial_1 \quad \partial_2 \quad \partial_3)\, \vec{B}_\star = (1 + V\, P(\partial))\, \overline{\tau} = \overline{\tau}.$$

2) Find a general solution of $\operatorname{div} \vec{A} = 0$. However, it is well-known that we have ([3], [11], [24]):

$$\operatorname{div} \vec{B} = 0 \;\Leftrightarrow\; \vec{B} = \operatorname{curl} \vec{\Psi}, \qquad \vec{\Psi} \in \mathcal{F}^3.$$

We finally obtain the following parametrization of (15):

$$\vec{B} = (S_1 \quad S_2 \quad S_3)^T\, \overline{\tau} + \operatorname{curl} \vec{\Psi}, \quad \forall\, \vec{\Psi} \in \mathcal{F}^3.$$

Finally, we note that if $P(\partial) = \partial_1$, then $P(0) = 0$, and thus, (13) does not hold over the ring $D = \mathbb{R}[\partial_1, \partial_2, \partial_3]$. However, if we consider the non-commutative ring $A_3(\mathbb{R}) = \mathbb{R}[x_1, x_2, x_3][\partial_1, \partial_2, \partial_3]$ instead of D, we easily check that $S = (x_1 \quad 0 \quad 0)^T$ satisfies

$$R' - R'\, S\, R' = x_1\, R,$$

which proves that (13) holds over $A_3(\mathbb{R})$. By Corollary 1, a particular solution of the inhomogeneous linear system $\operatorname{div} \vec{B} = \Phi(x_2, x_3)$, where $\Phi \in C^\infty(\mathbb{R}^2)$ is a fundamental solution of $\partial_1\, \tau = 0$, is given by:

$$\vec{B}_\star = (x_1\, \Phi(x_2, x_3) \quad 0 \quad 0)^T.$$

All $\mathcal{F}$-solutions of $\partial_1 \operatorname{div} \vec{B} = 0$ are then given by:

$$\vec{B} = \vec{B}_\star + \operatorname{curl} \vec{\Psi}, \quad \forall\, \vec{\Psi} \in \mathcal{F}^3.$$

Finally, let us finish by giving another example appearing in linear elasticity [7].

Example 8: In linear elasticity, we sometimes need to solve the following PDE

$$\Delta\, \Delta\, A = c\, \Delta\, V, \quad \Delta = \partial_1^2 + \partial_2^2, \quad c \in \mathbb{R} \setminus \{0\}, \tag{17}$$

where A is the *Airy function* and V a *potential*. See [19] for more details. Let us parametrize all $\mathcal{F} = \mathcal{D}'(\mathbb{R}^2)$-solutions of (17).

Let us introduce the ring $D=\mathbb{R}[\partial_1,\partial_2]$ and the matrix $R=(\Delta\,\Delta \quad -c\,\Delta)\in D^{1\times 2}$. Using the algorithms developed in [3], [15], [16], we easily obtain the following matrices:

$$R''=\Delta, \quad R'=(\Delta \quad -c), \quad T=0, \quad Q=(1 \quad \Delta/c)^T.$$

In particular, (17) admits the following trivial autonomous element:

$$\begin{cases} \tau=\Delta\,A-c\,V, \\ \Delta\,\tau=0, \end{cases}$$

A fundamental solution of $\Delta\,\tau=0$ in $\mathcal{F}$ is given by:

$$\overline{\tau}=\ln\left(1/\sqrt{x_1^2+x_2^2}\right).$$

Now, the matrix $S=(0 \quad -1/c)^T$ satisfies

$$R'\,S=1\ \Rightarrow\ R'-R'\,S\,R'=0\ \Rightarrow\ V=0$$

which shows that

$$\left(\begin{array}{c} A_\star \\ V_\star \end{array}\right)=S\,\overline{\tau}=\left(\begin{array}{c} 0 \\ -\dfrac{\overline{\tau}}{c} \end{array}\right)$$

is a particular solution of the inhomogeneous linear equation $\Delta\,A-c\,V=\overline{\tau}$. Finally, all $\mathcal{F}$-solutions of (17) are then given by:

$$\begin{cases} A=A, \\ V=\dfrac{1}{c}\,(\Delta\,A-\overline{\tau}), \end{cases} \qquad \forall\,A\in\mathcal{F}.$$

We note that the D-module $M/t(M)=D^{1\times 2}/(D\,R')$ is free (i.e., projective), explaining why a right-inverse S of R' gives a particular solution of the inhomogeneous equation. See [21] for more details.

