Extremal Lattices and Hilbert Modular Forms

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Introduction

Dense Sphere Packings, Extremal Lattices, and Modular Forms
The problem of the densest sphere packing is an interesting problem of both geometry and number theory.

The densest way to cover the plane with equal-radius circles is to put six circles around one circle, and then continue in the same way.

Figure 1: Cover of the plane with circles.

In three dimensions, the analogous problem is to pack as many equal-sized spheres as possible in the space. A configuration is called a *sphere packing*. In 1611 Johannes Keppler claimed in his paper "Strena seu de nive sexangula" ("On the six-cornered snowflake", [Kep11]) that the sphere packing A_3 is the densest. It took a long time to prove Kepler's conjecture. Finally at the beginning of the millennium, Thomas Hales [Hal05] gave a computational proof and Hales et al. [Hal$^+$15] gave a formal proof.

Figure 2: Kepler or A_3-packing.

In higher dimensions, little is known about dense packings of generalized spheres. Recently, Maryna Viazovska [Via16] proved that the $\mathbb{E}_8$-packing is the densest in

8 dimensions, and Cohn et al. [Coh$^+$16] proved that the Leech packing is the densest in 24 dimensions.

All these packings are in some sense regular. In each case, the centers of the circles, which determine the packing, form a $\mathbb{Z}$-lattice (i.e. a $\mathbb{Z}$-span of a basis). Such packings defined by lattices are called *lattice packings*. Many of the best known packings are lattice packings.

If L is a lattice, then its theta series

$$\Theta_L(q) = \sum_{m=0}^{\infty} |L(m)|\, q^m \ \in \mathbb{C}[[q]]$$

is the generating function of the "layers" $L(m) = \{\ell \in L \mid (\ell, \ell) = 2m\}$ of L. A fascinating property of theta series is that

$$\mathbb{H} \to \mathbb{C},\ z \mapsto \Theta_L\left(e^{2\pi i\, z}\right)$$

is an elliptic modular form of level one if L is even and unimodular (i.e. $(\ell, \ell) \in 2\mathbb{Z}$ for all $\ell \in L$ and $L^\# = L$). Since much is known about elliptic modular forms, this is a very useful property. Carl Ludwig Siegel [Sie69] showed for instance that the minimum of L satisfies

$$\min L := \min\{m > 0 \mid L(m) \neq \emptyset\} \leq 1 + \lfloor \tfrac{\dim L}{24} \rfloor.$$

Lattices achieving this bound are called *extremal*. Especially interesting are extremal lattices in the "jump dimensions" 24, 48, 72, etc. There are only finitely many isometry classes of extremal lattices of each dimension, because there are only finitely many isometry classes of unimodular lattices (see for instance [O'M63]). In many dimensions no extremal lattice is known, not to mention then classification of all extremal lattices.

Table 1: Number of known extremal even unimodular lattices.

dim.	8	16	24[1]	32[2]	...	48[3]	...	72[4]	80[5]	...	$\geq 163\,264$[6]
#	1	2	1	$\geq 10^7$	...	≥ 4	...	≥ 1	≥ 4	?	0

Similar to unimodular lattices are p-modular lattices (i.e. $\sqrt{p}L^\# \cong L$), where $p + 1$ divides 24. Heinz-Georg Quebbemann studied p-modular lattices in [Que95] and gave similar bounds to the minima as the bound above. Modular lattices achieving the bound in question are called *extremal*. Important examples are the 2-modular

[1] [Nie73]
[2] [Kin03]
[3] [ConSlo99, p. 195], [Neb98a], and [Neb14]
[4] [Neb12]
[5] [BacNeb98], [SteWat10] and [Wat12]
[6] The theta series of an extremal lattice (i.e. an extremal modular form) would have some negative coefficients, which is impossible, cf. [JenRou11]

Barnes-Wall lattice in dimension 16 and the 3-modular Coxeter-Todd lattice in dimension 12. They each define the densest known sphere packing in dimension 16 or 12.

Lattices over Real Quadratic Number Fields and Hilbert Modular Forms
In this thesis we look at lattices over real quadratic number fields F. The theory of lattices over totally real number fields is similar to the theory over the rationals. But one cannot give an estimate for the minimum of a lattice, because the notion of a minimum depends on the ordering of F. A general number field does not have a canonical total ordering, so we will define a total ordering $\leq_A$ which is suitable and sensible for our applications:

$$\mu \leq_A \nu :\Longleftrightarrow (\mathrm{tr}(\alpha_1\mu), \mathrm{tr}(\alpha_2\mu)) \leq (\mathrm{tr}(\alpha_1\nu), \mathrm{tr}(\alpha_2\nu)),$$

where $A = (\alpha_1, \alpha_2) \in F^2$ is a $\mathbb{Q}$-basis of F, α_1 and α_2 are totally positive and $\leq$ is the lexicographic ordering. The ambition of this thesis is to find and classify lattices whose minimum with respect to $\leq_A$ is extremal, by which we mean that they meet the bound obtained from the theory of Hilbert modular forms (see below).

In a similar manner to the classical case, one can define the theta series of a lattice. If Λ is a lattice, then

$$\Theta_\Lambda(q_1, q_2) = \sum_{n,m \in \mathbb{Z}_{\geq 0}} a_{n,m}\, q_1^n q_2^m \ \in \mathbb{C}[[q_1, q_2]]$$

is called the *theta series* of Λ, where

$$a_{n,m} = |\{\lambda \in \Lambda \mid \mathrm{tr}(\alpha_1(\lambda, \lambda)) = 2n \text{ and } \mathrm{tr}(\alpha_2(\lambda, \lambda)) = 2m\}|.$$

Since $\mathbb{Z}_F$ is a free $\mathbb{Z}$-module, Λ is also a $\mathbb{Z}$-lattice with both the forms

$$(\ ,\ )_1 := \mathrm{tr}(\alpha_1(\ ,\ )) \text{ and } (\ ,\ )_2 := \mathrm{tr}(\alpha_2(\ ,\ )),$$

where $A = (\alpha_1, \alpha_2)$ is as before. These two lattices, denoted by Λ_1 and Λ_2, are called first and second trace lattices of Λ, and Λ is uniquely described by them. The theta series of first and second trace lattice are $\Theta_L(q_1, 1)$ and $\Theta_L(1, q_2)$, respectively. So the theta series of Λ is the "merged" theta series of the trace lattices.

In some cases (if Λ is even and unimodular or trace even and trace unimodular[7]), the theta series is a modular form of two variables. First, David Hilbert thought about a generalization of modular forms from one to two variables by using real quadratic number fields. His ideas were further developed by Otto Blumenthal[8] and published in his Habilitationsschrift in 1901 [Blu03, Blu04]. One can define, in general, modular forms over totally real number fields in the same way. These modular forms are referred to as Hilbert (Blumenthal) modular forms.

[7]Definition 1.4.
[8]Professor at RWTH Aachen University 1905-1933.

This thesis develops a method to describe a Hilbert modular form as a "merger" of two elliptic modular forms of different levels. We can write a Hilbert modular form f in a q-expansion:[9]

$$f(q_1, q_2) = \sum_{n,m \in \mathbb{Z}_{\geq 0}} a_{n,m}(f)\, q_1^n q_2^m \ \in \mathbb{C}[[q_1, q_2]].$$

Then $f(q_1, 1)$ and $f(1, q_2)$ are Fourier expansions of modular forms of one variable and different levels.

Additionally, I developed an algorithm which computes the q-expansion of Hecke eigenforms for fields with narrow class number one.

Most importantly, the lexicographic ordering on $\mathbb{Z}^2_{\geq 0}$ defines an ordering of the coefficients $a_{n,m}(f)$. So we can define extremal Hilbert modular forms, i.e. forms such that the order of 1 at ∞ is as large as possible. Then lattices whose theta series are extremal are called extremal lattices. These lattices have an extremal minimum among all lattices of the same dimension, where the minimum is taken with respect to $\leq_A$.

We construct some extremal lattices. Their trace lattices often have interesting structures. For instance, the trace lattices of an extremal lattice of dimension 8 over $\mathbb{Q}[\sqrt{2}]$ are $\mathbb{E}_8 \perp \mathbb{E}_8$ and the Barnes-Wall lattice BW_{16}.

For computations, we often restrict to fields with class number one, and especially to the fields

$$\mathbb{Q}[\sqrt{5}],\ \mathbb{Q}[\sqrt{2}],\ \text{and}\ \mathbb{Q}[\sqrt{3}].$$ [10]

Table 2: Some extremal lattices.

Field	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{Q}[\sqrt{2}]$	$\mathbb{Q}[\sqrt{3}]$[11]	
Dim. 2	-	-	1	-
4	1	1	2	1
6	-	-	1	-
8	2	1	3	≥ 1
10	-	-	21	-
12	1	5		≥ 1
16	≥ 2	≥ 1	0	0
20		≥ 1	0	0
24	≥ 1	≥ 1	0	0

[9] Definition 3.21.
[10] Ordered by their discriminant.
[11] There are two different Types of extremal lattices, see Chapter 2.

Spherical Theta Series, Spherical Designs, and Classifications
Additionally, one can define spherical theta series. If L is an even unimodular lattice over the rationals and P is a harmonic homogeneous polynomial, then

$$\theta_{L,P}(q) = \sum_{\ell \in L} P(\ell) q^{(\ell,\ell)/2}$$

is an elliptic modular form. Analogously, if Λ is an even unimodular or trace even unimodular lattice over a real quadratic number field, then

$$\Theta_{\Lambda,P}(q_1, q_2) = \sum_{\lambda \in \Lambda} \operatorname{Norm}(P(\lambda))\, q_1^{(\lambda,\lambda)_1/2} q_2^{(\lambda,\lambda)_2/2}$$

is a Hilbert modular form.

In some cases, one can even classify all isometry classes of unimodular lattices in a given dimension. So Venkov classified in [ConSlo99, Chapter 18] all 24 even unimodular lattices in dimension 24 (The Niemeier lattices) – by using spherical theta series. Venkov's proof is more sophisticated than Niemeier's original computing-intensive proof (by Kneser's neighbor method).

Furthermore, Bachoc and Venkov developed the method to classify extremal unimodular or p-modular lattices or to show non-existence, see [BacVen01]. Their method uses spherical theta series and modular forms, but no heavy computer calculations.

In this thesis I extend the Bachoc-Venkov method to real quadratic number fields. To an extremal lattice Λ and a vector $\lambda \in \Lambda$ of given length I give a method to compute the so-called configuration numbers

$$n(\beta, \iota; \lambda) := |\{\mu \in \Lambda \mid (\mu,\mu) = \beta,\ (\mu,\lambda) = \iota\ \}|\,.$$

We use knowledge about Hilbert modular forms, but no knowledge about the concrete lattice. Often we can classify all lattices which have given configuration numbers.

For instance, we will see that the mentioned extremal lattice over $\mathbb{Q}[\sqrt{2}]$ of dimension 8 is up to isometry the only extremal lattice over $\mathbb{Q}[\sqrt{2}]$ of dimension 8. This was already shown by John Hsia and David Hung in [HsiHun89] by extensive case analysis (Kneser's neighbor method and Siegel's mass formula).

Another application of the Bachoc-Venkov method is that one finds many interesting spherical t-designs as layers of extremal lattices. We observe the same for layers of extremal lattices over number fields. For example, the minimal vectors of the even unimodular lattice over $\mathbb{Q}[\sqrt{5}]$ of dimension 4, which first and second trace lattices are $\mathbb{E}_8$ and H_4, form a spherical 11-design in $\mathbb{R}^4$. This design is the best 11-design in $\mathbb{R}^4$, belonging to the so-called 600-cell, see [BoyDan01].

The minimal vectors of the extremal lattice of dimension 8 over $\mathbb{Q}[\sqrt{2}]$ form a spherical 7-design.

Computations

Another way to tackle constructions or even classifications of lattices is to use computer calculations. First and foremost, there is the famous Kneser neighbor algorithm, see [Kne57], [Kne02], [Sch98], or [Kir14] (the last two cover Kneser's method over algebraic number fields). The algorithm lists all isometry classes in one genus. Therefore one can enumerate all unimodular lattices in principle, but this often takes a lot of computer time and is not feasible in higher dimensions.

Alternatively, one may want to find all (unimodular) lattices over a real quadratic number field which have a common fixed trace lattice. I explore two ways to do that.

First, a real quadratic number field F is a subfield of the $d_F th$ cyclotomic number field, where d_F is the field discriminant. So each automorphism of the trace lattice which has the $d_F th$ cyclotomic polynomial as minimal polynomial defines a lattice over F.

Secondly, I give an algorithm which lists all lattices over $\mathbb{Q}[\sqrt{D}]$ to a fixed D-modular trace lattice, where $D > 1$ is square-free.

These computational approaches are useful to partly answer the classification question, and they often answer the question completely in small dimensions and for small field discriminants. For higher dimensions or higher field discriminants, they fail in practice.

The method using modular forms has the potential to classify (extremal unimodular) lattices in higher dimensions, because it does not require intensive computations. Instead one uses the knowledge of the ring of Hilbert modular forms and the Fourier coefficients of modular forms. A problem is that one has to generate many equations from modular forms. If there are many forms, then this is difficult – if not impossible. Unfortunately, this seems often to be the case for real quadratic number fields.

Outline

In Chapter 1 lattices over totally real number fields are introduced. We give basic definitions, define the minimum of lattices, and two types of dualities, e.g. unimodular and trace unimodular lattices. Last, we introduce trace lattices, which are lattices over the rationals associated to a lattice over the number field by using the field trace.

In Chapter 2 and thereafter we consider just real quadratic number fields. We define Galois and fundamentally invariant lattices and prove a result about the modularity of trace lattices. Further we define three types of lattices which will be investigated further. We prove estimates of lattices over the fields $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$. As short excursus, we summarize the theories of roots and genera of lattices over real quadratic fields with class number one.

The theory of Hilbert modular forms is summarized in Chapter 3. To minimize notations, we just consider the case of real quadratic number fields. Further we recall

the structures of the rings of Hilbert modular forms over $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$. The theory of Hecke operators (in the case that the narrow class number is one) is summarized to give a method to compute Fourier coefficients of Hilbert modular forms. Some algorithms using Hecke eigenvalues are given in Appendix A. Last, we define
q-expansions of Hilbert modular forms in a general way. These q-expansions order the Fourier coefficients of Hilbert modular forms, so this is a way to do computations with Hilbert modular forms. Properties of q-expansions, especially the notion of extremal Hilbert modular forms, and the q-expansions of the forms over $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$ are given.

Chapter 4 ties together the two concepts by defining the theta series of a lattice over a real quadratic number field and proving that the theta series is a Hilbert modular form. Actually, three types of theta series are defined, according to the three types of lattices defined in Chapter 2. Moreover q-expansions of theta series are explored, and we define the central notion of this thesis, extremal lattices over real quadratic number fields.

In Chapter 5 we extend the theory of theta series to spherical theta series. These are theta series whose coefficients come from norms of values of harmonic polynomials. For that we define harmonic polynomials, and for applications we define Gegenbauer polynomials and spherical designs. We discuss spherical theta series in great detail. Finally we use them to extend the Bachoc-Venkov method to compute configuration numbers of extremal lattices.

The Chapters 6 to 9 deal with concrete fields. We construct and classify

Chapter 6: extremal Type (i) lattices over $\mathbb{Q}[\sqrt{5}]$,
Chapter 7: extremal Type (i) lattices over $\mathbb{Q}[\sqrt{2}]$,
Chapter 8: extremal Type (ii) lattices over $\mathbb{Q}[\sqrt{3}]$,
Chapter 9: extremal fundamentally invariant[12] Type (iii) lattices over $\mathbb{Q}[\sqrt{3}]$.

Finally, in Chapter 10 we give short descriptions of the two computational methods to find lattices over real quadratic number field to a fixed trace lattice.

The Algorithm to compute q-expansions of Hecke eigenforms, some extremal Hilbert modular forms, and extremal lattices are listed in the Appendix. They are also available on my website

http://www.math.rwth-aachen.de/~David.Dursthoff/.

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[12]Definition 1.4

Notation

Real number fields Throughout this thesis F is a totally real number field, i.e. a finite field extension of the rational field such that all embeddings into the complex field are real-valued. The degree of F over $\mathbb{Q}$ will be denoted by r. The r real embeddings $\sigma_1, \ldots, \sigma_r : F \hookrightarrow \mathbb{R}$ are ordered in some fixed way. If $\alpha \in F$, we write $\alpha^{(j)} := \sigma_j(\alpha)$ for $j = 1, \ldots, r$. Usually, we fix an ordering of the embeddings and identify α with $\alpha^{(1)}$.

The norm and trace of F are

$$\mathcal{N} : F \to \mathbb{Q}, \alpha \mapsto \prod_{j=1}^{r} \alpha^{(j)} \text{ and } \mathrm{tr} : F \to \mathbb{Q}, \alpha \mapsto \sum_{j=1}^{r} \alpha^{(j)}.$$

An element $\alpha \in F$ is called totally positive (notation: $\alpha \gg 0$) if $\alpha^{(j)} > 0$ for all $j = 1, \ldots, r$. We write $\alpha \gtrless 0$ if $\alpha^{(j)} > 0$ for $j = 1, \ldots, r-1$ and $\alpha^{(r)} < 0$.

The ring of integer elements of F is

$$\mathbb{Z}_F = \{\alpha \in F \mid p(\alpha) = 0 \text{ for a monic polynomial } p \in \mathbb{Z}[t]\}.$$

The unit group of $\mathbb{Z}_F$ is isomorphic to $\{\pm 1\} \times \mathbb{Z}^{r-1}$. The generators besides -1 are called fundamental units. We fix a fundamental unit ε_0, with $\varepsilon_0^{(1)}, \ldots, \varepsilon_0^{(r-1)} > 1$ (by Dirichlet's unit theorem there is such a unit).

An ideal of $\mathbb{Z}_F$ which is generated by $S \subseteq \mathbb{Z}_F$ is denoted by (S), or if $S = \{\alpha\}$, by $\alpha\mathbb{Z}_F$. The norm of an ideal $\mathfrak{a} \subseteq \mathbb{Z}_F$ is $\mathcal{N}(\mathfrak{a}) = [\mathbb{Z}_F : \mathfrak{a}]$. A fractional ideal is a finitely generated $\mathbb{Z}_F$-submodule of F. The fractional ideal generated by $S \subseteq F$ is denoted by (S). The fractional ideals form the *ideal group* of F.

The class group CL_F of F is the quotient group of all fractional ideals modulo the principal ideals. Its order $h_F := |CL_F|$ is always finite and called the class number of F. The narrow class group CL_F^+ is the quotient group of all fractional ideals modulo the principal ideals which are generated by a totally positive field element. Its order is the narrow class number $h_F^+ = |CL_F^+|$.

The inverse different is the fractional ideal

$$\mathbb{Z}_F^{\#} = \{\alpha \in F \mid \mathrm{tr}(\alpha\mu) \in \mathbb{Z} \text{ for all } \mu \in \mathbb{Z}_F\}.$$

The different ideal is $(\mathbb{Z}_F^{\#})^{-1} \subseteq \mathbb{Z}_F$. Its norm is the *discriminant* d_F of F. If the different ideal is generated by one element, the generator will always be denoted by δ. If possible, we will assume that δ is totally positive.

Often we will give concrete computations and formulas only for real quadratic number fields with class number one. So $F = \mathbb{Q}[\sqrt{D}]$ is the number field, where D is a square-free integer greater than one. The number D is related to the discriminant,

$$d_F = \begin{cases} D & \text{if } D \equiv 1 \pmod 4 \\ 4D & \text{if } D \equiv 2, 3 \pmod 4. \end{cases}$$

Also, we write $e := \sqrt{\frac{d_F}{D}} \in \{1, 2\}$. If $h_F = h_F^+ = 1$, then we may choose ε_0 and δ in such a way that $\delta/e = \sqrt{D}\varepsilon_0$.

Complex numbers and functions The upper half plane of the complex plane is

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

For $z \in \mathbb{C}^r$, $z_j \neq 0$ for all $j = 1, \ldots, r$, and $k \in \mathbb{Z}^r$ let $z^k := \prod_{j=1}^r z_j^{k_j}$.

The multiplicative and additive groups of F act on $\mathbb{C}^r$ via the r embeddings. If $\alpha \in F$ and $z \in \mathbb{C}^r$ let $\alpha z := (\alpha^{(1)} z_1, \ldots, \alpha^{(r)} z_r)$ and $z + \alpha = (z_1 + \alpha^{(1)}, \ldots, z_r + \alpha^{(r)})$. An $\alpha \in F^*$ preserves $\mathbb{H}^r$ by multiplication if and only if $\alpha \gg 0$. We define the $\mathbb{C}-$linear analogue of the field trace:

$$\mathrm{Tr} : \mathbb{C}^n \to \mathbb{C},\ z \mapsto \sum_{j=1}^r z_j.$$

With these notations, for example, it makes perfect sense to write α^k instead of $\prod_{j=1}^r \alpha^{(j) k_j}$ for $\alpha \in F \backslash \{0\}$ and $k \in \mathbb{Z}^r$. If there is no danger of confusion, we may also write α^k instead of $\mathcal{N}(\alpha)^k$ for $\alpha \in F$ and $k \in \mathbb{Z}$.

ζ_F is the Dedekind zeta function of F. the value of $s \in \mathbb{C}$ with real part $Re(s) > 1$ is

$$\zeta_F(s) = \sum_{\mathfrak{n} \subseteq \mathbb{Z}_F \text{ ideal}} \mathcal{N}(\mathfrak{n})^{-s}.$$

So ζ_F can be extended to a meromorphic function of the whole complex plane.

Lattices over the integers A $\mathbb{Z}$-lattice $(L, \mathfrak{q})$ is a free $\mathbb{Z}-$module $L \subseteq \mathbb{Q}^N$ of rank N together with a positive definite quadratic form $\mathfrak{q} : \mathbb{Q}^N \to \mathbb{Q}$. So all our lattices have full rank. The quadratic form $\mathfrak{q}$ has an associated bilinear form or scalar product $\mathfrak{b}$ or $(\ ,\ )$, i.e. $\mathfrak{b}(\lambda, \mu) := (\lambda, \mu) := \mathfrak{q}(\lambda + \mu) - \mathfrak{q}(\lambda) - \mathfrak{q}(\mu)$. Often we write L instead of $(L, \mathfrak{q})$. The Gram matrix of L is $\mathcal{G}(L)$, assuming that a single lattice basis is chosen. The determinant of L is $\det L := |\det \mathcal{G}(L)|$. We also call $\min L := \min\{\mathfrak{q}(\lambda) \mid \lambda \in L \backslash \{0\}\}$ the minimum of the lattice L.

An isomorphism $f : \mathbb{Q}^N \to \mathbb{Q}^N$ is called a *similarity* of norm $\ell \in \mathbb{N}$ if for all $x, y \in \mathbb{Q}^N$:

$$\mathfrak{b}(f(x), f(y)) = \ell \mathfrak{b}(x, y).$$

A lattice L is called *ℓ-modular* if there is a similarity f of norm ℓ with $f(L^{\#}) = L$.

The theta series of L is

$$\Theta_L(z) = \sum_{\lambda \in L} q^{\mathfrak{q}(\lambda)},$$

where $q := \exp(2\pi i z)$ and $z \in \mathbb{H}$. It is the generating function of the "layers"

$$L(m) = |\{\lambda \in L \mid \mathfrak{q}(\lambda) = m\}|$$

of L.

Special lattices like the $\mathbb{E}_8$-lattice, the Coxeter-Todd lattice, the Barnes-Wall lattice, and the Leech lattice are referred to by $\mathbb{E}_8$, K_{12}, BW_{16}, and Λ_{24}, respectively. Other symbols are used for different lattices. If not otherwise stated, they refer to the names in the Lattice Data Base [NebSlo].

Chapter 1

Lattices over Totally Real Number Fields

1.1 Basic Definitions

First, we develop the basic notions for lattices over number fields. A reference is [O'M63].

Let F be a totally real number field of degree r.

(i) Let V be a vector space over F of dimension $n \in \mathbb{N}$. We will often identify V with F^n, i.e. choose a fixed basis.

(ii) A *quadratic form* is a map $Q : V \to F$ such that $Q(\alpha\lambda) = \alpha^2 Q(\lambda)$ for all $\alpha \in F$ and $\lambda \in V$, and Q defines a symmetric bilinear form

$$B : V \times V \to F,\ (\lambda, \mu) \mapsto Q(\lambda + \mu) - Q(\lambda) - Q(\mu).$$

This form is called the polar form of Q.

(iii) We call Q *totally positive definite* if $Q(\lambda) \gg 0$ for all $\lambda \in V\backslash\{0\}$.

(iv) Let $E = (e_1, \ldots, e_n)$ be a basis of V. The Gram matrix of Q with respect to E is $(B(e_i, e_j))_{i,j} \in F^{n\times n}$.

(v) A *lattice* over F or a $\mathbb{Z}_F$*-lattice* is a pair (Λ, Q) of a finitely generated $\mathbb{Z}_F$-module $\Lambda \subset F^n$ which contains an F-basis of V and a totally positive definite quadratic form $Q : V \to F$. Often we omit the quadratic form and write Λ instead of (Λ, Q).

(vi) If Λ is a free $\mathbb{Z}_F$-module (which is always the case if $h_F = 1$), then we call a basis of V which generates Λ as a $\mathbb{Z}_F$-module a *lattice basis* of Λ.

(vii) The rank of Λ is defined as the dimension of V.

(viii) If Λ is free, then the *determinant* of Λ (notation $\det\Lambda$) is the determinant of any Gram matrix. The determinant is unique up to multiplication by squares of units.

(ix) An isometry between two lattices (Λ, Q) and (Λ', Q') is an isomorphism $f : \Lambda \to \Lambda'$ with $Q'(f(\lambda)) = Q(\lambda)$ for all $\lambda \in \Lambda$. If there is an isometry, the lattices are called isometric (notation: $(\Lambda, Q) \cong (\Lambda', Q')$). An *automorphism* of (Λ, Q) is an isometry $(\Lambda, Q) \to (\Lambda, Q)$. The group of all automorphisms of (Λ, Q) is $\mathrm{Aut}(\Lambda, Q)$.

(x) For $\lambda \in V$ the value $Q(\lambda)$ is called the norm of λ and $B(\lambda, \lambda)$ is called the square length of λ. The trivial case $V = F$ is generally not considered, so this notation should not cause any confusion.

(xi) Let $\alpha \in F$ and Λ be a lattice. We call

$$\Lambda(\alpha) := \{\lambda \in \Lambda \mid Q(\lambda) = \alpha\}.$$

the *layer* of Λ at α.

1.2 The Minimum

The minimum of a $\mathbb{Z}$-lattice is the minimal norm of lattice vectors besides zero. Over a number field, one does not have an obvious ordering. We use the following total ordering on a number field F.

Definition 1.1 (A Total Ordering on F)
Let $\leq$ be the lexicographic ordering on $\mathbb{Q}^r$, $A = (\alpha_1, \ldots, \alpha_r) \in F^r$ be a $\mathbb{Q}$-basis of F, and let

$$\varphi_A : F \to \mathbb{Q}^r,\ \nu \mapsto (\mathrm{tr}(\alpha_1\nu), \ldots, \mathrm{tr}(\alpha_r\nu))^{\mathrm{tr}}.$$

We define the ordering $\leq_A$ on F:

$$\mu \leq_A \nu \iff \varphi_A(\mu) \leq \varphi_A(\nu).$$

Proposition 1.2 *φ_A is bijective. Especially, $\leq_A$ is a total ordering.*

Proof. Let $\nu \in F$. There are $\nu_1, \ldots \nu_r \in \mathbb{Q}$ such that $\nu = \sum_{i=1}^r \nu_i\alpha_i$. Then

$$\varphi_A(\nu) = (\mathrm{tr}(\alpha_i\alpha_j))_{i,j}\,(\nu_1, \ldots, \nu_r)^{\mathrm{tr}}.$$

The trace bilinear form of F over $\mathbb{Q}$ is non-degenerated, so the Gram matrix $(\mathrm{tr}(\alpha_i\alpha_j))_{i,j}$ is non-singular. Hence φ is bijective.

Since the lexicographic ordering on $\mathbb{Q}^r$ is total, $\leq_A$ is a total ordering. $\square$

Of course, instead of the lexicographic ordering one may use any other total ordering of $\mathbb{Q}^r$. We use the minimum with respect to $\leq_A$ to define the minimum of a lattice. We will specify A in sensible way in the next chapter.

Definition 1.3 (The Minimum of a Lattice)
Let (Λ, Q) be lattice of rank n and $A = (\alpha_1, \dots, \alpha_r) \in F^r$ be a $\mathbb{Q}$-basis of F such that $\alpha_1, \dots, \alpha_r \gg 0$.

*The A-*minimum *of Λ (notation $\min_A(\Lambda)$ or $\min(\Lambda)$) is the $\leq_A$-minimum of $\{Q(\lambda) \mid \lambda \in \Lambda \backslash \{0\}\}$. We call*

$$\mathrm{Min}(\Lambda) := \mathrm{Min}_A(\Lambda) := \{\lambda \in \Lambda \mid Q(\lambda) = \min \Lambda\}$$

*the A-*minimal vectors *of Λ.*

1.3 Duality of Lattices

The dual of a $\mathbb{Z}$-lattice is the lattice generated by the dual basis of any lattice basis. Over number fields one can think of two different dualities, see also [O'M63] and [Ebe02].

Definition 1.4 *Let (Λ, Q) be a lattice in the vector space V.*

(i) *We call (Λ, Q)* integral *if $B(\lambda, \mu) \in \mathbb{Z}_F$ for all $\lambda, \mu \in \Lambda$ and* even *if $Q(\lambda) \in \mathbb{Z}_F$ for all $\lambda \in \Lambda$.*

(ii) *The* dual lattice *of (Λ, Q) is*

$$(\Lambda, Q)^{\#} := \Lambda^{\#} := \{x \in V \mid B(x, \lambda) \in \mathbb{Z}_F \text{ for all } \lambda \in \Lambda\}.$$

We call (Λ, Q) unimodular *if $(\Lambda, Q)^{\#} = (\Lambda, Q)$.*

(iii) *We call (Λ, Q)* trace integral *if $\mathrm{tr}(B(\lambda, \mu)) \in \mathbb{Z}$ for all $\lambda, \mu \in \Lambda$ and* trace even *if $\mathrm{tr}(\mu Q(\lambda)) \subseteq \mathbb{Z}$ for all $\lambda \in \Lambda$ and $\mu \in \mathbb{Z}_F$.*

(iv) *The* trace dual lattice *of (Λ, Q) is*

$$(\Lambda, Q)^* := \Lambda^* := \{x \in V \mid \mathrm{tr}(B(x, \lambda)) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

We call trace unimodular *if $(\Lambda, Q)^* = (\Lambda, Q)$. If additionally (Λ, Q) is trace even, we call (Λ, Q)* trace even unimodular*.*

The first definition is a direct transfer of the rational concept. On the other hand, the second one uses the structure of the number field over the rationals.

Lemma 1.5 *Let (Λ, Q) be a lattice.*

(i) (Λ, Q) is integral if and only if $(\Lambda, Q) \subseteq (\Lambda, Q)^{\#}$.

(ii) (Λ, Q) is trace integral if and only if $(\Lambda, Q) \subseteq (\Lambda, Q)^*$.

(iii) $(\Lambda, Q)^{\#}$ and $(\Lambda, Q)^*$ are lattices.

Proof. (i) and (ii) are clear by definition.

(iii). Without loss of generality, Λ is integral. Clearly $\Lambda^{\#}$ is a torsion-free $\mathbb{Z}_F$-module. Let $E \subseteq \Lambda$ be a basis, and let $E^{\#}$ be its dual basis. Then $\Lambda \subseteq \Lambda^{\#} \subseteq \langle E^{\#} \rangle$. So $\Lambda^{\#}$ has the same rank as Λ and $\langle E \rangle$ and hence is a lattice. One shows analogously that Λ^* is a lattice. □

For some fields, duality and trace duality are similar.

Proposition 1.6 *Let (Λ, Q) be a lattice. Then $\Lambda^* = \mathbb{Z}_F^{\#} \Lambda^{\#}$.*

Assume that $\mathbb{Z}_F^{\#} = \delta^{-1}\mathbb{Z}_F$ for some $\delta \gg 0$. Let $\delta^{-1}Q$ be the quadratic form defined via $(\delta^{-1}Q)(\lambda) := \delta^{-1} \cdot Q(\lambda)$. Then (Λ, Q) is unimodular if and only if $(\Lambda, \delta^{-1}Q)$ is trace unimodular.

If the inverse different has no total positive generator, then the rescaling is not possible. Thus unimodularity and trace unimodularity are really not similar.

Proof. Let $x \in \Lambda^{\#}$. Then $B(x, \Lambda) \subseteq \mathbb{Z}_F$. Since $\mathrm{tr}(\mathbb{Z}_F^{\#} B(x, \Lambda)) \subseteq \mathbb{Z}$, we have $\mathbb{Z}_F^* x \subseteq \Lambda^*$.

For the other direction let $x \in \Lambda^*$. Hence $\mathrm{tr}(B(x, \Lambda)) \subseteq \mathbb{Z}$ and thus

$$\mathrm{tr}(\mathbb{Z}_F B(x, \Lambda)) \subseteq \mathbb{Z}.$$

So $B(x, \Lambda) \subseteq \mathbb{Z}_F^{\#}$ and hence $B(\alpha x, \Lambda) \subseteq \mathbb{Z}_F$ for all $\alpha \in (\mathbb{Z}_F^{\#})^{-1}$.

If $\mathbb{Z}_F^{\#} = \delta^{-1}\mathbb{Z}_F$, then $(\Lambda, Q)^* = \delta^{-1}(\Lambda, Q)^{\#}$. So $(\Lambda, \delta^{-1}Q)^* = \delta^{-1}(\Lambda, \delta^{-1}Q)^{\#} = (\Lambda, Q)^{\#}$. Hence (Λ, Q) is unimodular if and only if (Λ, Q) is trace unimodular.

□

1.4 Trace lattices

Let (Λ, Q) be a $\mathbb{Z}_F$-lattice of rank n. Then, since $\mathbb{Z}_F$ is a free $\mathbb{Z}$-module of rank $r = [F : \mathbb{Q}]$, also Λ is a free $\mathbb{Z}$-module of rank rn (cf. [Ebe02, Prop. 6.5]).

We will define quadratic forms on Λ over $\mathbb{Q}$, parameterized by $\alpha \in F_{\gg 0}$. So we will get lattices L_α over $\mathbb{Z}$, called trace lattices.

Definition 1.7 *Let (Λ, Q) be a $\mathbb{Z}_F$-lattice of rank n. Let $\alpha \in F$ be totally positive. Then Λ is a $\mathbb{Z}$-module of dimension rn and*

$$\mathfrak{q}_\alpha : \mathbb{Q} \otimes \Lambda \to \mathbb{Z},\ \lambda \mapsto \mathrm{tr}(\alpha Q(\lambda)).$$

is a rational quadratic form. Then $L_\alpha := L_\alpha(\Lambda, Q) := (\Lambda, \mathfrak{q}_\alpha)$ is a $\mathbb{Z}$-lattice, called the trace lattice *of (Λ, Q) with respect to α. We call $\mathfrak{q}_\alpha$ the trace quadratic form of Q with respect to α. Its polar form is $\mathfrak{b}_\alpha : (\lambda, \mu) \mapsto \mathrm{tr}(\alpha B(\lambda, \mu))$.*

Lemma 1.8 *Let (Λ, Q) be a lattice and $\alpha \in F$ be totally positive.*

(i) The dual lattice of L_α is $L_\alpha^\# = (\alpha^{-1}\mathbb{Z}_F^\#\Lambda^\#, q_\alpha) = (\alpha^{-1}\Lambda^*, q_\alpha)$.

(ii) If Λ is even and $\alpha \in \mathbb{Z}_F^\#$ or Λ is trace even and $\alpha \in \mathbb{Z}_F$, then L_α is even.

(iii) If Λ is integral, then $\det(L_\alpha) = (\mathcal{N}(\alpha)d_F)^n|\Lambda^\#/\Lambda|$. *If Λ is trace integral, then* $\det(L_\alpha) = \mathcal{N}(\alpha)^n|\Lambda^*/\Lambda|$.

Proof. (i). Since $\mathfrak{b}_\alpha(\lambda, \mu) = \mathrm{tr}(\alpha B(\lambda, \mu))$,

$$L_\alpha^\# = \{\lambda \in F^n \mid \mathrm{tr}(\alpha B(\lambda, \Lambda)) \subseteq \mathbb{Z}\} = \alpha^{-1}\mathbb{Z}_F^\#\Lambda^\#.$$

Since $\Lambda^* = \mathbb{Z}_F^\#\Lambda^\#$ (see Lemma 1.5(iii)), $(\alpha^{-1}\mathbb{Z}_F^\#\Lambda^\#, q_\alpha) = (\alpha^{-1}\Lambda^*, q_\alpha)$.

(ii). If Λ is an even lattice, then $Q(\lambda) \in \mathbb{Z}_F$ for all $\lambda \in \Lambda$. If $\alpha \in \mathbb{Z}_F^\#$, then by the definition of $\mathbb{Z}_F^\#$, we have $\mathfrak{q}_\alpha(\lambda) = \mathrm{tr}(\alpha Q(\lambda)) \in \mathbb{Z}$ for all $\lambda \in \Lambda$. If Λ is trace even, then $\mathrm{tr}(\nu Q(\lambda)) \in \mathbb{Z}$ for all $\nu \in \mathbb{Z}_F$ and $\lambda \in \Lambda$. So this is especially true for $\nu = \alpha$, and hence L_α is even.

(iii). First assume that Λ is integral, i.e. $\Lambda \subseteq \Lambda^\#$. Without loss of generality, $\alpha \in \mathbb{Z}_F^\#$. Hence L_α is even by (ii). So

$$L_\alpha^\#/L_\alpha = (\alpha^{-1}\mathbb{Z}_F^\#\Lambda^\#)/\Lambda$$

as Abelian groups. Hence

$$\det L_\alpha = |L_\alpha^\#/L_\alpha| = |\alpha^{-1}\mathbb{Z}_F^\#/\mathbb{Z}_F|^n|\Lambda^\#/\Lambda|.$$

The proof for Λ trace even is analogue. □

Lemma 1.9 *Let (Λ, Q) be a lattice of rank n.*

(i) Let (Λ, Q) be unimodular. If $(\mathbb{Z}_F^\#)^{-1} = \delta\mathbb{Z}_F$ for some $\delta \gg 0$, then the trace lattice $L_{\delta^{-1}}$ is even unimodular. In particular, rn is a multiple of 8.