IV. Applications to multidimensional optimal control

The second main purpose of this paper is to present the following application of Theorem 4 to multidimensional optimal control.

Theorem 6: Let $R\in D^{q\times p}$, $\mathcal{F}$ a left D-module and $\ker_{\mathcal{F}}(R.)=\{\eta\in\mathcal{F}^p \mid R\,\eta=0\}$ a linear system of PDEs of order r. Let us suppose that we have the general parametrization

$$\eta=\overline{\eta}+Q\,\xi, \quad \forall\,\xi\in\mathcal{F}^m,$$

of $\ker_{\mathcal{F}}(R.)$ given by Theorem 4. We consider the problem of extremizing the quadratic cost

$$I=\int\frac{1}{2}\,\eta_r^T\,L\,\eta_r\,dx,$$

where

$$\eta_r=(\partial^\alpha\,\eta=\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}\,\eta,\ 0\leq|\alpha|=\alpha_1+\ldots+\alpha_n\leq r)$$

and L is a symmetric matrix with entries in A, under the differential constraint $R\,\eta=0$. The optimal system is then defined by

$$\begin{cases} \eta=\overline{\eta}+Q\,\xi, \\ \mathcal{A}\,\xi+(\widetilde{Q}\cdot\mathcal{B})\,\overline{\eta}=0, \end{cases} \qquad \forall\,\xi\in\mathcal{F}^m,$$

where $\eta=(\eta_l)_{1\leq l\leq p}$ and:

$$\pi_k^\alpha=\sum_{1\leq l\leq p,0\leq|\beta|\leq r}L_{k,l}^{\alpha,\beta}\,\partial^\beta\,\eta_l, \quad \mathcal{A}=\widetilde{Q}\cdot\mathcal{B}\cdot Q$$

and

$$\mathcal{B}\,\eta=\left(\sum_{0\leq|\alpha|\leq r}(-1)^{|\alpha|}\partial^\alpha\,\pi_k^\alpha\right)_{1\leq k\leq p}^T.$$

Proof: In [19], it was shown that the optimal system is given by

$$\begin{cases} R\,\eta=0, \\ \mathcal{B}\,\eta-\widetilde{R}\,\lambda=0, \end{cases} \tag{18}$$

where $\widetilde{R}$ is the formal adjoint of R and λ is Lagrange multiplier. We denote by $\widetilde{Q}$ the formal adjoint of Q. Due to step 2) of the algorithm described by Theorem 4, we have $R'\,Q=0$, where the factorization $R=R''\,R'$ is obtained in step 1) of the same algorithm. Therefore, we have

$$\widetilde{Q}\,\widetilde{R}=\widetilde{Q}\,\widetilde{R''\,R'}=\widetilde{Q}\,\widetilde{R'}\,\widetilde{R''}=\widetilde{R'\,Q}\,\widetilde{R''}=0.$$

By multiplying $\widetilde{Q}$ on the left of the second equation in (18), we thus find

$$\widetilde{Q}\,\mathcal{B}\,\eta=0. \tag{19}$$

Now, we have

$$R\,\eta=0 \quad\Leftrightarrow\quad \eta=\overline{\eta}+Q\,\xi, \quad \xi\in\mathcal{F}^m. \tag{20}$$

Substitution easily shows that (19) together with the first equation in (18) is equivalent to

$$\begin{cases} \eta=\overline{\eta}+Q\,\xi, \\ \widetilde{Q}\cdot\mathcal{B}\,(\overline{\eta}+Q\,\xi)=0, \end{cases}$$

which proves the theorem. ■

Remark 4:

1) By substituting the parametrization of the behaviour $\ker_{\mathcal{F}}(R.)$, the Lagrange multiplier λ is eliminated.