(ii) If (Λ, Q) is trace even unimodular, then the trace lattice L_1 is even unimodular. Thus rn is a multiple of 8.

Proof. $L_{\delta^{-1}}$ or L_1, respectively, are even by (ii) and unimodular by (iii) of the previous lemma. So the $\dim(L_{\delta^{-1}}) = \dim(L_1) = rn$ is a multiple of 8. □

It is easy to see that the quadratic form Q is determined by r trace quadratic forms.

Lemma 1.10 *Let V be an F-vector space, hence V is also a $\mathbb{Q}$-vector space. Let $(\alpha_1, \dots, \alpha_r) \in F^r_{\gg 0}$ be a $\mathbb{Q}$-basis of F. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_r : V \to \mathbb{Q}$ be positive definite quadratic forms over $\mathbb{Q}$.*

Then there is a unique quadratic form $Q : V \to F$ over F such that for all $j = 1, \dots, r$ the trace quadratic form with respect to α_j is $\mathfrak{q}_j$, i.e.

$$\mathfrak{q}_j(\lambda) = \mathrm{tr}(\alpha_j Q(\lambda)).$$

Proof. This follows from the same argument as in the proof of Proposition 1.2 (i.e. because the trace bilinear form is non-degenerated). □

Endomorphisms of Trace Lattices Let $\nu \in \mathbb{Z}_F$ and let Λ be a lattice. Then $\nu\Lambda \subseteq \Lambda$, and so $\lambda \mapsto \nu\lambda$ is an endomorphism of each trace lattice L_α. We also write ν for this endomorphism. The endomorphism is self-adjoint with respect to the bilinear form $\mathfrak{b}_\alpha$.

The existence of such endomorphisms is necessary for a $\mathbb{Z}$-lattice to be a trace lattice and sufficient if $\mathbb{Z}_F = \mathbb{Z}[\widetilde{\nu}]$, which we will assume in the following.

Definition 1.11 *Let F be a totally real number field and $\widetilde{\nu} \in F$ such that $\mathbb{Z}_F = \mathbb{Z}[\widetilde{\nu}]$. Let $\alpha \in F$ be totally positive. Let $(L, \mathfrak{q})$ be a $\mathbb{Z}$-lattice and let $\nu \in \mathrm{End}(L)$ be a self-adjoint endomorphism with the same minimal polynomial as $\widetilde{\nu}$. Then $V := \mathbb{Q} \otimes L$ is an F-vector space, where multiplication is given by*

$$\widetilde{\nu} \cdot \lambda := \nu(\lambda) \text{ for } \lambda \in V.$$

Since $\nu(L) \subseteq L$, L is a $\mathbb{Z}_F$-module. We denote this module by Λ.

If there is a totally positive definite quadratic form $Q : V \to F$ such that $\mathfrak{q}$ is the trace quadratic form of Q with respect to α, then (Λ, Q) is a $\mathbb{Z}_F$-lattice and $L_\alpha = (L, \mathfrak{q})$. We call (Λ, Q) an F-structure of $(L, \mathfrak{q})$ to α and ν.

An F-structure is given by a field element α and an endomorphism ν. Endomorphisms in the same conjugacy class over $\mathrm{Aut}(L)$ define isometric $\mathbb{Z}_F$-lattices.

Proposition 1.12 *Let $\widetilde{\nu}, \alpha \in F$ be as before. Let $(L, \mathfrak{q})$ and $(L', \mathfrak{q}')$ be $\mathbb{Z}$-lattices, and let $\nu \in \mathrm{End}(L, \mathfrak{q})$ and $\nu' \in \mathrm{End}(L', \mathfrak{q}')$ be self-adjoint and have the same minimal polynomial as $\widetilde{\nu}$. Let (Λ, Q) be an F-structure of $(L, \mathfrak{q})$ to α and ν, and let (Λ', Q') be am F-structure of $(\Lambda, \mathfrak{q})$ to α and ν'.*

Then $f : (\Lambda, Q) \to (\Lambda', Q')$ *is an isometry if and only if* $f : (L, \mathfrak{q}) \to (L', \mathfrak{q}')$ *is an isometry with* $f\nu f^{-1} = \nu'$.

Proof. If $f : (\Lambda, Q) \to (\Lambda', Q')$ is an isometry, then $Q'(f(\lambda)) = Q(\lambda)$ for all $\lambda \in \Lambda$. Especially

$$\mathfrak{q}'(f(\lambda)) = \operatorname{tr}\left(\alpha Q'(f(\lambda))\right) = \operatorname{tr}\left(\alpha Q(\lambda)\right) = \mathfrak{q}(\lambda).$$

So $f : (L, \mathfrak{q}) \to (L', \mathfrak{q}')$ is an isometry. And for all $\lambda \in L$:

$$f\nu(\lambda) = f(\widetilde{\nu}\lambda) = \widetilde{\nu} f(\lambda) = \nu' f(\lambda).$$

Hence $f\nu f^{-1} = \nu'$.

For the other direction, we assume without loss of generality that $(L, \mathfrak{q}) = (L', \mathfrak{q}')$. So $f \in \operatorname{Aut}(L, \mathfrak{q})$. For $\lambda \in \Lambda$ we have

$$f(\widetilde{\nu}\lambda) = f(\nu\lambda) = \nu' f(\lambda) = \widetilde{\nu} f(\lambda).$$

Hence $f : \Lambda \to \Lambda'$ is F-linear.

We choose some basis $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ of F such that $\alpha = \alpha_1$ and $\alpha_j \gg 0$. Let $2 \leq j \leq r$. Let L_j be the trace lattice of (Λ, Q) with respect to α_j, and let L'_j be the trace lattice of (Λ', Q') with respect to α_j . Let $\frac{\alpha_j}{\alpha_1} = \sum_{i=0}^{r-1} c_i \widetilde{\nu}^i$, $c_i \in \mathbb{Q}$. Set

$$\eta_j := \sum_{i=0}^{r-1} c_i \nu^i \text{ and } \eta'_j := \sum_{i=0}^{r-1} c_i \nu'^i.$$

Then $\eta'_j f = f\eta_j$. Let $\mathfrak{b}_j$ and $\mathfrak{b}'_j$ be the bilinear forms of L_j and L'_j, respectively. Then

$$\mathfrak{b}_j(\lambda, \mu) = \mathfrak{b}(\eta_j(\lambda), \mu) \text{ and } \mathfrak{b}'_j(\lambda, \mu) = \mathfrak{b}(\eta'_j(\lambda), \mu).$$

So

$$\mathfrak{b}'_j\left(f(\lambda), f(\mu)\right) = \mathfrak{b}\left(\eta'_j(f(\lambda)), f(\mu)\right) = \mathfrak{b}\left(f(\eta_j(\lambda)), f(\mu)\right) = \mathfrak{b}(\eta_j(\lambda), \mu) = \mathfrak{b}_j(\lambda, \mu).$$

Therefore f is an isometry between L_j and L'_j for all $j = 2, \ldots, r$, and so $Q'(f(\lambda)) = Q(\lambda)$ for all $\lambda \in \Lambda$, by Lemma 1.10. Hence $f : \Lambda \to \Lambda'$ is an isometry. $\square$

Chapter 2

Lattices over Real Quadratic Number Fields

In this thesis we are mainly interested in real quadratic number fields. Therefore we restrict to real quadratic number fields for further details. Especially, we consider the fields $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$.

In general, let $F = \mathbb{Q}[\sqrt{D}]$ with $D \geq 2$ square-free. We use the notations of the following table. Let also ε_0 be the fundamental unit with the property $\varepsilon_0^{(1)} > 1$.

Table 2.1: Notations

D *mod* 4	1	$2, 3$
d_F (discriminant)	D	$4D$
ω	$\frac{1+\sqrt{D}}{2}$	$\sqrt{D}$
$e = \sqrt{\frac{d_F}{D}}$	1	2

For concrete fields,

- if $F = \mathbb{Q}[\sqrt{5}]$, then $d_F = 5$, $\omega = \varepsilon_0 = \frac{1+\sqrt{5}}{2}$, and $e = 1$,
- if $F = \mathbb{Q}[\sqrt{2}]$, then $d_F = 8$, $\omega = \sqrt{2}$, $\varepsilon_0 = 1 + \sqrt{2}$, and $e = 2$,
- if $F = \mathbb{Q}[\sqrt{3}]$, then $d_F = 12$, $\omega = \sqrt{3}$, $\varepsilon_0 = 2 + \sqrt{3}$, and $e = 2$.

2.1 Galois and Fundamental Invariance

Definition 2.1 *Let* (Λ, Q) *be a* $\mathbb{Z}_F$*-lattice in a vector space* V*.*

(i) Let $\alpha \mapsto \overline{\alpha}$ *be the non-trivial Galois automorphism of* F*. It defines a multiplication*

$$* : F \times V \to V,\ (\alpha, \lambda) \mapsto \alpha * \lambda := \overline{\alpha} \cdot \lambda.$$

Hence V is a vector space, where the action of F is given by $*$. We denote this vector space by $\overline{V}$. Then $\Lambda \subset \overline{V}$ is a $\mathbb{Z}_F$-module, denoted by $\overline{\Lambda}$. And

$$\overline{Q} : \overline{V} \to F, \lambda \mapsto \overline{Q(\lambda)}.$$

is a quadratic form. The lattice $(\overline{\Lambda}, \overline{Q})$ is called the Galois-conjugate lattice *of (Λ, Q).*

(Λ, Q) is called Galois-invariant if $(\Lambda, Q) \cong (\overline{\Lambda}, \overline{Q})$.

(ii) Assume that the fundamental unit ε_0 has norm 1. The fundamental-conjugate lattice *of (Λ, Q) is $\Lambda^{\varepsilon_0} := (\Lambda, \varepsilon_0 Q)$, where*

$$\varepsilon_0 Q : F^n \to F, \lambda \mapsto \varepsilon_0 Q(\lambda).$$

We call a lattice (Λ, Q) fundamentally invariant *if $\Lambda \cong \Lambda^{\varepsilon_0}$.*

Remark 2.2 *A lattice (Λ, Q) is Galois invariant if and only if there is a semi-endomorphism $\sigma : \Lambda \to \Lambda$ (i.e. $\sigma(\alpha\lambda + \mu) = \overline{\alpha}\sigma(\lambda) + \sigma(\mu)$ for $\alpha \in F$, $\lambda, \mu \in \Lambda$) with the property*

$$Q(\sigma(\lambda)) = \overline{Q(\lambda)} \text{ for all } \lambda \in \Lambda.$$

A lattice (Λ, Q) is fundamentally invariant if and only if that there is an endomorphism $\tau : \Lambda \to \Lambda$ with $Q(\tau(\lambda)) = \varepsilon_0 Q(\lambda)$ for all $\lambda \in \Lambda$.

Proposition 2.3 (Modularity of Trace Lattices)

(i) Let (Λ, Q) be an even unimodular lattice, and let e, d_F, etc. be as in Table 2.1.

(a) The trace lattice $L_{e^{-1}}$ is even D-modular.

(b) Assume that (Λ, Q) is Galois-invariant. Let $\alpha \in \mathbb{Z}_F^{\#}$ be totally positive. Then the trace lattice L_α is even $\mathcal{N}(\alpha)d_F$-modular.

(ii) Let (Λ, Q) be a Galois-invariant trace even unimodular lattice. Let $\alpha \in \mathbb{Z}_F^{\#}$ be totally positive. Then the trace lattice L_α is even $\mathcal{N}(\alpha)$-modular.

Proof. (i)(a). By Lemma 1.8(i), $L_{e^{-1}}^{\#} = e\mathbb{Z}_F^{\#} L_{e^{-1}}$ and $e\mathbb{Z}_F^{\#} = \frac{1}{\sqrt{D}}\mathbb{Z}_F$ by definition. So as an endomorphism of $\mathbb{Q} \otimes L_{e^{-1}}$, $\sqrt{D}$ maps $L_{e^{-1}}^{\#}$ onto $L_{e^{-1}}$. For $\lambda, \mu \in \mathbb{Q} \otimes L_{e^{-1}}$ we have

$$\mathfrak{b}_{e^{-1}}(\sqrt{D}(\lambda), \sqrt{D}(\mu)) = \mathrm{tr}(e^{-1}B(\sqrt{D}\lambda, \sqrt{D}\mu)) = D\,\mathfrak{b}_{e^{-1}}(\lambda, \mu),$$

and hence $\sqrt{D}$ is a similarity of norm D. So $L_{e^{-1}}$ is D-modular.

(i)(b). Let σ be a semi-endomorphism of (Λ, Q) such that $\mathbb{Q}(\sigma(\lambda)) = \overline{Q(\lambda)}$ for all $\lambda \in \Lambda$. Write also σ for the induced endomorphism on L_α. Again by Lemma 1.8(i) we have $L_\alpha^\# = (\alpha\sqrt{d_F})^{-1} L_\alpha$. Put $f := \sigma\alpha\sqrt{d_F} \in \mathrm{End}(\mathbb{Q}\otimes L_\alpha)$. Then $f(L_\alpha^\#) = L_\alpha$. For $\lambda, \mu \in \mathbb{Q}\otimes L_{e^{-1}}$ we have

$$\begin{aligned}
\mathfrak{b}_\alpha(f(\lambda), f(\mu)) &= \mathrm{tr}\left(\alpha\ B\left(\sigma(\alpha\sqrt{d_F}\lambda), \sigma(\alpha\sqrt{d_F}\mu)\right)\right) \\
&= \mathrm{tr}\left(\alpha\ \overline{B\left(\alpha\sqrt{d_F}\lambda, \alpha\sqrt{d_F}\mu\right)}\right) \\
&= \mathrm{tr}\left(\overline{\alpha}\ B\left(\alpha\sqrt{d_F}\lambda, \alpha\sqrt{d_F}\mu\right)\right) = \mathcal{N}(\alpha)d_F\ \mathfrak{b}_\alpha(\lambda, \mu).
\end{aligned}$$

Hence f is a similarity of norm $\mathcal{N}(\alpha)d_F$.

One proves (ii) analogously to (i)(b). □

2.2 Three Types of Lattices

We define three types of lattices and describe their trace lattices. We distinguish three types because the situation is different whether the fundamental unit has norm -1 (Type (i)) or 1 (Type (ii) and (iii)).

Definition 2.4

Type (i): *Suppose that the fundamental unit ε_0 has norm -1. Then the different is generated by $\delta = \sqrt{d_F}\varepsilon_0 \gg 0$. We define $\alpha_1 := \delta^{-1}$ and $\alpha_2 := e^{-1}$.*

An even unimodular lattice (Λ, Q) is called a lattice of Type (i).

Type (ii): *Suppose that the fundamental unit ε_0 has norm 1. We define $\alpha_1 := e^{-1}$ and $\alpha_2 = \frac{\sqrt{D}-1}{\sqrt{d_F}}$.*

A Galois-invariant even unimodular lattice (Λ, Q) is called a lattice of Type (ii).

Type (iii): *Suppose that the fundamental unit ε_0 has norm 1. We define $\alpha_1 := 1$ and $\alpha_2 := D + \sqrt{D}$.*

A Galois-invariant trace even unimodular lattice (Λ, Q) is called a lattice of Type (iii).

For each type, we denote the trace lattice L_{α_1} by (Λ_1, Q_1) and call it the first trace lattice. *We denote the trace lattice L_{α_2} by (Λ_2, Q_2) and call it the* second trace lattice. *And B_1 and B_2 denote the bilinear forms of Λ_1 and Λ_2, respectively.*

For concrete fields, we have the following elements α_1 and α_2.

Type (i): (a) If $F = \mathbb{Q}[\sqrt{5}]$, then $\alpha_1 = \frac{5-\sqrt{5}}{10}$ and $\alpha_2 = 1$,

(b) If $F = \mathbb{Q}[\sqrt{2}]$, then $\alpha_1 = \frac{2-\sqrt{2}}{4}$ and $\alpha_2 = \frac{1}{2}$.

Type (ii): If $F = \mathbb{Q}[\sqrt{3}]$, then $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{3-\sqrt{3}}{6}$.

Type (iii): If $F = \mathbb{Q}[\sqrt{3}]$, then $\alpha_1 = 1$ and $\alpha_2 = 3 + \sqrt{3}$.

Theorem 2.5 *Let* (Λ, Q) *be a lattice.*

Type (i): (Λ_1, Q_1) *is even unimodular and* (Λ_2, Q_2) *is even* D*-modular.*

Type (ii): (Λ_1, Q_1) *is even* D*-modular and* (Λ_2, Q_2) *is even* $(D-1)$*-modular.*

Type (iii): (Λ_1, Q_1) *is even unimodular and* (Λ_2, Q_2) *is even* $D(D-1)$*-modular.*

Proof. By Lemma 1.8 (ii), the trace lattices are even. Their determinants are given by Lemma 1.8 (iii). And by Proposition 2.3, they are modular. □

A natural question is whether the reverse of the theorem is true. So one starts with a lattice over $\mathbb{Z}$ and studies the F-structures of this lattice.

Structures of Rational Lattices Let $F = \mathbb{Q}[\sqrt{D}]$ be a real quadratic number field and $\mathbb{Z}_F = \mathbb{Z}[\omega]$, where ω is defined as in Table 2.1. Let $(L, \mathfrak{q})$ be an even $\mathbb{Z}$-lattice of even dimension N. Assume that there is a self-adjoint $\nu \in \operatorname{End}(L, \mathfrak{q})$ having the same minimal polynomial as ω.

Therefore L is an $\mathbb{Z}_F$ module, where $\omega \cdot \lambda := \nu(\lambda)$ for $\lambda \in L$. We denote this module by Λ. We want to find the F-structures of L which are lattices of Type (i), (ii), or (iii) and whose first trace lattice is L. So suppose that L is either unimodular (for Types (i) and (iii)) or D-modular (for Type (ii)).

Let $\eta = a + b\nu$, where $a, b \in \mathbb{Q}$ such that $\alpha_1^{-1}\alpha_2 = a + b\omega$. If L is the first trace lattice of an lattice Λ of Type (i), (ii), or (iii), then $B_2(\lambda, \mu) = B_1(\eta(\lambda), \mu)$ for all $\lambda, \mu \in \Lambda$, where B_1 and B_2 are the bilinear forms of the first and second, respectively, trace lattices.

So we define a lattice $L' = (L, \mathfrak{q}')$ via the bilinear form

$$\mathfrak{b}' : L \times L \to \mathbb{Q},\ (\lambda, \mu) \mapsto \mathfrak{b}(\eta(\lambda), \mu),$$

where $\mathfrak{b}$ is the polar form of $(L, \mathfrak{q})$.

And let $\overline{\nu} \in \operatorname{End}(L)$ such that $\overline{\omega} \cdot \lambda = \overline{\nu}(\lambda)$ for all $\lambda \in L$.

Theorem 2.6 Type (i): *Suppose that* $\mathcal{N}(\varepsilon_0) = -1$ *and that* L *is unimodular.*

(a) If $D \equiv 1 \pmod 4$*, then there is a unique lattice* (Λ, Q) *of Type (i) such that* L *and* L' *are the first and second trace lattices.*

(b) If $D \equiv 2 \pmod 4$ *and* L' *is even, then there is a unique lattice* (Λ, Q) *of Type (i) such that* L *and* L' *are the first and second trace lattices.*

If there is additionally an $f \in \mathrm{Aut}(L)$ with $f\nu = \overline{\nu} f$, then (Λ, Q) is Galois invariant.

Type (ii): *Suppose that $\mathcal{N}(\varepsilon_0) = 1$ and that L is D-modular. If L' is even and there is an $f \in \mathrm{Aut}(L)$ with $f\nu = \overline{\nu} f$, then there is a unique lattice (Λ, Q) of Type (ii) such that L and L' are the first and second trace lattices.*

Type (iii): *Suppose that $\mathcal{N}(\varepsilon_0) = 1$ and that L is unimodular.*

(a) If $D \equiv 1 \pmod 4$ and there is an $f \in \mathrm{Aut}(L)$ with $f\nu = \overline{\nu} f$, then there is a unique lattice (Λ, Q) of Type (iii) such that L and L' are the first and second trace lattices.

(b) If $D \equiv 2, 3 \pmod 4$, L' is even and there is an $f \in \mathrm{Aut}(L)$ with $f\nu = \overline{\nu} f$, then there is a unique lattice (Λ, Q) of Type (iii) such that L and L' are the first and second trace lattices.

Proof. (i). By Lemma 1.10, there is a unique lattice (Λ, Q) such that $L_{\alpha_1} = L$ and $L_{\alpha_2} = L'$. By Lemma 1.8(iii), (Λ, Q) is unimodular. We have to show that (Λ, Q) is even.

Since L is even, we have

$$\mathrm{tr}(\alpha_1 Q(\lambda)) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda.$$

Let $\lambda \in \Lambda$. So $\mathrm{tr}(\alpha_1 Q(\beta\lambda)) = \mathrm{tr}(\alpha_1 \beta^2 Q(\lambda)) \in \mathbb{Z}$ for all $\beta \in \mathbb{Z}_F$, and $Q(\lambda) \in (\alpha_1 \langle Z_F^2 \rangle_{\mathbb{Z}})^{\#}$.

If $D \equiv 1 \pmod 4$, then $\mathbb{Z}_F = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$. Hence $\langle \mathbb{Z}_F^2 \rangle_{\mathbb{Z}} = \mathbb{Z}_F$. Since $(\alpha_1 \mathbb{Z}_F)^{\#} = \mathbb{Z}_F$, $Q(\lambda) \in \mathbb{Z}_F$ for all $\lambda \in \Lambda$, and (Λ, Q) is thus even.

If $D \equiv 2 \pmod 4$, then $\langle \mathbb{Z}_F^2 \rangle_{\mathbb{Z}} = \mathbb{Z}[2\sqrt{D}]$ has index 2 in $\mathbb{Z}_F$. Therefore $\mathbb{Z}_F = (\alpha_1 \mathbb{Z}_F)^{\#}$ has index 2 in $(\alpha_1 \mathbb{Z}[2\sqrt{D}])^{\#}$, and hence

$$2Q(\lambda) \in \mathbb{Z}_F.$$

So $Q(\lambda) = a + b\sqrt{D}$, $a, b \in \frac{1}{2}\mathbb{Z}$ and $\varepsilon_0 = x + y\sqrt{D}$, $x, y \in \mathbb{Z}$. Then $\alpha_1 = \frac{x - y\sqrt{D}}{-2\sqrt{D}}$ and $\alpha_2 = 1/2$. Since L and L' are even,

$$\mathrm{tr}(\alpha_1 Q(\lambda)) = ya - xb \in \mathbb{Z} \text{ and } \mathrm{tr}(\tfrac{Q(\lambda)}{2}) = a \in \mathbb{Z}.$$

So $a \in \mathbb{Z}$ and $xb \in \mathbb{Z}$. Since $-1 = \mathcal{N}(\varepsilon_0) = x^2 - Dy^2$, x is odd. Hence $b \in \mathbb{Z}$. Therefore $Q(\lambda) \in \mathbb{Z}_F$ and (Λ, Q) is even.

If there is an $f \in \mathrm{Aut}(L, \mathfrak{q})$ with $f\nu = \overline{\nu} f$, then by Proposition 1.12 $(\Lambda, Q) \cong (\overline{\Lambda}, \overline{Q})$ is true. Hence Λ is Galois-invariant.

(ii). Again, there is a unique unimodular lattice (Λ, Q) such that $L_{\alpha_1} = L$ and $L_{\alpha_2} = L'$. Let $\lambda \in \Lambda$ and $Q(\lambda) = a + b\sqrt{D}$. So

$$\mathrm{tr}(\alpha_1 Q(\lambda)) = \tfrac{2}{e} a \in \mathbb{Z} \text{ and } \mathrm{tr}(\alpha_2 Q(\lambda)) = \tfrac{2}{e}(a - b) \in \mathbb{Z},$$

because L and L' are even. It follows that $Q(\lambda) \in \mathbb{Z}_F$ and therefore Λ is even. And since there is an $f \in \mathrm{Aut}(L, \mathfrak{q})$ with $f\nu = \overline{\nu} f$, $(\Lambda, Q) \cong (\overline{\Lambda}, \overline{Q})$ by Proposition 1.12. The lattice (Λ, Q) is even, unimodular, and Galois-invariant, hence of Type (ii).

(iii). Again, there is a unique trace unimodular lattice (Λ, Q) with $L_{\alpha_1} = L$ and $L_{\alpha_2} = L'$. (a). Let $\lambda \in \Lambda$. Analogously to (i)(a), $Q(\lambda) \in \mathbb{Z}_F^{\#}$. Hence Λ is trace even. (b). Let $Q(\lambda) = \frac{a+b\sqrt{D}}{2\sqrt{D}}$ with $a, b \in \mathbb{Q}$. So

$$\mathrm{tr}(Q(\lambda)) = b \in \mathbb{Z} \text{ and } \mathrm{tr}((D + \sqrt{D})Q(\lambda)) = a + Db \in \mathbb{Z}.$$

Therefore $a, b \in \mathbb{Z}$, hence $Q(\lambda) \in \mathbb{Z}_F^{\#}$, and Λ is trace even. Again, Λ is Galois invariant because there is an $f \in \mathrm{Aut}(L, \mathfrak{q})$ with $\nu f = f\overline{\nu}$. □

Minima estimates for trace lattices We conclude this section by remarking minima estimates of trace lattices for concrete number fields.

Proposition 2.7

(i) [Neb13, 2,4] Suppose $F = \mathbb{Q}[\sqrt{5}]$. Let Λ be an even unimodular lattice over F. For the minima of the first and second trace lattices Λ_1 and Λ_2, respectively, we have estimates

$$2 \min \Lambda_1 \;\leq\; \min \Lambda_2 \;\leq\; \tfrac{5}{2} \min \Lambda_1.$$

(ii) SupposeF $= \mathbb{Q}[\sqrt{2}]$. Let Λ be an even unimodular lattice over F. For the minima of the first and second trace lattices Λ_1 and Λ_2, respectively, we have estimates

$$\min \Lambda_1 \;\leq\; \min \Lambda_2 \;\leq\; 2 \min \Lambda_1.$$

(iii) Suppose $F = \mathbb{Q}[\sqrt{3}]$. Let Λ be an even unimodular lattice over F. For the minima of the first and second trace lattices Λ_1 and Λ_2, respectively, we have estimates

$$\tfrac{1}{2} \min \Lambda_1 \;\leq\; \min \Lambda_2 \;\leq\; \tfrac{4}{3} \min \Lambda_1.$$

(iv) Suppose $F = \mathbb{Q}[\sqrt{3}]$. Let Λ be a trace even unimodular lattice over F. For the minima of the first and second trace lattices Λ_1 and Λ_2, respectively, we have estimates

$$\frac{3}{2} \min \Lambda_1 \;\leq\; \min \Lambda_2 \;\leq\; 4 \min \Lambda_1.$$

Proof. See also [Neb99] and, for (i), [Neb13, 2.4]. Let $n_0 := \min \Lambda_1$ and $m_0 := \min \Lambda_2$. Let $\lambda, \mu \in \Lambda$ such that $Q_1(\lambda) = n_0$ and $Q_2(\mu) = m_0$.

(i) Λ_1 is the trace lattice with respect to $(\sqrt{5}\varepsilon_0)^{-1} = \frac{1+\overline{\varepsilon_0}^2}{5}$ and Λ_2 is the trace lattice with respect to 1, where $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ is a fundamental unit. Hence

$$n_0 = Q_1(\lambda) = \mathrm{tr}\left(\frac{1+\overline{\varepsilon_0}^2}{5}Q(\lambda)\right) = \frac{1}{5}\left(Q_2(\lambda) + Q_2(\overline{\varepsilon_0}\lambda)\right) \geq \frac{2}{5}m_0,$$

$$m_0 = Q_2(\mu) = \mathrm{tr}\left((\sqrt{5}\varepsilon_0)^{-1}(1+\varepsilon_0^2)Q(\mu)\right) = (Q_1(\mu) + Q_1(\varepsilon_0\mu)) \geq 2n_0.$$

(ii) Λ_1 is the trace lattice with respect to $(2\sqrt{2}\varepsilon_0)^{-1} = \frac{1+\overline{\varepsilon_0}^2}{8}$ and Λ_2 is the trace lattice with respect to $\frac{1}{2}$, where $\varepsilon_0 = 1+\sqrt{2}$ is a fundamental unit. Therefore

$$n_0 = Q_1(\lambda) = \mathrm{tr}\left(\frac{1+\overline{\varepsilon_0}^2}{8}Q(\lambda)\right) = \frac{1}{4}\left(Q_2(\lambda) + Q_2(\overline{\varepsilon_0}\lambda)\right) \geq \frac{1}{2}m_0,$$

$$m_0 = Q_2(\mu) = \mathrm{tr}\left((2\sqrt{2}\varepsilon_0)^{-1}\frac{1+\varepsilon_0^2}{2}Q(\mu)\right) = \frac{1}{2}\left(Q_1(\mu) + Q_1(\varepsilon_0\mu)\right) \geq n_0.$$

(iii) Λ_1 is the trace lattice with respect to $1/2$, Λ_2 is the trace lattice with respect to $\frac{3-\sqrt{3}}{6}$, and $\varepsilon_0 = 2+\sqrt{3}$ is a fundamental unit. Therefore

$$n_0 = Q_1(\lambda) = \mathrm{tr}\left(\frac{3-\sqrt{3}}{6}\frac{5+\varepsilon_0^2}{8}Q(\lambda)\right) = \frac{1}{8}\left(5Q_2(\lambda) + Q_2(\varepsilon_0\lambda)\right) \geq \frac{3}{4}m_0,$$

$$m_0 = Q_2(\mu) = \mathrm{tr}\left(\frac{1}{2}\frac{5+\overline{\varepsilon_0}^2}{12}Q(\mu)\right) = \frac{1}{12}\left(5Q_1(\mu) + Q_1(\overline{\varepsilon_0}\mu)\right) \geq \frac{1}{2}n_0.$$

(iv) Λ_1 is the trace lattice with respect to 1, Λ_2 is the trace lattice with respect to $3+\sqrt{3}$, and $\varepsilon_0 = 2+\sqrt{3}$ is a fundamental unit. Therefore

$$n_0 = Q_1(\lambda) = \tfrac{1}{6}\mathrm{tr}\left((3+\sqrt{3})(1+\tfrac{(1-\sqrt{3})^2}{2})Q(\lambda)\right) = \tfrac{1}{6}Q_2(\lambda) + \tfrac{1}{12}Q_2((1-\sqrt{3})\lambda) \geq \frac{m_0}{4},$$

$$m_0 = Q_2(\mu) = \mathrm{tr}\left((3+\sqrt{3})Q(\mu)\right) = Q_1(\mu) + \tfrac{1}{2}Q_1((1+\sqrt{3})\mu) \geq \frac{3}{2}n_0.$$

□

Remark 2.8 *The trace lattices are modular lattices. With the theory of modular forms, one finds upper bounds for minima of such lattices. If a lattice is p-modular, where $p \in \mathbb{N}$ such that $p+1$ divides 24, then*

$$\min L \leq 1 + \left\lfloor \frac{\dim L}{48/(p+1)} \right\rfloor.$$

For $p = 1$ this is due to Siegel, compare [Sie69], for $p > 1$ this was first observed by Quebbemann in [Que95]. Lattices achieving this bound are called extremal.

For $F = \mathbb{Q}[\sqrt{5}]$, the second trace lattice is 5-modular. The bound for $p = 5$ is strictly bigger than the upper bound from the proposition for $n \geq 40$. Hence the second trace lattice cannot be extremal for $\dim \Lambda \geq 20$, cf. [Neb13, Corollary 2.9]. Actually, Nebe showed that both first and second trace lattices could be extremal only for dimensions 4 and 12.

For the fields $F = \mathbb{Q}[\sqrt{2}]$ and $F = \mathbb{Q}[\sqrt{3}]$, there is no such discrepancy, the upper bounds of the proposition are always bigger than the bound of modular lattices. Also, the lower bound does not yield a contradiction.

2.3 Root lattices

Root lattices are important examples of lattices. A reference of the following is [Sch94].

We assume that F has class number one and only consider even unimodular lattices. Let ε_0 be the fundamental unit of F, as before.

Definition 2.9 *Let (Λ, Q) be an even unimodular lattice in a vector space V. A vector $\lambda \in V$ defines an isometry, called* reflection *along λ:*

$$s_\lambda : V \to V, \ \mu \mapsto \mu - \frac{B(\lambda, \mu)}{Q(\lambda)}\lambda.$$

A root *of Λ is a vector $\lambda \in \Lambda$ such that s_λ is an automorphism of Λ.*

Write $\overline{R}(\Lambda)$ for the set of all roots of Λ.

Λ is called a root lattice *if $\overline{R}(\Lambda)$ generates Λ and* rootless *if $\overline{R}(\Lambda) = \emptyset$.*

If $\lambda \in V$ and $\alpha \in F^*$, then $s_\lambda = s_{\alpha\lambda}$. So, if $\alpha\lambda \in \Lambda$ and $\lambda \in \Lambda$ is a root, then $\alpha\lambda$ is also a root. We call $\lambda \in \Lambda$ *primitive* if $\Lambda \cap F\lambda = \mathbb{Z}_F\lambda$.

Lemma 2.10 *If $\mu \in \Lambda$ is a root, then there are $\alpha \in \mathbb{Z}_F$ and $\lambda \in \Lambda$ such that $\mu = \alpha\lambda$, λ is primitive, and $Q(\lambda) \in \{1, \varepsilon_0\}$.*

Proof. Since $h_F = 1$, there are a primitive $\lambda' \in \Lambda$ and an $\alpha' \in \mathbb{Z}_F$ such that $\mu = \alpha'\lambda'$. Since the reflection $s_{\lambda'}$ is an automorphism of Λ, we have

$$\frac{B(\lambda', \Lambda)}{Q(\lambda')}\lambda' \subseteq \Lambda.$$

Since λ' is primitive,

$$\frac{B(\lambda', \Lambda)}{Q(\lambda')} \subseteq \mathbb{Z}_F.$$

So $\frac{1}{Q(\lambda')}\lambda' \in \Lambda^\#$. Since $\Lambda^\# = \Lambda$ and λ' is primitive, we have $Q(\lambda') \in \mathbb{Z}_F^*$. Hence there is an $\varepsilon \in \mathbb{Z}_F^*$ such that $Q(\varepsilon\lambda') = \varepsilon^2 Q(\lambda') \in \{1, \varepsilon_0\}$. Set $\lambda := \varepsilon\lambda'$. Then $Q(\lambda) \in \{1, \varepsilon_0\}$ and $\mu = \alpha'\varepsilon\lambda$. $\square$

If ε_0 has norm -1, then of course $Q(\lambda) = \varepsilon_0$ is not possible. We define the set of reduced roots

$$R(\Lambda) := \Lambda(1) \cup \Lambda(\varepsilon_0).$$

Example 2.11 *(i) For $F = \mathbb{Q}[\sqrt{5}]$ there is the even unimodular root lattice F_4, cf. [CosHsi87]. It is determined by its Gram matrix*

$$\begin{pmatrix} 2 & \varepsilon_0 & \varepsilon_0 & \varepsilon_0 \\ \varepsilon_0 & 2 & 1 & 1 \\ \varepsilon_0 & 1 & 2 & 1 \\ \varepsilon_0 & 1 & 1 & 2 \end{pmatrix}.$$

It is up to isometry the only even unimodular lattice of dimension 4, cf. [Maa40].

(ii) For $F = \mathbb{Q}[\sqrt{2}]$ there is the root lattice Δ_4', which is sometimes also called F_4, cf. [HsiHun89]. It has the Gram matrix

$$\begin{pmatrix} 2 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 2 \end{pmatrix}.$$

Δ_4' is up to isometry the only even unimodular lattice in dimension 4. In fact, $\Delta_4' = D_4 + \sqrt{2}D_4^{\#}$, where D_4 is the lattice generated by the root system D_4, see [Sch94].

(iii) If $F = \mathbb{Q}[\sqrt{3}]$, then reduced roots are lattice vectors of norm 1 or $2+\sqrt{3}$. There are the even unimodular root lattices G_2 and F_4, see [Sch94] and [Hun91]. They have the Gram matrices

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 & 1+\sqrt{3} & 1+\sqrt{3} \\ 1 & 2 & 1+\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} & 4+2\sqrt{2} & 2+\sqrt{2} \\ 1+\sqrt{3} & 1+\sqrt{3} & 2+\sqrt{2} & 4+2\sqrt{2} \end{pmatrix},$$

respectively.

2.4 Genera and the Mass Formula

In this section we assume that the class number of the real quadratic number field $F = \mathbb{Q}[\sqrt{D}]$ is one. So all $\mathbb{Z}_F$-lattices are free $\mathbb{Z}_F$-modules.

Let Λ be a full lattice in (V,Q), where V is an F-vector space of dimension n and $Q : F \to V$ is a totally positive definite quadratic form. The genus of Λ is

$$\operatorname{Gen}(\Lambda) = \{M \subseteq V \text{ lattice} \mid \Lambda_{\mathfrak{p}} \cong M_{\mathfrak{p}} \text{ for all prime ideals } \mathfrak{p} \subseteq \mathbb{Z}_F\}.$$

Here $\Lambda_{\mathfrak{p}}$ is the lattice over the local ring $\mathbb{Z}_{F_{\mathfrak{p}}}$, i.e. $\Lambda_{\mathfrak{p}} = \mathbb{Z}_{F\mathfrak{p}} \otimes_{\mathbb{Z}_F} \Lambda$. A genus contains only finitely many isometry classes of lattices, see for example [O'M63, Theorem 103:4]. There is an algorithm to list all of them, Kneser's famous neighboring method, see [Kne57],[Kne02], [Sch98], or [Kir14] (the last two cover Kneser's method over general algebraic number fields).

In the even unimodular case there is exactly one genus.

Theorem 2.12

(i) Assume that the narrow class number is one. If 4 divides n, then there is precisely one genus of even unimodular lattices.

(ii) Assume that the narrow class number is two. If n is even, then there is at least one genus of even unimodular lattices. Over $\mathbb{Q}[\sqrt{3}]$, there is precisely one genus.