2) If all solutions of $R\,\eta=0$ in $\mathcal{F}^m$ are parametrized by means of the matrix of differential operators $Q\in D^{p\times m}$, i.e.,

$$R\,\eta=0 \quad\Leftrightarrow\quad \eta=Q\,\xi, \quad \xi\in\mathcal{F}^m$$

holds instead of (20), then the same argument as in the previous proof gives a proof of Theorem 1, which is a particular case of Theorem 6.

We point out that no controllability hypothesis is required in Theorem 6, as it was the case in [11], [14], [19], [20]. Hence, Theorem 6 generalizes for non-controllable multidimensional linear systems with constant coefficients the results obtained in [11], [14], [19], [20] for controllable ones. In particular, in the 1-D case, this result allows us to remove the controllability condition used in the behavioural approach literature in the study of optimal control problems with a finite horizon and replace it by the stabilizability condition for the ones with an infinite horizon. In particular, this last result shows that, within the behavioural approach, we can recover the general results previously developed in the literature on optimal control of 1-D linear systems ([1], [4]) and generalize them to multidimensional linear systems defined by partial differential equations.

Let us illustrate Theorem 6 on two examples.

Example 9: Let us consider the following quadratic optimal problem

$$I = \int_0^T \frac{1}{2}\,(x_1^2(t) + x_2^2(t) + u^2(t))\,dt, \tag{21}$$

under the differential constraint defined by the Kalman system:

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + u, \\ \dot{x}_2 = x_1 + u, \\ x_1(0) = x_1^0, \\ x_2(0) = x_2^0. \end{array} \right. \tag{22}$$

Let us denote by $\mathcal{F} = C^\infty(]0, +\infty[)$. Using Theorem 4, we can prove that all the $\mathcal{F}$-solutions of (22) are parametrized by:

$$\left\{ \begin{array}{l} x_1(t) = (x_1^0 - x_2^0)\,e^{-t} + \xi(t), \\ x_2(t) = \xi(t), \qquad\qquad \forall\, \xi \in \mathcal{F}. \\ u(t) = -(x_1^0 - x_2^0)\,e^{-t} + \dot{\xi}(t) - \xi(t). \end{array} \right. \tag{23}$$

If we substitute (23) into (21), we then obtain a variational problem without differential constraint and computing the corresponding Euler-Lagrange equations, we finally get the following optimal system:

$$\left\{ \begin{array}{l} \ddot{\xi}(t) - 3\,\xi(t) = (x_1^0 - x_2^0)\,e^{-t}, \\ \dot{\xi}(T) - \xi(T) = (x_1^0 - x_2^0)\,e^{-T}, \\ \xi(0) = x_2^0. \end{array} \right. \tag{24}$$

The integration of (24) yields $\xi(t)$ given in (25). Hence, if we substitute ξ into (23), then we obtain

$$\left\{ \begin{array}{lcl} \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right) & = & P(t) \left(\begin{array}{c} x_1^0 - x_2^0 \\ x_2^0 \end{array} \right), \\ u(t) & = & Q(t) \left(\begin{array}{c} x_1^0 - x_2^0 \\ x_2^0 \end{array} \right), \end{array} \right. \tag{26}$$

where $P(t)$ and $Q(t)$ are given in (27) resp. (28). Eliminating the initial conditions $x_1^0 - x_2^0$ and x_2^0 from (26), we finally obtain the optimal controller

$$u(t) = K(t) \left(\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right),$$

where $K(t)$ is given in (29). Finally, we note that if T tends to $+\infty$, we only need the system to be stabilizable and not controllable as it is usually required in the behavioural approach to optimal control. See [11] and the references therein.

The computations given in Theorem 6 have been implemented in the 1-D case in the library OREMODULES. See [2] for more details and examples.

Example 10: Let us consider the following multidimensional linear system:

$$(\partial_1 + 1)\,(\partial_1\, y_1(x) + \partial_2\, y_2(x)) = 0. \tag{30}$$

Let us extremize the cost defined by

$$I = \frac{1}{2} \int (y_1^2(x) + y_2^2(x))\,dx_1\,dx_2, \tag{31}$$

under the differential constraint formed by the system (30).

All $\mathcal{F} = C^\infty(\mathbb{R}^2)$-solutions of (30) are given by

$$\left\{ \begin{array}{l} y_1(x) = \Phi(x_2)\,e^{-x_1} - \partial_2\,\xi(x), \\ y_2(x) = \partial_1\,\xi(x), \end{array} \right. \tag{32}$$

for all $\xi \in \mathcal{F}$ and $\Phi \in C^\infty(\mathbb{R})$. Hence, by substituting (32) into (31), we obtain a variational problem without differential constraint. Euler-Lagrange equations then give:

$$\left\{ \begin{array}{l} \Delta\,\xi(x) = \dot{\Phi}(x_2)\,e^{-x_1}, \\ y_1(x) = \Phi(x_2)\,e^{-x_1} - \partial_2\,\xi(x), \\ y_2(x) = \partial_1\,\xi(x). \end{array} \right.$$

REFERENCES

[1] M. Bergounioux, *Optimisation et contrôle des systèmes linéaires*, Dunod, 2001.

[2] F. Chyzak, A. Quadrat, D. Robertz, OREMODULES project, `http://wwwb.math.rwth-aachen.de/OreModules`.

[3] F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Appl. Algebra Engrg. Comm. Comput.* 16 (2005), pp. 319–376.

[4] H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, Wiley, 1972.

[5] S. Fröhler, U. Oberst, "Continuous time-varying linear systems", *Systems & Control Letters* 35 (1998), pp. 97–110.

[6] L. Landau, E. Lifschitz, *Physique théorique. Tome 2: Théorie des champs*, MIR, fourth edition, 1989.

[7] L. Landau, E. Lifschitz, *Physique théorique. Tome 7: Elasticité*, MIR, second edition, 1990.

[8] B. Malgrange, "Systèmes à coefficients constants", Séminaire Bourbaki 1962/63, pp. 1–11.

[9] J. C. McConnell, J. C. Robson, *Noncommutative Noetherian Rings*, American Mathematical Society, 2000.

[10] U. Oberst, "Multidimensional constant linear systems", *Acta Appl. Math.* 20 (1990), pp. 1–175.

[11] H. K. Pillai, S. Shankar, "A behavioural approach to control of distributed systems", *SIAM Journal on Control and Optimization* 37 (1999), pp. 388–408.

$$\xi(t) = -\frac{1}{2}\,\frac{(-2\,e^{-\sqrt{3}\,t} + \sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + e^{-t-2\,\sqrt{3}\,T} + 2\,e^{-t} - e^{-t}\,\sqrt{3})}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}}\,(x_1^0 - x_2^0) - \frac{1}{2}\,\frac{(-4\,e^{-\sqrt{3}\,t} - 2\,e^{\sqrt{3}\,(t-2\,T)} + 2\,\sqrt{3}\,e^{-\sqrt{3}\,t})}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}}\,x_2^0. \tag{25}$$