Proof. O'Meara shows in [O'M63, 102:3] that all even unimodular lattices of a quadratic space lie in the same genus. For the existence of even unimodular lattices, see [Sch94, Proposition 3.1] for the cases $n \in 4\mathbb{N}$. For (ii) in the case $n \in 2\mathbb{N}$, it is sufficient to construct an even unimodular lattice of dimension 2 and determinant one. If $D \equiv 3 \pmod 4$, then $D = -1 + 4\ell$ for some $\ell \in \mathbb{N}$. The lattice with Gram matrix

$$\begin{pmatrix} 2 & \sqrt{D} \\ \sqrt{D} & 2\ell \end{pmatrix}$$

is even unimodular.

The fact that for $\mathbb{Q}[\sqrt{3}]$ there are no other genera follows from [O'M63, §93D]. □

Some genera contain only one isometry class. Over the rationals they were partly classified by Watson ([Wat84]) and recently completed by Kirschmer and Lorch ([LorKir13]). One-class genera appear up to dimension 10. For totally real number fields the one-class genera have been classified by Kirschmer in [Kir14], see also [Kir16]. In the case of real quadratic fields and even unimodular lattices, every one-class genus is of rank 4 and over the field $\mathbb{Q}[\sqrt{2}]$, $\mathbb{Q}[\sqrt{3}]$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{13}]$, or $\mathbb{Q}[\sqrt{17}]$.

One way to estimate how many isometry classes are contained in a genus is to use the mass formula. If the isometry classes of a genus $\mathcal{G}$ are represented by the lattices $\Lambda_1, \ldots, \Lambda_h$, then the mass of the genus or of Λ_i is

$$\mathrm{Mass}(\mathcal{G}) := \mathrm{Mass}(\Lambda_i) := \sum_{j=1}^{h} \frac{1}{|\mathrm{Aut}(\Lambda_j)|}.$$

The mass of a genus can be calculated directly by using Siegel's mass formula. For even unimodular lattices the formula is the following.

Theorem 2.13 (Siegel's mass formula for even unimodular lattices) *Let Λ be an even unimodular lattice. Then*

$$\mathrm{Mass}(\Lambda) = 4^{1-n} \cdot L_F(n/2, \chi_n) \cdot \prod_{\ell=1}^{\frac{n}{2}-1} \zeta_F(2\ell) \cdot d_F^{n(n-1)/4} \cdot \pi^{1-n(n+1)/2} \cdot \prod_{\ell=1}^{\frac{n}{2}-1} \frac{\pi(2\ell)!}{4^{2\ell}}.$$

Here $\zeta_F(s) = \prod_{\mathfrak{p}}(1 - \mathcal{N}(\mathfrak{p})^{-s})^{-1}$, where the product is over all prime ideals $\mathfrak{p}$, is the Dedekind zeta function, and $L_F(s, \chi_n) = \prod_{\mathfrak{p}}(1 - \chi_n(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$ is the L-series attached to F with character

$$\chi_n(\mathfrak{p}) = \left(\frac{-1}{\mathfrak{p}}\right) \text{ (Legendre symbol).}$$

Proof. See [Sie37], [Hsi89], [Hun91], or [Kir16]. □

Using Kneser's neighboring method to list all isometry classes of a genus is only practicable if the mass of the genus is small. So their mass may be relatively small although there are many isometry classes. M. Kirschmer classified all genera of unimodular lattices with mass at most 1/2 over totally real number fields other than $\mathbb{Q}$. In the greatest possible rank, 12, there is only one genus. It is the genus of even unimodular lattices of dimension 12 over $\mathbb{Q}[\sqrt{5}]$. It has 15 isometry classes and the mass is

$$\frac{668874965279}{579400335360000000} \approx 1.2 \cdot 10^{-6}.$$

The isometry classes were classified by P. Costello and J. Hsia in [CosHsi87].

The following table shows approximations of the masses of even unimodular lattices over the fields $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$ in small dimensions.

With Kneser's neighboring method one can find all even unimodular lattices. The highest cases I could compute in reasonable time are the following.

Theorem 2.14 *(i) The mass of the even unimodular lattices (Type (i)) of dimension 12 over $F = \mathbb{Q}[\sqrt{2}]$ is*

$$\frac{9214790966898371}{1078738078924800} \approx 8.5.$$

Note that J.S. Hsia in [Hsi89] estimates the mass to be about 7.1, this is an error.

dim.	$\mathbb{Q}[\sqrt{5}]$	$\mathbb{Q}[\sqrt{2}]$	*dim.*	$\mathbb{Q}[\sqrt{3}]$
4	$6.9 \cdot 10^{-5}$	$4.3 \cdot 10^{-4}$	2	$4.2 \cdot 10^{-2}$
8	$3.8 \cdot 10^{-9}$	$3.9 \cdot 10^{-6}$	4	$1.7 \cdot 10^{-3}$
12	$1.2 \cdot 10^{-6}$	8.5	6	$1.1 \cdot 10^{-4}$
16	$1.8 \cdot 10^{6}$	$4.5 \cdot 10^{18}$	8	$1.3 \cdot 10^{-3}$
20	$2.0 \cdot 10^{30}$	$6.9 \cdot 10^{49}$	10	3.0

With Kneser's neighboring method for the prime $\sqrt{2}\mathbb{Z}_F$ one can compute representatives of all 577 isometry classes. Among these 577 lattices, 99 lattices have no vectors with norm in $\mathbb{Z}_F^*$, hence they have no roots. Among these lattices, 5 lattices also have no elements of norm in $\sqrt{2}\mathbb{Z}_F^*$.

These 5 lattices have an extremal $\leq_A$-minimum among all lattices of Type (i), where $A = \left(\frac{2-\sqrt{2}}{4}, \frac{1}{2}\right)$. Their Gram matrices are listed in the Appendix.

(ii) The mass of the even unimodular lattices of dimension 10 over $F = \mathbb{Q}[\sqrt{3}]$ is

$$\frac{9957385009}{3344302080} \approx 3.$$

Using Kneser's neighboring method for the prime $\sqrt{3}\mathbb{Z}_F$, we find representatives of all 430 isometry classes. Among these 430 lattices, 99 lattices have no vectors of norm in $(\mathbb{Z}_F^*)^2$ and 21 lattices have no vectors of norm in $\mathbb{Z}_F^*$, i.e. have no roots. These 21 lattices without roots also do not have vectors of norm $2(\mathbb{Z}_F^*)^2$.

So they have an extremal $\leq_A$-minimum, where $A = \left(\frac{1}{2}, \frac{3-\sqrt{3}}{6}\right)$. Their Gram matrices are given in the Appendix.

Even in the last cases, it is really difficult to list all lattices in the genus. We see from the table that it is not possible to list the whole genus for dimensions greater than 12. For fields of higher discriminant, even lower dimensions are hopeless. For example, the mass of even unimodular lattices over $\mathbb{Q}[\sqrt{29}]$ in dimensions 8 is approximately 202.

So one has to use another way to find all even unimodular lattices, or to find all "interesting" even unimodular lattices (in some sense). We are interested in lattices of Type (i), (ii), or (iii) which have an extremal $\leq_A$-minimum, where $A = (\alpha_1, \alpha_2)$ is given by the type.

Chapter 3

Hilbert Modular Forms

The theory of Hilbert modular forms was introduced by Otto Blumenthal in his Habilitationsschrift in 1901 following unpublished notes by Hilbert, see [Blu03] and [Blu04]. Good introductions to Hilbert modular forms are the books by Garrett [Gar90], Freitag [Fre90] and van der Geer [vdG88]. These books are the main references for the sections about classical Hilbert modular forms, Eisenstein series, and Hecke operators.

We restrict to real quadratic number fields, although one may define Hilbert modular forms for any totally real number field. Then for real quadratic number fields, Hilbert modular forms are holomorphic functions on the Cartesian product of two copies of the complex upper half plane or one copy of the upper half plane and one of the lower half plan. The later was for example done for $\mathbb{Q}[\sqrt{5}]$ in [Maa40]. The reason why one needs to consider the lower half plane is that the modular forms of fields F with $h_F^+ = h_F$ on $\mathbb{H}\times\mathbb{H}$ and $\mathbb{H}\times\overline{\mathbb{H}}$ are not the same. Gundlach observed this when he constructed the modular forms of $\mathbb{Q}[\sqrt{3}]$, see [Gun65].

3.1 Classical Hilbert Modular Forms

Let $F = \mathbb{Q}[\sqrt{D}]$ be a real quadratic number field, where $D > 1$ is square-free.

The two embeddings of F into $\mathbb{R}$ are $\alpha \mapsto \alpha^{(j)}$, $j = 1, 2$. The algebraic group $\mathrm{GL}_2(F)$ is a discrete subgroup of $\mathrm{GL}_2(\mathbb{R})^2$,

$$\mathrm{GL}_2(F) \hookrightarrow \mathrm{GL}_2(\mathbb{R})^2,\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto \left(\begin{pmatrix}\alpha^{(1)} & \beta^{(1)}\\ \gamma^{(1)} & \delta^{(1)}\end{pmatrix}, \begin{pmatrix}\alpha^{(2)} & \beta^{(2)}\\ \gamma^{(2)} & \delta^{(2)}\end{pmatrix}\right).$$

Hence we identify $\mathrm{GL}_2(F)$ as a subgroup of $\mathrm{GL}_2(\mathbb{R})^2$. If $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(F)$, we write $g^{(j)}$ for $\begin{pmatrix}\alpha^{(j)} & \beta^{(j)}\\ \gamma^{(j)} & \delta^{(j)}\end{pmatrix}$, $j = 1, 2$.

Definition 3.1

$$\mathrm{GL}_2^+(\mathbb{R}) := \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g > 0\}$$

is the connected component of 1*, and*

$$\mathrm{GL}_2^+(F) := \{g \in \mathrm{GL}_2(F) \mid \det g \gg 0\}$$

can be embedded into $\mathrm{GL}_2^+(\mathbb{R})^2$*.*

Also we define

$$\begin{aligned}
\mathrm{GL}_2^+(\mathbb{Z}_F) &:= \{g \in \mathrm{GL}_2^+(F) \mid g, g^{-1} \in \mathbb{Z}_F^{2\times 2}\},\\
Z(F) &:= Z(\mathrm{GL}_2^+(F)) = \{\left(\begin{smallmatrix}\alpha & \\ & \alpha\end{smallmatrix}\right) \mid \alpha \in F^*\}, \text{ and}\\
Z(\mathbb{Z}_F) &:= Z(\mathrm{GL}_2^+(\mathbb{Z}_F)) = \{\left(\begin{smallmatrix}\alpha & \\ & \alpha\end{smallmatrix}\right) \mid \alpha \in \mathbb{Z}_F^*\}.
\end{aligned}$$

The group $\mathrm{GL}_2(\mathbb{R})$ acts on $\mathbb{C}$ via Moebius transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Hence $\mathrm{GL}_2(\mathbb{R})^2$ acts diagonally on $\mathbb{C}^2$ via coordinatewise Moebius transformations.

The group $\mathrm{GL}_2^+(\mathbb{R})$ preserves the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and the lower half plane $\overline{\mathbb{H}} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$. Therefore $\mathrm{GL}_2^+(\mathbb{R})^2$ preserves $\mathbb{H} \times \mathbb{H}$ and $\mathbb{H} \times \overline{\mathbb{H}}$. If $g = (g_1, g_2) \in \mathrm{GL}_2(\mathbb{R})^2$ with $\det(g_1) > 0$ and $\det(g_2) < 0$, then g interchanges $\mathbb{H} \times \mathbb{H}$ and $\mathbb{H} \times \overline{\mathbb{H}}$.

The theory of modular forms works for certain subgroups of $\mathrm{GL}_2^+(F)$, the so-called congruence subgroups.

Definition 3.2 *Let* $\mathfrak{n} \subseteq \mathbb{Z}_F$ *be an integral ideal. The* principal congruence subgroup *of level* $\mathfrak{n}$ *is*

$$\Gamma(\mathfrak{n}) := \left\{ g \in \mathrm{GL}_2^+(\mathbb{Z}_F) \mid g \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \pmod{\mathfrak{n}} \right\}.$$

A subgroup $\Gamma \leq \mathrm{GL}_2^+(F)$ *is called a* congruence subgroup *if there is an ideal* $\mathfrak{n} \subseteq \mathbb{Z}_F$ *such that* $\Gamma(\mathfrak{n}) \leq \Gamma Z(\mathbb{Z}_F)$ *and the index is finite.*

Lemma 3.3 *Let* Γ, Γ' *be congruence subgroups.*

(i) $\mathrm{SL}_2(\mathbb{Z}_F)$ *is a congruence subgroup.*

(ii) $\Gamma \cap \Gamma'$ *is a congruence subgroup with finite index in both* Γ *and* Γ'*.*

(iii) $g\Gamma g^{-1}$ *is a congruence subgroup for all* $g \in \mathrm{GL}_2(F)$*.*

Proof. Suppose that $\Gamma(\mathfrak{n})$ and $\Gamma(\mathfrak{n}')$ have finite index in $Z(\mathbb{Z}_F)\Gamma$ and $Z(\mathbb{Z}_F)\Gamma'$, respectively. (i) In fact $[\Gamma(\mathbb{Z}_F) : \mathbb{Z}(\mathbb{Z}_F)\mathrm{SL}_2(\mathbb{Z}_F)] = |(\mathbb{Z}_F^*)_{\gg 0}/(\mathbb{Z}_F^*)^2|$. (ii) The principal congruence subgroup $\Gamma(\mathfrak{n} \cap \mathfrak{n}')$ has finite index in $\Gamma(\mathfrak{n})$ and $\Gamma(\mathfrak{n}')$, hence in $Z(\mathbb{Z}_F)(\Gamma \cap \Gamma')$. (iii) Let $\ell \in \mathbb{N}$ such that ℓg and ℓg^{-1} have entries in $\mathbb{Z}_F$. Thus $\Gamma(\ell^2\mathfrak{n})$ has finite index in $g\Gamma(\mathfrak{n})g^{-1}$ and hence in $g\Gamma g^{-1}$. □

Definition 3.4 *The group* $\mathrm{GL}_2^+(F)$ *acts on the holomorphic functions of* $\mathbb{H}\times\mathbb{H}$ *and* $\mathbb{H}\times\overline{\mathbb{H}}$ *in the following way.*

Let $z\in\mathbb{H}\times\mathbb{H}$ *or* $z\in\mathbb{H}\times\overline{\mathbb{H}}$ *and* $g=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\mathrm{GL}_2^+(F)$. *We define the* factors of automorphy $j_i(g,z):=(\gamma^{(i)}z_i+\delta^{(i)})$, $j=1,2$, *and*

$$j(g,z):=(\gamma^{(1)}z_1+\delta^{(1)},\gamma^{(2)}z_2+\delta^{(2)}).$$

Let $k\in\mathbb{Z}^2$, *called the* weight vector. *For a holomorphic functions* $f:\mathbb{H}\times\mathbb{H}\to\mathbb{C}$ *or* $f:\mathbb{H}\times\overline{\mathbb{H}}\to\mathbb{C}$ *and* $g\in\mathrm{GL}_2^+(F)$ *we define a holomorphic function* $f|_kg$ *via*

$$f|_kg(z):=\det(g)^{k/2}j(g,z)^{-k}f(gz).$$

We use our standard notation $\alpha^k=(\alpha^{(1)})^{k_1}(\alpha^{(2)})^{k_2}$ *and* $a^k=a_1^{k_1}a_2^{k_2}$ *for* $\alpha\in F$ *and* $a\in\mathbb{C}^2$, $a_1,a_2\neq 0$.

$\mathrm{GL}_2^+(F)$ *acts on the holomorphic functions of* $\mathbb{H}\times\mathbb{H}$ *or of* $\mathbb{H}\times\overline{\mathbb{H}}$ *via*

$$(g,f)\mapsto f|_kg.$$

If the weight vector is (k,k) *for* $k\in\mathbb{Z}$ *(*parallel weight*), we also write* $k\in\mathbb{Z}$ *instead of* (k,k).

Definition 3.5 (Hilbert Modular Forms)
Let Γ *be a congruence subgroup and* $k\in\mathbb{Z}^2$ *be a weight vector or* $k=(k,k)$ *a parallel weight.*

(i) A holomorphic function $f:\mathbb{H}\times\mathbb{H}\to\mathbb{C}$ *or* $f:\mathbb{H}\times\overline{\mathbb{H}}\to\mathbb{C}$ *is called a* Hilbert modular form *of* level Γ *and weight* k *if*

$$f|_kg=f \text{ for all } g\in\Gamma.$$

If $\Gamma=\Gamma(\mathfrak{n})$, *we also call* $\mathfrak{n}$ *the level of* f. *And if* $\eta\in\mathbb{Z}_F$ *generates* $\mathfrak{n}$ *we call* η *the level of* f.

(ii) The spaces of Hilbert modular forms of level Γ *and weight* k *are*

$$M_k(\Gamma):=\{f:\mathbb{H}\times\mathbb{H}\to\mathbb{C}\text{ holomorphic}\mid f|_kg=f\text{ for all }g\in\Gamma\},$$

$$\overline{M}_k(\Gamma):=\{f:\mathbb{H}\times\overline{\mathbb{H}}\to\mathbb{C}\text{ holomorphic}\mid f|_kg=f\text{ for all }g\in\Gamma\}.$$

(iii) For Hilbert modular forms of parallel weight, there are the graded algebras

$$M(\Gamma)=\bigoplus_{k\in\mathbb{Z}}M_k(\Gamma),\ \overline{M}(\Gamma)=\bigoplus_{k\in\mathbb{Z}}\overline{M}_k(\Gamma),$$

and

$$M(\Gamma)_{ev}=\bigoplus_{k\in 2\mathbb{Z}}M_k(\Gamma),\ \overline{M}(\Gamma)_{ev}=\bigoplus_{k\in 2\mathbb{Z}}\overline{M}_k(\Gamma).$$

(iv) *The Galois automorphism of F yields a transformation*

$$\mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\mathbb{H},\ (z_1,z_2)\mapsto(z_2,z_1).$$

A Hilbert modular form $f\in M(\Gamma)$ is called (Galois) symmetric *if $f(z_2,z_1)=f(z_1,z_2)$ for all $z\in\mathbb{H}\times\mathbb{H}$ and* anti-symmetric *if $f(z_2,z_1)=-f(z_1,z_2)$ for all $z\in\mathbb{H}\times\mathbb{H}$. Also*

$$M^+(\Gamma):=\{f\in M(\Gamma)\mid f \textit{ is sym.}\},\ M^-(\Gamma):=\{f\in M(\Gamma)\mid f \textit{ is anti-sym.}\},$$

$$M^+(\Gamma):=\sum_{k\in\mathbb{Z}}M_k^+(\Gamma) \textit{ and } M_{ev}^+(\Gamma):=\sum_{k\in 2\mathbb{Z}}M_k^+(\Gamma).$$

(v) *Analogously, there is the transformation*

$$\mathbb{H}\times\overline{\mathbb{H}}\to\mathbb{H}\times\overline{\mathbb{H}},\ (z_1,z_2)\mapsto(-z_2,-z_1).$$

$f\in\overline{M}(\Gamma)$ is called (Galois) symmetric *if $f(-z_2,-z_1)=f(z_1,z_2)$ for all $z\in\mathbb{H}\times\overline{\mathbb{H}}$ and* anti-symmetric *if $f(-z_2,-z_1)=-f(z_1,z_2)$ for all $z\in\mathbb{H}\times\overline{\mathbb{H}}$. Also*

$$\overline{M}^+(\Gamma):=\{f\in\overline{M}(\Gamma)\mid f \textit{ is sym.}\},\ \overline{M}^-(\Gamma):=\{f\in\overline{M}(\Gamma)\mid f \textit{ is anti-sym.}\},$$

$$\overline{M}^+(\Gamma):=\sum_{k\in\mathbb{Z}}\overline{M}_k^+(\Gamma) \textit{ and } \overline{M}_{ev}^+(\Gamma):=\sum_{k\in 2\mathbb{Z}}\overline{M}_k^+(\Gamma).$$

If Γ is a principal congruence subgroup, i.e. $\Gamma=\Gamma(\mathfrak{m})$, we often write $M_k(\mathfrak{m})$ instead of $M_k(\Gamma)$, $M(\mathfrak{m})$ instead of $M(\Gamma)$, etc. And, if $\mathfrak{m}=\eta\mathbb{Z}_F$ is principal, we write $M_k(\eta)$ instead of $M_k(\Gamma(\eta\mathbb{Z}_F))$, $M(\eta)$ instead of $M(\Gamma(\eta\mathbb{Z}_F))$, etc.

We are especially interested in the following groups and their Hilbert modular forms.

Definition 3.6 (i) *The special linear group $\Gamma_F:=\mathrm{SL}_2(\mathbb{Z}_F)$ is called the* Hilbert modular group*.*

(ii) *Let $g_0:=\begin{pmatrix}\sqrt{d_F} & \\ & 1\end{pmatrix}\in\mathrm{GL}_2(F)$, where $\sqrt{d_F}\mathbb{Z}_F$ is the different ideal of F. We call*

$$\overline{\Gamma}_F:=g_0\Gamma_F g_0^{-1}=\left\{\begin{pmatrix}\alpha & \beta\sqrt{d_F}\\ \gamma/\sqrt{d_F} & \delta\end{pmatrix}\;\middle|\;\begin{array}{l}\alpha,\beta,\gamma,\delta\in\mathbb{Z}_F\\ \det g=1\end{array}\right\}$$

the conjugated Hilbert modular group*.*

(iii) *Let ε_0 be a fundamental unit and assume that $\mathcal{N}(\varepsilon_0)=1$ (i.e. $h_F^+=2h_F$). We define the extension groups $\Gamma_F^{\varepsilon_0}:=\langle\Gamma_F,\left(\begin{smallmatrix}\varepsilon_0 & \\ & 1\end{smallmatrix}\right)\rangle$ and $\overline{\Gamma}_F^{\varepsilon_0}:=\langle\overline{\Gamma}_F,\left(\begin{smallmatrix}\varepsilon_0 & \\ & 1\end{smallmatrix}\right)\rangle$. The matrix $\left(\begin{smallmatrix}\varepsilon_0 & \\ & 1\end{smallmatrix}\right)$ acts as $z\mapsto\varepsilon_0 z$. A Hilbert modular form of level $\Gamma_F^{\varepsilon_0}$ or $\overline{\Gamma}_F^{\varepsilon_0}$ is also called a* fundamentally symmetric *Hilbert modular form of level Γ_F or $\overline{\Gamma}_F$.*

The group $\overline{\Gamma}_F$ is a congruence subgroup. It acts on $\mathbb{H}\times\mathbb{H}$ as Γ_F acts on $\mathbb{H}\times\overline{\mathbb{H}}$. More precisely, there is a direct connection between modular forms of $\mathbb{H}\times\overline{\mathbb{H}}$ and $\mathbb{H}\times\mathbb{H}$.

Proposition 3.7 *Let Γ be a congruence subgroup. Let $g_0 := \left(\begin{smallmatrix}\sqrt{d_F} & \\ & 1\end{smallmatrix}\right)$.*

(i) $f : \mathbb{H}\times\overline{\mathbb{H}} \to \mathbb{C}$ is a Hilbert modular form of level Γ if and only if

$$f\circ g_0^{-1} : \mathbb{H}\times\mathbb{H}\to\mathbb{C},\ z\mapsto f(g_0^{-1}z)$$

is a Hilbert modular form of level $g_0\Gamma g_0^{-1}$. Additionally, f is symmetric if and only if $f\circ g_0$ is symmetric. Therefore

$$\overline{M}_k(\Gamma)\cong M_k(g_0\Gamma g_0^{-1}),\ \overline{M}_k^{+}(\Gamma)\cong M_k^{+}(g_0\Gamma g_0^{-1})$$

(ii) If $\varepsilon\in\mathbb{Z}_F^*$ with $\mathcal{N}(\varepsilon)=-1$, then $\left(\begin{smallmatrix}\varepsilon & \\ & 1\end{smallmatrix}\right)$ defines the map $\mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\overline{\mathbb{H}}$, $z\mapsto\varepsilon z$. Assume that $\left(\begin{smallmatrix}\varepsilon & \\ & 1\end{smallmatrix}\right)$ normalizes Γ and that $\left(\begin{smallmatrix}\varepsilon & \\ & \varepsilon^{-1}\end{smallmatrix}\right)\in\Gamma$. Then $M_k(\Gamma)\cong\overline{M}_k(\Gamma)$ by the isomorphism $f\mapsto f\circ\left(\begin{smallmatrix}\varepsilon & \\ & 1\end{smallmatrix}\right)$.

(iii) Make the same assumptions as in (ii) and additionally that the weight k is parallel. If k is odd, then

$$M_k^{+}(\Gamma)\cong\overline{M}_k^{-}(\Gamma),\ M_k^{-}(\Gamma)\cong\overline{M}_k^{+}(\Gamma),$$

and if k is even, then

$$M_k^{+}(\Gamma)\cong\overline{M}_k^{+}(\Gamma),\ M_k^{-}(\Gamma)\cong\overline{M}_k^{-}(\Gamma).$$

Part (i) is especially interesting for $\Gamma=\Gamma_F$ and $g_0\Gamma_F g_0^{-1}=\overline{\Gamma}_F$. In other words, Hilbert modular forms on $\mathbb{H}\times\overline{\mathbb{H}}$ for the Hilbert modular group are the same as Hilbert modular forms on $\mathbb{H}\times\mathbb{H}$ for the conjugated Hilbert modular group.

Proof. (i). Let $f:\mathbb{H}\times\overline{\mathbb{H}}\to\mathbb{C}$ be a Hilbert modular form of level Γ and weight k, and let $g=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\Gamma$. For $z\in\mathbb{H}\times\mathbb{H}$ we compute

$$f\circ g_0^{-1}(g_0 g g_0^{-1}z) = f(g\underbrace{g_0^{-1}z}_{\in\mathbb{H}\times\overline{\mathbb{H}}}) = j(g,g_0^{-1}z)^k f(g_0^{-1}z).$$

Since $j(g,g_0^{-1}z) = (\gamma^{(1)}\frac{z_1}{\sqrt{d_F}}+\delta^{(1)},\gamma^{(2)}\frac{z_2}{-\sqrt{d_F}}+\delta^{(2)}) = j(g_0gg_0^{-1},z)$, f is a Hilbert modular form for $g_0\Gamma g_0^{-1}$. If f is symmetric, then $f(-z_2,-z_1)=f(z_1,z_2)$ for all $\mathbb{H}\times\overline{\mathbb{H}}$. Hence for all $z\in\mathbb{H}\times\mathbb{H}$:

$$f\circ g_0^{-1}(z_2,z_1) = f(\tfrac{z_2}{\sqrt{d_F}},\tfrac{z_1}{-\sqrt{d_F}}) = f(\tfrac{z_1}{\sqrt{d_F}},\tfrac{z_2}{-\sqrt{d_F}}) = f\circ g_0^{-1}(z_1,z_2).$$

One proves the other direction analogously.

(ii). One shows analogously to (i) that $f\in M_k(\Gamma)$ if and only if $f\circ\left(\begin{smallmatrix}\varepsilon & \\ & 1\end{smallmatrix}\right)\in\overline{M}_k(\Gamma)$.

(iii). Let $f \in M_k(\Gamma)$ be symmetric, i.e. $f(z_2, z_1) = f(z_1, z_2)$ for all $z \in \mathbb{H} \times \mathbb{H}$. Then for $z \in \mathbb{H} \times \overline{\mathbb{H}}$:

$$\begin{aligned} f \circ \left(\begin{smallmatrix} \varepsilon & \\ & 1 \end{smallmatrix}\right)(-z_2, -z_1) &= f(-\varepsilon z_2, -\overline{\varepsilon} z_2) = f(-\overline{\varepsilon} z_2, -\varepsilon z_1) \\ = f\left(\left(\begin{smallmatrix} \varepsilon^{-1} & \\ & \varepsilon \end{smallmatrix}\right)(\varepsilon z_1, \overline{\varepsilon} z_2)\right) &= \mathcal{N}(\varepsilon)^k \, f \circ \left(\begin{smallmatrix} \varepsilon & \\ & 1 \end{smallmatrix}\right)(z_1, z_2). \end{aligned}$$

Since $\mathcal{N}(\varepsilon) = (-1)^k$, this proves (iii). □

The projective plane $\mathbb{P}^1(F)$ can be identified with F and the exceptional point ∞, and analogously $\mathbb{P}^1(\mathbb{R}) \subseteq \mathbb{P}^1(\mathbb{C})$ with $\mathbb{R} \subseteq \mathbb{C}$, respectively, and the exceptional point $i\infty$. With the two real embeddings we get an injection

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R})^2 \hookrightarrow \mathbb{P}^1(\mathbb{C})^2.$$

The point $\infty \in \mathbb{P}^1(F)$ is mapped to $(i\infty, i\infty)$. The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ with fractional transformations, so $\mathrm{GL}_2(\mathbb{C})^2$ acts on $\mathbb{P}_1^2(\mathbb{C})$ componentwise. The action of $\mathrm{GL}_2^+(F) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})$ is transitively on $\mathbb{P}^1(F)$.

Definition 3.8 (Cusps)
Let $\Gamma \subset \mathrm{GL}_2^+(F)$ be a congruence subgroup. The orbits in $\Gamma\backslash\mathbb{P}^1(F)$ are called the cusps *of Γ. In abuse of notation, the representatives of the orbits are called cusps as well.*

For level one, the number of cusps is equal to the class number of F. This was first proved by Maaß in [Maa40], in which he corrected a famous error of Blumenthal concerning the cusps of the fundamental domain.

Theorem 3.9 (Number of Cusps, Maaß)
Let Γ be a congruence subgroup with $\mathrm{SL}_2(\mathbb{Z}_F) \leq \Gamma \leq \mathrm{GL}_2^+(\mathbb{Z}_F)$. Then we have a bijection between the cusps and the class group of F,

$$\Gamma\backslash\mathbb{P}^1(F) \to CL(F), \; \Gamma(a:b) \mapsto [(a,b)].$$

Proof. [Maa40], [Fre90, Lemma 3.5], or [Gar90, 1.3].
This follows from the fact that a (fractional) ideal in a Dedekind domain can be generated by two elements. So if (x, y) is a fractional ideal, then its ideal class is

$$\left\{(x', y') \mid \begin{pmatrix} x' \\ y' \end{pmatrix} = \nu \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ for some } \nu \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Z(F)\Gamma \right\}.$$

But Γ acts on the same way on $\mathbb{P}^1(F)$, i.e. the orbit of $(x:y) \in \mathbb{P}^1(F)$ is

$$\left\{(x' : y') \mid \begin{pmatrix} x' \\ y' \end{pmatrix} = \nu \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ for some } \nu \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Z(F)\Gamma \right\}.$$

□

Corollary 3.10 *If Γ is any congruence subgroup, then Γ has only a finite number of cusps. Denote this number by s_Γ.*

Let $\kappa_1, \ldots, \kappa_s$ be representatives of the cusps and $g_1, \ldots, g_s \in \mathrm{SL}_2(F)$ such that $g_j(\infty) = \kappa_j$. Let $B = \mathrm{GL}_2^+(F)_\infty$ be the stabilizer of ∞, i.e. the upper triangular matrices. Then we have a disjoint union

$$\mathrm{GL}_2^+(F) = \bigcup_{j=1}^{s_\Gamma} \Gamma g_j B.$$

Proposition 3.11 (Fourier expansion)
Let Γ be a congruence subgroup and $f \in M_k(\Gamma)$ or $f \in \overline{M}_k(\Gamma)$. Especially $f(z+\nu) = f(z)$ for all $\nu \in F$ with $\left(\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.[1]

Hence $\mathfrak{m} := \{\nu \in F \mid \left(\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\}$ is a $\mathbb{Z}$-module and f has a Fourier expansion at $i\infty$ over the dual module $\mathfrak{m}^\# = \{\nu \in F \mid \mathrm{tr}(\nu\mathfrak{m}) \subseteq \mathbb{Z}\}$:

$$f(z) = \sum_{\nu \in \mathfrak{m}^\#} a_\nu(f) e^{2\pi i \operatorname{Tr}(\nu z)2}$$

(absolutely convergent on $\mathbb{H} \times \mathbb{H}$ or $\mathbb{H} \times \overline{\mathbb{H}}$, respectively; uniformly absolutely convergent in any compact subset). The coefficient $a_\nu(f) \in \mathbb{C}$ is called the Fourier coefficient *of f at ν.*

Proof. [Gar90, 1.2] $\square$

Theorem 3.12 (Koecher's principle)
Let f be a Hilbert modular form. Then f is holomorphic at $i\infty$, i.e. for the Fourier coefficients we have

$$a_\nu(f) \neq 0 \implies \nu = 0 \ \text{or} \begin{cases} \nu \gg 0 & \text{if } f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}, \\ \nu \gtrless 0 & \text{if } f : \mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{C}. \end{cases}$$

Proof. For $\mathbb{H} \times \mathbb{H}$ see [Gar90, 1.4]. This was actually first proved by Götzky in [Göt28]. Koecher proved a similar result for Siegel modular forms.

For $\mathbb{H} \times \overline{\mathbb{H}}$ the proof works the same way. Or one uses the connection between Γ and $g_0\Gamma g_0^{-1}$, see Theorem 3.7. Let $f : \mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{C}$ be a Hilbert modular form for Γ, then f has the Fourier expansion $f(z) = \sum_{\nu \in \mathbb{Z}_F^\#} a_\nu(f) \exp(2\pi i \operatorname{Tr}(z\nu))$. Also $f \circ g_0^{-1} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ is a Hilbert modular form for $g_0\Gamma g_0^{-1}$. Hence it has a Fourier expansion over $\mathfrak{m} = \sqrt{d_F}\mathbb{Z}_F$,

$$f \circ g_0^{-1}(z') = a_0(f \circ g_0^{-1}) + \sum_{\nu' \in \mathfrak{m}^\#_{\gg 0}} a_{\nu'}(f \circ g_0^{-1}) \exp(2\pi i \operatorname{Tr}(\nu' z')).$$

[1] $z + \nu = (z_1 + \nu^{(1)}, z_2 + \nu^{(2)})$.
[2] $\operatorname{Tr}(\nu z) = \nu^{(1)} z_1 + \nu^{(2)} z_2$.

Here $\mathfrak{m}^{\#} = \sqrt{d_F}^{-1}\mathbb{Z}_F^{\#}$. Hence f also has the Fourier expansion

$$f(z) = f(g_0^{-1}g_0 z) = \sum_{\nu' \in \sqrt{d_F}^{-1}\mathbb{Z}_F^{\#}} a_{\nu'}(f \circ g_0) \exp(2\pi i \operatorname{Tr}(z\sqrt{d_F}\nu')),$$

where $a_{\nu'}(f \circ g_0) \neq 0$ only if $\nu' = 0$ or $\nu' \gg 0$. By comparing the Fourier expansions, we get $\nu' = \sqrt{d_F}\nu$ and $a_{\nu'}(f \circ g_0^{-1}) = a_\nu(f)$. Hence $a_\nu(f) \neq 0$ only if $\nu = 0$ or $\nu \gtrless 0$. $\square$

Definition 3.13 *A Hilbert modular form f of level Γ and weight k is called a* cusp form *if $a_0(f|_k(g)) = 0$ for all $g \in \mathrm{GL}_2^+(F)$. The* space of cusp forms *of level Γ and weight k is denoted by $S_k(\Gamma)$ or $\overline{S}_k(\Gamma)$.*

Proposition 3.14

(i) If $M_k(\Gamma) \neq 0$ or $\overline{M}_k(\Gamma) \neq 0$, then $k = (0,0)$ or $k_1, k_2 > 0$.

(ii) $M_0(\Gamma) = \overline{M}_0(\Gamma) = \mathbb{C}$ and $S_0(\Gamma) = \overline{S}_0(\Gamma) = 0$.

(iii) If $S_k(\Gamma) \neq M_k(\Gamma)$ or $\overline{S}_k(\Gamma) \neq \overline{M}_k(\Gamma)$, then k is parallel.

Proof. [vdG88, Lemma (6.3)] and [Gar90, 1.4 and 1.7]. $\square$

Theorem 3.15 *The spaces $M_k(\Gamma)$ and $\overline{M}_k(\Gamma)$ have finite dimension over $\mathbb{C}$. Also*

$$\Big(\dim M_k(\Gamma) - \dim S_k(\Gamma)\Big) \leq s_\Gamma \text{ and } \Big(\dim \overline{M}_k(\Gamma) - \dim \overline{S}_k(\Gamma)\Big) \leq s_\Gamma,$$

where s_Γ is the number of cusps of Γ.

Proof. [Fre90, Theorem 6.1] or [Gar90, 1.7, 1.8]. $\square$

Hence one needs only finitely many Fourier coefficients to describe a Hilbert modular form uniquely. This is very useful for computations with Hilbert modular forms.

Example 3.16 (Eisenstein series)
Very important (non-cusp) modular forms are Eisenstein series. The classical Eisenstein series over $\mathbb{Q}$,

$$\sum_{(m,n) \in \mathbb{Z} \setminus \{0\}} (mz+n)^{-k}$$

is absolutely convergent for $k \geq 3$ and an elliptic modular form of weight k (for even $k \geq 4$).

Over totally real number fields other than $\mathbb{Q}$, the generalization of these Eisenstein series are also absolutely convergent and modular forms. But one may also define Eisenstein series for $k = 1$ and $k = 2$. These are also holomorphic functions and modular forms.

For simplicity reasons, we assume that $h_F = 1$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z}_F)$.

Eisenstein series for Hilbert modular forms were introduced by Hecke in [Hec24] and generalized by Kloosterman in [Klo28]. The following also uses [Gar90, Section 1.5 and 1.8], [Fre90, Chapter I §5 and III §4], and [Gun65].

Let B be the stabilizer of ∞ under Γ, i.e. the group of all triangular matrices in Γ. Let R be a set of representatives of $B\backslash\Gamma$.

Let $k \in \mathbb{N}$. If $\mathcal{N}(\varepsilon_0) = -1$, then assume that k is even.

First, let $k > 2$. We call

$$E_k(z) := \sum_{g \in R} j(g, z)^{-k}$$

the Eisenstein series *of weight k, where $z \in \mathbb{H} \times \mathbb{H}$ or $z \in \mathbb{H} \times \overline{\mathbb{H}}$.*

E_k is absolutely convergent and independent form the choice of R, because

$$j\left(\left(\begin{smallmatrix} \varepsilon^{-1} & \mu \\ 0 & \varepsilon \end{smallmatrix}\right), z\right)^{-k} = \mathcal{N}(\varepsilon)^{-k} = 1$$

for all $k \in \mathbb{N}$ (if $\mathcal{N}(\varepsilon_0) = 1$) or all $k \in 2\mathbb{N}$ (if $\mathcal{N}(\varepsilon_0) = -1$) and all $\varepsilon \in \mathbb{Z}_F^*$ and $\mu \in \mathbb{Z}_F$.