$$P(t) = \frac{1}{2}\begin{pmatrix} \frac{2\,e^{-t} - e^{-t}\,\sqrt{3} + 2\,e^{-\sqrt{3}\,t} - \sqrt{3}\,e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,(t-2\,T)} + e^{-t-2\,\sqrt{3}\,T}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} & \frac{2\,e^{\sqrt{3}\,(t-2\,T)} + 4\,e^{-\sqrt{3}\,t} - 2\,\sqrt{3}\,e^{-\sqrt{3}\,t}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} \\ \frac{2\,e^{-\sqrt{3}\,t} - \sqrt{3}\,e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,(t-2\,T)} - e^{-t-2\,\sqrt{3}\,T} - 2\,e^{-t} + e^{-t}\,\sqrt{3}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} & \frac{4\,e^{-\sqrt{3}\,t} + 2\,e^{\sqrt{3}\,(t-2\,T)} - 2\,\sqrt{3}\,e^{-\sqrt{3}\,t}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} \end{pmatrix}, \tag{27}$$

$$Q(t) = \frac{1}{2}\begin{pmatrix} \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} & \\ & \frac{-2\,\sqrt{3}\,e^{-\sqrt{3}\,t} - 2\,e^{\sqrt{3}\,(t-2\,T)} + 2\,e^{-\sqrt{3}\,t} + 2\,\sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)}}{e^{-2\,\sqrt{3}\,T} + 2 - \sqrt{3}} \end{pmatrix}. \tag{28}$$

$$K(t) = Q(t)\,P(t)^{-1} = -\frac{1}{2}\begin{pmatrix} \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{-2\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{-\sqrt{3}\,t}} & \\ & \frac{-\sqrt{3}\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{\sqrt{3}\,(t-2\,T)} + e^{-\sqrt{3}\,t}}{-2\,e^{-\sqrt{3}\,t} - e^{\sqrt{3}\,(t-2\,T)} + \sqrt{3}\,e^{-\sqrt{3}\,t}} \end{pmatrix}. \tag{29}$$

[12] H. K. Pillai, J. C. Willems, "Lossless and dissipative distributed systems", *SIAM Journal of Control and Optimization* 40 (2002), pp. 1406–1430.

[13] J. W. Polderman, J. C. Willems, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, TAM 26, Springer, 1998.

[14] J.-F. Pommaret, *Partial Differential Control Theory*, Kluwer, 2001.

[15] J.-F. Pommaret, A. Quadrat, "Generalized Bezout Identity", *Appl. Algebra Engrg. Comm. Comput.* 9 (1998), pp. 91–116.

[16] J.-F. Pommaret, A. Quadrat, "Localization and parametrization of linear multidimensional control systems", *Systems & Control Letters* 37 (1999), pp. 247–260.

[17] J.-F. Pommaret, A. Quadrat, "Algebraic analysis of linear multidimensional control systems", *IMA J. Math. Control and Inform.* 16 (1999), pp. 275–297.

[18] J.-F. Pommaret, A. Quadrat, "A functorial approach to the behaviour of multidimensional control systems", *Int. J. Appl. Math. Comput. Sci.* 13 (2003), pp. 7–13.

[19] J.-F. Pommaret, A. Quadrat, "A differential operator approach to multidimensional optimal control", *Int. J. Control* 77 (2004), pp. 821–836.

[20] A. Quadrat, *Analyse algébrique des systèmes de contrôle linéaires multidimensionnels*, PhD Thesis, Ecole Nationale des Ponts et Chaussées, 23/09/1999.

[21] A. Quadrat, D. Robertz, "Parametrizing all solutions of uncontrollable multidimensional linear systems", *Proceedings of the* 16^{th} *IFAC World Congress*, Prague (Czech Republic) (04-08/07/05).

[22] A. Quadrat, D. Robertz, "Constructive computation of flat outputs of a class of multidimensional linear systems with variable coefficients", to appear in Proceedings of MTNS06, Kyoto (Japan) (24-28/07/06).

[23] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.

[24] J. Wood, "Modules and behaviours in nD systems theory", *Multidimens. Systems Signal Process.* 11 (2000), pp. 11–48.

[25] P. Zervos, *Le problème de Monge*, Mémorial des sciences mathématiques 53, Gauthier-Villars, 1932.

[26] E. Zerz, "An algebraic analysis approach to linear time-varying systems", *IMA J. Math. Control Inform.* 23 (2006), pp. 113–126.