Since $E_k(gz) = j(g, z)^k E_k(z)$ for all $g \in \Gamma$, E_k is a Hilbert modular form of weight k and level Γ.

Secondly, let $k = 2$ or, if $\mathcal{N}(\varepsilon_0) = 1$ and $z \in \mathbb{H} \times \overline{\mathbb{H}}$, let $k = 1, 2$. We can also define Eisenstein series, which are Hilbert modular forms, but we have to use Hecke summation, see [Hec24]. Let

$$E_k(s; z) := \sum_{g \in R} j(g, z)^{-k} |j(g, z)|^s.$$

This function is holomorphic for $z \in \mathbb{H} \times \mathbb{H}$ or $z \in \mathbb{H} \times \overline{\mathbb{H}}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2 - k$. *Its continuation to all $s \in \mathbb{C}$ is meromorphic and regular in 0.*

$E_k(s; z)$ is independent of the choice of R and

$$E_k(s; gz) = j(g, z)^k |j(g, z)|^{-s} E_k(s; z) \text{ for all } g \in \Gamma \text{ (and fixed } s\text{).}$$

So we call

$$E_k(z) := E_k(0; z)$$

the Eisenstein series *of weight k, and $E_k(\tau)$ is a Hilbert modular form of weight k and level Γ.*

Another important examples of modular forms are theta series (of lattices). We will define these in later chapter, and compute some theta series.

3.2 Hilbert Modular Forms for $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$

Hilbert modular forms for $\mathbb{Q}[\sqrt{5}]$. The ring of Hilbert Modular forms of level one and parallel weight for the field $\mathbb{Q}[\sqrt{5}]$ was partly described by Maaß in [Maa41] and completely constructed by Gundlach in [Gun63]. Resnikoff [Res74], Hirzebruch [Hir73, Hir76, Ebe02], Müller [Mül85], and Mayer [May07] contributed other proofs using different structures such as algebraic geometric methods, embeddings into spaces of Siegel modular forms, or Borcherds products.

The ring of Hilbert modular forms is generated by the Eisenstein series A_2 of weight 2 and by the cusp forms s_5, B_6, and s_{15} (We use the notation of Ebeling's book [Ebe02]). The forms A_2, B_6, and s_{15} are symmetric, and s_5 can be chosen anti-symmetric. The index denotes in each case the weight of the form. The generators admit one relation, the square of s_{15} can be expressed by the other generators, i.e. there is a polynomial p is 3 variables such that $s_{15}^2 = p(A_2, s_5, B_6)$. Hence

$$M\left(\mathrm{SL}_2(\mathbb{Z}[\tfrac{1+\sqrt{5}}{2}])\right) = \mathbb{C}[A_2, s_5, B_6, s_{15}]\Big/(s_{15}^2 - p(A_2, s_5, B_6)).$$

The ring of symmetric forms of even weight is the polynomial ring

$$M\left(\mathrm{SL}_2(\mathbb{Z}[\tfrac{1+\sqrt{5}}{2}])\right)_{ev}^{+} = \mathbb{C}[A_2, B_6, C_{10}],$$

where $C_{10} = s_5^2$.

Hilbert modular forms for $\mathbb{Q}[\sqrt{2}]$. Again, Gundlach constructed the ring of Hilbert modular forms of level one for the field $\mathbb{Q}[\sqrt{2}]$, see [Gun65]. Müller gave in [Mül84] an elementary description of the modular forms. The ring of Hilbert modular forms is generated by the Eisenstein series g_2 and by the cusp forms s_4, s_5, s_6 and s_9. The weight is given by the index. The forms g_2, s_4, s_6, and s_9 are symmetric, and s_5 is anti-symmetric. The generators admit two relations, $s_5^2 = s_4 s_6$ and there is a polynomial p is 3 variables such that $s_9^2 = p(g_2, s_4, s_6)$. Hence

$$M\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])\right) = \mathbb{C}[g_2, s_4, s_5, s_6, s_9]\Big/(s_5^2 - s_4 s_6, s_9^2 - p(g_2, s_4, s_6)).$$

The ring of symmetric forms of even weight is

$$M_{ev}^{+}\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])\right) = \mathbb{C}[g_2, s_4, s_6].$$

Hilbert modular forms for $\mathbb{Q}[\sqrt{3}]$. Again, Gundlach constructed the rings of Hilbert modular forms of level one for the field $\mathbb{Q}[\sqrt{3}]$, see [Gun65]. One has to distinguish between modular forms on $\mathbb{H} \times \mathbb{H}$ and $\mathbb{H} \times \overline{\mathbb{H}}$.

Gundlach only considered modular forms which are Galois and fundamentally symmetric, i.e. modular forms for the group

$$\Gamma_F^{\varepsilon_0} := \langle \mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]), \left(\begin{smallmatrix} 2+\sqrt{3} & 0 \\ 0 & 1 \end{smallmatrix}\right) \rangle.$$

The ring of Galois and fundamentally symmetric Hilbert modular forms of $\mathbb{H} \times \mathbb{H}$ is a polynomial ring in the Eisenstein series g_2, g_3, and g_4 of weight 2, 3, and 4, respectively,

$$M^{\varepsilon_0 +}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])) = \mathbb{C}[g_2, g_3, g_4].$$

The ring of Galois and fundamentally symmetric Hilbert modular forms of $\mathbb{H} \times \overline{\mathbb{H}}$ is is a polynomial ring in the Eisenstein series g_1, g_4, and g_6 of weight 1, 4, and 6, respectively,

$$\overline{M}^{\varepsilon_0 +}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])) = \mathbb{C}[g_1, g_4, g_6].$$

We will construct $\overline{M}^+(\mathrm{SL}_2[\sqrt{3}])$ in Theorem 8.1. There is a symmetric cusp form s_3 of weight 3, such that

$$\overline{M}^+(\mathrm{SL}_2[\sqrt{3}]) = \mathbb{C}[g_1, s_3, g_4].$$

3.3 Hecke Operators for Class Number One

Assume that the narrow class number is one, $h_F^+ = 1$. In this case, and only in this case, we can define Hecke operators on classical Hilbert modular forms. These are similar to the Hecke operators of elliptic modular forms. If $h_F^+ > 1$, the same definitions would provide operators which would not map Hilbert modular forms to Hilbert modular forms. A solution is to consider adelic automorphic forms instead of modular forms. For these the theory of Hecke operators works nicely, but the Fourier coefficients are harder to compute. Since we are mainly interested in the Fourier coefficients, we introduce Hecke operators on classical Hilbert modular forms. In consequence we have $h_F^+ = 1$, and hence we only consider modular forms on $\mathbb{H} \times \mathbb{H}$.

We also only consider level one. So let $\Gamma := \mathrm{SL}_2(\mathbb{Z}_F)$.

Let $k \in \mathbb{Z}^r$ be a weight vector with each $k_j > 2$. This section is a recap of [Gar90, 1.15].

Let $\mathfrak{n} \subseteq \mathbb{Z}_F$ be an ideal and $\eta \in \mathbb{Z}_F$ be a totally positive generator of $\mathfrak{n}$. We define

$$\Delta(\mathfrak{n}) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) \cap \mathbb{Z}_F^{2\times 2} \;\middle|\; (\det g) = \mathfrak{n} \right\},$$

and let $R(\mathfrak{n})$ be a transversal of ${}_{\Gamma Z(\mathbb{Z}_F)}\backslash\Delta(\mathfrak{n})$.

Recall that $Z(\mathbb{Z}_F)$ is the center of $\mathrm{GL}_2^+(\mathbb{Z}_F)$. The $\mathfrak{n}$-th Hecke operator is

$$T_\mathfrak{n} : M_k(\Gamma) \to M_k(\Gamma),\ f(z) \mapsto \sum_{g \in R(\mathfrak{n})} f|_k g\,(z).$$

Proposition 3.17 *Let $f = \sum_{\nu \in \mathbb{Z}_F^\#} a_\nu(f) q^\nu \in M_k(\Gamma)$. Then $T_\mathfrak{n} f$ is a Hilbert modular form and has the Fourier coefficients*

$$a_\nu(T_\mathfrak{n} f) = \eta^{1-k/2} \sum_\delta \delta^{k-1} a_{\nu\eta/\delta^2}(f),$$

where the sum is over $\mathbb{Z}_F^* \backslash \{\delta \gg 0 \mid \delta|\eta,\ \nu/\delta \in \mathbb{Z}_F^\#\}$. Especially, $T_\mathfrak{n}$ maps cusp forms to cusp forms.

Lemma 3.18 *Let $\mathfrak{m}, \mathfrak{n} \subseteq \mathbb{Z}_F$ be coprime ideals and $\mathfrak{p} \subseteq \mathbb{Z}_F$ a prime ideal. Then for the restriction of their Hecke operators we have*

(i) $T_\mathfrak{n}$ is self-adjoint with respect to the Petersson inner product $<,>$.

(ii) $T_\mathfrak{m} T_\mathfrak{n} = T_\mathfrak{n} T_\mathfrak{m}$.

(iii) $T_\mathfrak{p} T_{\mathfrak{p}^\ell} = T_\mathfrak{p}^{\ell+1} + \mathcal{N}(\mathfrak{p}) T_{\mathfrak{p}^{\ell-1}}$.

With these results by using elementary linear algebra, the following theorem follows immediately.

Theorem 3.19 *There is an orthogonal basis of $S_k(\Gamma)$ consisting of simultaneous eigenvectors for all the Hecke operators. These eigenvectors are called* Hecke eigenforms.

If $f = \sum_{\nu \in \mathbb{Z}_F^\#} a_\nu(f) q^\nu \in S_k(\Gamma)$ is such a Hecke eigenform, then the eigenvalue for an integral ideal $\mathfrak{n} = \eta \mathbb{Z}_F$ with $\eta \gg 0$ is

$$T_\mathfrak{n} f = \lambda(\mathfrak{n}) f \text{ with } \lambda(\mathfrak{n}) = \mathcal{N}(\eta) \eta^{-k/2} \frac{a_{\mu\eta}(f)}{a_\mu(f)},$$

where $\mu \gg 0$ generates $\mathbb{Z}_F^\#$.

Corollary 3.20 *Let $\mathbb{Z}_F^\# = \delta^{-1} \mathbb{Z}_F$ with $\delta \gg 0$. Let $f \in S_k(\Gamma)$ be a Hecke eigenform with $a_\delta^{-1}(f) = 1$ (we call f* normalized*).*

Let $\alpha, \beta \in \mathbb{Z}_F$ be totally positive and coprime, and let $\pi \in \mathbb{Z}_F$ be a totally positive prime. Then the following identities are true for all $\nu \in \mathbb{Z}_F^*$ with $\nu \gg 0$:

(i) $a_{\alpha\beta/\delta}(f) = a_{\alpha/\delta}(f) a_{\beta/\delta}(f)$ and

(ii) $a_{\pi/\delta}(f) a_{\pi^m/\delta}(f) = a_{\pi^{m+1}/\delta}(f) + \pi^{k-1} a_{\pi^{m-1}/\delta}(f)$.

Computational Implementation L. Dembélé and J. Voight provided a Magma[3] package to compute the spaces $S_k(\Gamma)$, see [DemVoi12]. For arbitrary class number, Dembélé and Voight's program computes the Hecke operators for given level and even weight. They use the adelic modular forms which are defined over the adelic ring of F.

For narrow class number one, I have written a MAGMA program which computes the so called q-expansion of Hecke eigenforms. The q-expansion is defined in the next section. The program is available on my website

http://www.math.rwth-aachen.de/∼David.Dursthoff/.

3.4 q-Expansion and Extremal Modular Forms

If f is a Hilbert modular form, then f has a Fourier expansion, i.e.

$$f = a_0(f) + \sum_{0 \ll \nu \in \mathbb{Z}_F^{\#}} a_\nu(f) q^\nu,$$

where $q := (\exp 2\pi i z_1, \exp 2\pi i z_2)$ and $q^\nu = e^{2\pi i\, \nu^{(1)} z_1} e^{2\pi i\, \nu^{(2)} z_2}$.

So we have an expression of modular forms in two variables q_1 and q_2, and we may order the coefficients in a nice way. This is an useful way to compute modular forms and describe them by their first coefficients. Since the space of modular forms of given weight is finite, finitely many coefficients describe a modular form uniquely.

On the other hand, restrictions of the variables yield modular forms over smaller fields. So, for example, a restriction of a Hilbert modular form $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$ of weight k to the diagonal line $\{(z_1, z_1) \mid z_1 \in \mathbb{H}\}$ is an elliptic modular form of degree $2k$ for the classical modular group $\mathrm{SL}_2(\mathbb{Z})$.

To combine these approaches we want to write a Hilbert modular form as

$$f(q_1, q_2) = \sum_\nu a_\nu(f) q_1^{\mathrm{tr}\beta_1\nu} q_2^{\mathrm{tr}\beta_2\nu},$$

where $\beta_1, \beta_2 \in \mathbb{Z}_F^{\#}$, such that $f(q_1, 1)$ is an elliptic modular form of level 1. The restriction to another line in $\mathbb{H} \times \mathbb{H}$ or $\mathbb{H} \times \overline{\mathbb{H}}$, given by q_2, should yield elliptic modular forms of higher level. To do that, we define arbitrary q-expansions.

Definition 3.21 (q-Expansion)
Let $B = (\beta_1, \beta_2) \in \mathbb{Z}_F^2$ be a $\mathbb{Q}$-basis of F. Set $T := (\beta_\ell^{(j)})_{\ell,j} \in \mathrm{GL}_2(\mathbb{C})$, $w := (w_1, w_2) := zT^{-1}$, $q_1 := \exp(2\pi i\, w_1)$, $q_2 := \exp(2\pi i\, w_2)$, and $q = (q_1, q_2)$.

Let Γ be a congruence subgroup and $f \in M_k(\Gamma)$ or $f \in \overline{M}_k(\Gamma)$. For $\nu \in \mathbb{Z}_F^{\#}$ the Fourier coefficient of f at ν is $a_\nu(f)$. For $n, m \in \mathbb{Z}$ we define

$$a_{n,m}(f) := \begin{cases} a_\nu(f) & \text{if there is a } \nu \in \mathbb{Z}_F^{\#} \text{ such that } n = \mathrm{tr}(\beta_1\nu) \text{ and } m = \mathrm{tr}(\beta_2\nu), \\ 0 & \text{else.} \end{cases}$$

[3] The Magma Algebra System. See [BosCanPla97] or http://magma.maths.usyd.edu.au/magma/

We call

$$f(q) = f(q_1, q_2) = \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m}(f)\, q_1^n q_2^m$$

the q-expansion *of* f *with respect to* B.

Note that the Fourier coefficients are non-zero only for $\nu = 0$ and $\nu \gg 0$ (in the case $f : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$) or $\nu \gtrless 0$ (in the case $f : \mathbb{H}\times\overline{\mathbb{H}} \to \mathbb{C}$).

Theorem 3.22

(i) Suppose that $\beta_1, \beta_2 \gg 0$. *Let* $f \in M_k(\Gamma)$. *Then* f *is equal to its* q*-expansion, i.e.* $f(z) = f(q_1, q_2)$ *for all* $z \in \mathbb{H}\times\mathbb{H}$, *and*

$$f(q_1, q_2) \in \mathbb{C}[q_1][[q_2]] \cap \mathbb{C}[q_2][[q_1]] \subseteq \mathbb{C}[[q_1, q_2]].$$

(ii) Suppose that $\beta_1, \beta_2 \gtrless 0$. *Let* $f \in \overline{M}_k(\Gamma)$. *Then* f *is equal to its* q*-expansion, i.e.* $f(z) = f(q_1, q_2)$ *for all* $z \in \mathbb{H}\times\overline{\mathbb{H}}$, *and*

$$f(q_1, q_2) \in \mathbb{C}[q_1][[q_2]] \cap \mathbb{C}[q_2][[q_1]] \subseteq \mathbb{C}[[q_1, q_2]].$$

Proof. (i). Let $z \in \mathbb{H}\times\mathbb{H}$. Since $\mathrm{Tr}(\nu z) = w_1 \mathrm{tr}(\beta_1\nu) + w_2 \mathrm{tr}(\beta_2\nu)$, we have

$$f(z) = \sum_{\nu\in(\mathbb{Z}_F^{\#})_{\gg 0}} a_\nu(f)\, e^{2\pi i \mathrm{Tr}(\nu z)} = \sum_{\nu\in(\mathbb{Z}_F^{\#})_{\gg 0}} a_\nu(f)\, q_1^{\mathrm{tr}(\beta_1\nu)} q_2^{\mathrm{tr}(\beta_2\nu)} = f(q_1, q_2).$$

Since $\beta_1, \beta_2 \in (\mathbb{Z}_F)_{\gg 0}$, we have $\mathrm{tr}(\beta_1\nu), \mathrm{tr}(\beta_2\nu) \in \mathbb{Z}_{>0}$ for all $\nu \in (\mathbb{Z}_F^{\#})_{\gg 0}$. So $f(q_1, q_2) \in \mathbb{C}[[q_1, q_2]]$.

To see that $f \in \mathbb{C}[q_2][[q_1]]$ we have to show that for given $n \in \mathbb{N}$ the set

$$S(n) := \{\nu \in \mathbb{Z}_F^{\#} \mid \nu \gg 0,\ \mathrm{tr}(\beta_1\nu) = n\}$$

is finite. We identify F with $F^{(1)}$ and set $x := \mathrm{tr}(\beta_1)$ and $y := \mathrm{tr}(\sqrt{D}\beta_1)$.

Let $\nu = a + b\sqrt{D} \in \mathbb{Z}_F^{\#}$ with $a, b \in \mathbb{Z}$ (if $D \equiv 2, 3 \pmod 4$) or $a, b \in \frac{1}{2}\mathbb{Z}$, $(a-b) \in \mathbb{Z}$ (if $D \equiv 1 \pmod 4$). Then $\nu \gg 0$ if and only if $|b|\sqrt{D} \leq a$. So

$$\nu \in S(n) \iff \mathrm{tr}(\beta_1\nu) = ax + by = n \iff a = \frac{n - b\,y}{x}.$$

Since $|b|\sqrt{D} < a$ we have

$$-n < b\,\underbrace{(x\sqrt{D} - y)}_{=2\overline{\beta_1}\sqrt{D}>0} \text{ and } b\,\underbrace{(x\sqrt{D} + y)}_{=2\beta_1\sqrt{D}>0} < n$$

and hence $\frac{-n}{2\sqrt{D}\overline{\beta_1}} < b < \frac{n}{2\sqrt{D}\beta_1}$. Therefore there are only finitely many choices for b and $S(n)$ is finite. One shows $f \in \mathbb{C}[q_1][[q_2]]$ analogously.

(ii). Analogously $f(z) = f(q_1, q_2)$ for all $z \in \mathbb{H} \times \overline{\mathbb{H}}$, note that we sum over $\nu = 0$ and all $\nu \in \mathbb{Z}_F^{\#}$ with $\nu \gtrless 0$ (see Theorem 3.12). Here we have $\beta_1, \beta_2 \in (\mathbb{Z}_F)_{\gtrless 0}$, hence $\mathrm{tr}(\beta_1\nu), \mathrm{tr}(\beta_2\nu) \in \mathbb{Z}_{>0}$ for all $\nu \in (\mathbb{Z}_F^{\#})_{\gtrless 0}$. The Fourier coefficients are non-trivial only for $\nu = 0$ and $\nu \in \mathbb{Z}_F^{\#}$ with $\nu \gtrless 0$, so again $f(q_1, q_2) \in \mathbb{C}[[q_1, q_2]]$. As for (i), one shows $f(q_1, q_2) \in \mathbb{C}[q_1][[q_2]] \cap \mathbb{C}[q_2][[q_1]]$. $\square$

So we consider Hilbert modular forms as formal power series with some conditions on the coefficients. This is useful for computer calculations. Only finitely many coefficients must be calculated, because the space of modular forms is finite dimensional. And the proof of the previous theorem gives a way to compute bounds for the powers of q_1 and q_2. We compute such bounds for concrete choices of B in the next section.

Some choices of β_1 and β_2 are especially interesting, because the restrictions $f(q_1, 1)$ and $f(1, q_2)$ define elliptic modular forms of interesting levels. The levels are the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid N \text{ divides } c \right\}.$$

Proposition 3.23 *Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}_F))$ and $\beta_1, \beta_2 \gg 0$ or $f \in \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F))$ and $\beta_1, \beta_2 \gtrless 0$. Then*

$$\mathbb{H} \to \mathbb{C},\ \tau \mapsto f\,(q, 1) = \sum_{n \geq 0} \sum_{m \geq 0} a_{n,m}(f)\, q^n,\ \textit{where } q = e^{2\pi i \tau},$$

is an elliptic modular form of weight $2k$ and level $\Gamma_0(\mathcal{N}(\beta_1))$.

Proof. First we consider the $\mathbb{H} \times \mathbb{H}$ case. We define the map

$$\omega : \mathbb{H} \to \mathbb{H} \times \mathbb{H},\ \tau \mapsto (\beta_1^{(1)}\tau, \beta_1^{(2)}\tau).$$

Let $\tau \in \mathbb{H} \times \mathbb{H}$. Put $z := \omega(\tau)$, then in the notation of the q-expansion we have $w = zT^{-1} = (\tau, 0)$. Hence $f(z) = f(q, 1)$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{N}(\beta_1))$. Since c is a multiple of $\mathcal{N}(\beta_1) = \beta_1\overline{\beta_1}$, we have $\frac{c}{\beta_1} \in \mathbb{Z}_F$. We have

$$\begin{pmatrix} \beta_1 & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1/\beta_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b\beta_1 \\ c/\beta_1 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_F).$$

Therefore

$$f(\omega(g\tau)) = f\left(\begin{pmatrix} \beta_1 & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1/\beta_1 & 0 \\ 0 & 1 \end{pmatrix} (\beta_1^{(1)}\tau, \beta_1^{(2)}\tau) \right) = (\frac{c}{\beta_1}\beta_1\tau + d)^k f(\omega(\tau)).$$

Since $(\frac{c}{\beta_1}\beta_1\tau + d)^k = (c\tau + d)^{2k}$, the restriction $f \circ \omega$ is an elliptic modular form of weight $2k$ and level $\Gamma_0(\mathcal{N}(\beta_1))$ (and trivial character).

In the $\mathbb{H} \times \overline{\mathbb{H}}$ case one has $\beta_1^{(2)} < 0$, hence one gets a map $\omega : \mathbb{H} \to \mathbb{H} \times \overline{\mathbb{H}}$. The proof works in the same way as for the $\mathbb{H} \times \mathbb{H}$ case. $\square$

Next we will define an ordering on the coefficients.

Definition 3.24
Let $B = (\beta_1, \beta_2) \in \mathbb{Z}_F$ form a $\mathbb{Q}$-basis of F. Assume that $\beta_1, \beta_2 \gg 0$ in the $\mathbb{H} \times \mathbb{H}$ case or $\beta_1, \beta_2 \gtrless 0$ in the $\mathbb{H} \times \overline{\mathbb{H}}$ case. The lexicographic ordering $\leq$ on $\mathbb{Z} \times \mathbb{Z}$ yields a total ordering $\leq_B$ of the monomials $\{q_1^n q_2^m \mid n, m \in \mathbb{Z}\}$:

$$q_1^n q_2^m \leq_B q_1^{n'} q_2^{'m} \;\Leftrightarrow\; (n, m) \leq (n', m').$$

And $\leq_B$ defines a valuation on $M_k(\mathrm{SL}_2(\mathbb{Z}_F))$ and $\overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F))$:

$$\nu_B(f) := \min\{(n, m) \mid a_{n,m}(f) \neq 0\}.$$

The total ordering $\leq_B$ of F, see Definition 1.1, gives the same valuation. More precisely, $\nu_B(f) = (n, m)$ if and only if there is some $\nu \in \mathbb{Z}_F^\#$ such that $n = \mathrm{tr}(\beta_1\nu)$, $m = \mathrm{tr}(\beta_2\nu)$, and ν is the $\leq_B$-minimum of $\{\nu' \in \mathbb{Z}_F^\# \mid a_{\nu'}(f) \neq 0\}$. So it is justified to use $\leq_B$ both for an ordering of the number field and the monomials.

With the valuation we can define extremal Hilbert modular forms. In this way, this was first done in [Neb13].

Definition 3.25 (Extremal Hilbert Modular Forms)
Let $k \in \mathbb{N}$ and $X \leq M_k(\mathrm{SL}_2(\mathbb{Z}_F))$ or $X \leq \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F))$.

A Hilbert modular form $f \in X$ is called extremal *in X (with respect to B) if*

$$\nu_B(f - 1) \geq \nu_B(f' - 1) \text{ for all } f' \in X.$$

In conclusion, the ordering $\leq_B$ and the valuation ν_B are useful when computing with modular forms. For example, by projecting to sufficient many coefficients and Gauss eliminations, one may compute a basis $(f_1, \ldots, f_s)$ of $M_k(\Gamma)$ or $\overline{M}_k(\Gamma)$ with

$$\nu(f_1) < \nu(f_2) < \cdots < \nu(f_s) \text{ and } a_{\nu(f_i)}(f_j) = \delta_{i,j}.$$

Especially, if $h_F = 1$, the forms $f_2, \ldots, f_s$ are cusp forms, and if f_1 is not a cusp form, then it is extremal.

For our purposes we want to fix β_1 and β_2. If not mentioned otherwise, we will always use the following three q-expansions, distinguished by the field and the modular forms.

Definition 3.26 (Standard q-Expansion)

(i) *Suppose that the fundamental unit ε_0 has norm -1. Set $e := \sqrt{\frac{d_F}{D}} \in \{1,2\}$, then $\delta = e\sqrt{D}\varepsilon_0$ is a totally positive generator of the different. We set*

$$\beta_1 := 1 \text{ and } \beta_2 := \sqrt{D}\varepsilon_0.$$

Hence, if $f \in M_k(\mathrm{SL}_2(\mathbb{Z}_F))$, then the q-expansion to $B = (\beta_1, \beta_2)$ is

$$f \;=\; a_0(f) + \sum_{\nu \in \mathbb{Z}^{\#}_{F \gg 0}} a_\nu(f) q_1^{\mathrm{tr}(\nu)} q_2^{\mathrm{tr}(\beta_2 \nu)} \;=\; a_0(f) + \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\mu/\delta}(f) q_1^{\mathrm{tr}(\mu/\delta)} q_2^{\mathrm{tr}(\mu/e)}.$$

We prefer the second sum, so we sum over totally positive elements of integer ring instead of the inverse different. The restriction $f(q_1, 1)$ is an elliptic modular form of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, and $f(1, q_2)$ is an elliptic modular form of weight $2k$ for the congruence subgroup $\Gamma_0(D)$.

Hilbert modular forms $f \in \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F))$ are not considered because of Proposition 3.7(ii).

(ii) *Suppose that $\mathcal{N}(\varepsilon_0) = 1$. We set $e := \sqrt{\frac{d_F}{D}}$,*

$$\beta_1 := \sqrt{D} \text{ and } \beta_2 := (\sqrt{D} - 1).$$

If $f \in \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F))$, then the q-expansion to $B = (\beta_1, \beta_2)$ is

$$f \;=\; a_0(f) + \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\mu/\sqrt{d_F}}(f) q_1^{\mathrm{tr}(\mu/e)} q_2^{\mathrm{tr}(\mu \frac{\sqrt{D}-1}{\sqrt{d_F}})}.$$

So the restriction $f(q_1, 1)$ is an elliptic modular form of weight $2k$ for $\Gamma_0(D)$, and $f(1, q_2)$ is an elliptic modular form of weight $2k$ for $\Gamma_0(D-1)$.

(iii) *Suppose $\mathcal{N}(\varepsilon_0) = 1$. We set $e := \sqrt{\frac{d_F}{D}}$,*

$$\beta_1 := 1 \text{ and } \beta_2 := D + \sqrt{D}.$$

If $f \in M_k(\mathrm{SL}_2(\mathbb{Z}_F))$, then the q-expansion to $B = (\beta_1, \beta_2)$ is

$$f \;=\; a_0(f) + \sum_{\nu \in \mathbb{Z}^{\#}_{F \gg 0}} a_\nu(f) q_1^{\mathrm{tr}(\nu)} q_2^{\mathrm{tr}(\nu(D+\sqrt{D}))}.$$

So the restriction $f(q_1, 1)$ is an elliptic modular form of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, and $f(1, q_2)$ is an elliptic modular form of weight $2k$ for $\Gamma_0(D(D-1))$.

3.5 q-Expansions over $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$

We use the following standard q-expansions given by $B = (\beta_1, \beta_2)$.

- If $F = \mathbb{Q}[\sqrt{5}]$ (Case (i)), then $\beta_1 = 1$ and $\beta_2 = \frac{5+\sqrt{5}}{2}$.
- If $F = \mathbb{Q}[\sqrt{2}]$ (Case (i)), then $\beta_1 = 1$ and $\beta_2 = 2 + \sqrt{2}$.
- If $F = \mathbb{Q}[\sqrt{3}]$ and modular forms $\mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{C}$ (Case (ii)), then $\beta_1 = \sqrt{3}$ and $\beta_2 = -1 + \sqrt{3}$.
- If $F = \mathbb{Q}[\sqrt{3}]$ and modular forms $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$ (Case (iii)), then $\beta_1 = 1$ and $\beta_2 = 3 + \sqrt{3}$.

Theorem 3.27 (q-expansions of the generators of $M\left(\mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}])\right)$)
The generators of $M\left(\mathrm{SL}_2(\mathbb{Z}[\frac{1+\sqrt{5}}{2}])\right)$ introduced in Section 3.2 have the following q-expansions.

$$
\begin{aligned}
A_2 &= 1 + 120q_1q_2^2 + 120q_1q_2^2 + 120q_1^2q_2^3 + 600q_1^2q_2^4 + 720q_1^2q_2^5 + 600q_1^2q_2^6 \\
&\quad +120q_1^2q_2^7 + 1440q_1^3q_2^7 + 1440q_1^3q_2^8 + 1200q_1^3q_2^9 + 720q_1^3q_2^{10} + O(q_1^4q_2^6) \\
s_5 &= q_1q_2^2 - q_1q_2^3 - q_1^2q_2^3 - 10q_1^2q_2^4 + 10q_1^2q_2^6 + q_1^2q_2^7 \\
&\quad +120q_1^3q_2^6 - 108q_1^3q_2^7 + 108q_1^3q_2^8 - 120q_1^3q_2^9 + O(q_1^4q_2^6) \\
B_6 &= q_1q_2^2q_1q_2^3 + q_1^2q_2^3 - 910q_1^2q_2^5 + q_1^2q_2^7 - 910q_1^3q_2^5 + 25650q_1^3q_2^6 + 24092q_1^3q_2^7 \\
&\quad +24092q_1^3q_2^8 + 25650q_1^3q_2^9 - 910q_1^3q_2^{10} + O(q_1^4q_2^7) \\
C_{10} &= q_1^2q_2^4 - 2q_1^2q_2^5 + q_1^2q_2^6 - 2q_1^3q_2^5 - 18q_1^3q_2^6 + 20q_1^3q_2^8 \\
&\quad -18q_1^3q_2^9 - 2q_1^3q_2^{10} + O(q_1^4q_2^6) \\
s_{15} &= q_1^2q_2^5 - q_1^3q_2^5 - 275q_1^3q_2^7 - 275q_1^3q_2^8 - q_1^3q_2^{10} + O(q_1^4q_2^7)
\end{aligned}
$$

Proof. There are several ways to compute the coefficients. One may apply the methods described in [Mül85] an [May07]. For A_2, B_6, and C_{10} one can use the Hilbert modular forms Magma package by Dembélé and Voight and my algorithms, see Appendix A. The forms A_2, s_5, B_6, and C_{10} can be constructed with lattices, see Chapter 6. For s_{15} see [Mül85]. $\square$

Theorem 3.28 (q-expansions of the generators of $M\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])\right)$)
The generators of $M\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])\right)$ introduced in Section 3.2 have the following q-

expansions.

$$\begin{aligned}
g_2 &= 1 + 48q_1q_2 + 144q_1q_2^2 + 48q_1q_2^3 + 336q_1^2q_2^2 + 720q_1^2q_2^4 + 384q_1^2q_2^5 + 336q_1^2q_2^6 \\
&\quad +144q_1^3q_2^2 + 480q_1^3q_2^3 + 1152q_1^3q_2^4 + 864q_1^3q_2^5 + 1440q_1^3q_2^6 + 864q_1^3q_2^7 \\
&\quad +1152q_1^3q_2^8 + 480q_1^3q_2^9 + 144q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_4 &= q_1q_2 - 2q_1q_2^2 + q_1q_2^3 - 4q_1^2q_2^2 - 8q_1^2q_2^3 + 24q_1^2q_2^4 - 8q_1^2q_2^5 - 4q_1^2q_2^6 - 2q_1^3q_2^2 \\
&\quad +26q_1^3q_2^3 + 16q_1^3q_2^4 - 14q_1^3q_2^5 - 52q_1^3q_2^6 - 14q_1^3q_2^7 + 16q_1^3q_2^8 + 26q_1^3q_2^9 \\
&\quad -2q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_5 &= q_1q_2 - q_1q_2^3 + 16q_1^2q_2^2 - 56q_1^2q_2^3 + 56q_1^2q_2^5 - 16q_1^2q_2^6 \\
&\quad -42q_1^3q_2^3 + 378q_1^3q_2^5 - 378q_1^3q_2^7 + 42q_1^3q_2^9 + O(q_1^4q_2^3) \\
s_6 &= q_1q_2^2 - 2q_1^2q_2^2 - 16q_1^2q_2^3 + 12q_1^2q_2^4 - 16q_1^2q_2^5 - 2q_1^2q_2^6 + q_1^3q_2^2 + 32q_1^3q_2^3 + 40q_1^3q_2^4 \\
&\quad -32q_1^3q_2^5 + 170q_1^3q_2^6 - 32q_1^3q_2^7 + 40q_1^3q_2^8 + 32q_1^3q_2^9 + q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_9 &= q_1q_2^2 - 96q_1^2q_2^3 - 336q_1^2q_2^4 - 96q_1^2q_2^5 + O(q_1^3q_2^2)
\end{aligned}$$

Proof. Again, one can use the Magma package by Dembélé and Voight and my algorithms of Appendix A. The forms g_2, s_4, s_5, and s_6 can be constructed with lattices, see Chapter 7. The coefficients of s_9 are given in [Mül84]. □

Theorem 3.29 (q-expansions of the generators for $\mathbb{Q}[\sqrt{3}]$)

(i) The generators of $M^{\varepsilon_0+}\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])\right)$ *introduced in Section 3.2 have the following q-expansions:*

$$\begin{aligned}
g_2 &= 1 + 72q_1q_2^2 + 96q_1q_2^3 + 72q_1q_2^4 + 96q_1^2q_2^3 + 360q_1^2q_2^4 + 288q_1^2q_2^5 \\
&\quad +672q_1^2q_2^6 + 288q_1^2q_2^7 + 360q_1^2q_2^8 + 96q_1^2q_2^9 + O(q_1^3q_2^4) \\
g_3 &= 1 - 108q_1q_2^2 - 288q_1q_2^3 - 108q_1q_2^4 - 288q_1^2q_2^3 - 1836q_1^2q_2^4 - 4320q_1^2q_2^5 \\
&\quad -3744q_1^2q_2^6 - 4320q_1^2q_2^7 - 1836q_1^2q_2^8 - 288q_1^2q_2^9 + O(q_1^3q_2^4) \\
g_4 &= 23 + 2160q_1q_2^2 + 6720q_1q_2^3 + 2160q_1q_2^4 + 6720q_1^2q_2^3 + 140400q_1^2q_2^4 \\
&\quad +319680q_1^2q_2^5 + 490560q_1^2q_2^6 + 319680q_1^2q_2^7 + 140400q_1^2q_2^8 \\
&\quad +6720q_1^2q_2^9 + O(q_1^3q_2^4)
\end{aligned}$$

(ii) The generators of $\overline{M}^{+}\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])\right)$ *introduced in Section 3.2 have the following q-expansions:*

$$\begin{aligned}
g_1 &= 1 + 12q_1q_2 + 12q_1^2q_2 + 12q_1^2q_2^2 + 12q_1^2q_2^3 + 12q_1^3q_2^3 + O(q_1^4q_2^2) \\
s_5 &= q_1q_2 - q_1^2q_2 - 4q_1^2q_2^2 - q_1^2q_2^3 + 9q_1^3q_2^3 + O(q_1^4q_2^2) \\
g_4 &= 23 + 240q_1q_2 + 240q_1^2q_2 + 17520q_1^2q_2^2 + 240q_1^2q_2^3 + 60480q_1^3q_2^2 \\
&\quad +181680q_1^3q_2^3 + 60480q_1^4q_2^3 + O(q_1^4q_2^2) \\
g_6 &= 1681 + 504q_1q_2 + 504q_1^2q_2 + 532728q_1^2q_2^3 + 504q_1^2q_2^3 + 4058208q_1^3q_2^2 \\
&\quad +29883672q_1^3q_2^3 + 4058208q_1^3q_2^4 + O(q_1^4q_2^2)
\end{aligned}$$

Proof. The coefficients of all forms except s_5 can be found in [CohDeu88]. The coefficients of s_5 and all other forms can be constructed with lattices, see Chapters 8 and 9. $\square$

Finally, we focus on estimates for the coefficients and the powers. They are useful both for computations and theoretical calculations. Also, we see how the extensions of the Galois automorphisms

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}, (z_1, z_2) \mapsto (z_2, z_1), \ \mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{H} \times \overline{\mathbb{H}}, (z_1, z_2) \mapsto (-z_2, -z_1)$$

translate to the q-expansion.

Lemma 3.30 **($F = \mathbb{Q}[\sqrt{5}]$)** *Let*

$$f = \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\mu/\delta}(f) q_1^{\mathrm{tr}(\mu/\delta)} q_2^{\mathrm{tr}(\mu)} \in M_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Let $\mu \in \mathbb{Z}_F$ be totally positive and set $n := \mathrm{tr}(\mu\delta)$ and $m := \mathrm{tr}(\mu)$, i.e. $a_{\mu/\delta}(f) = a_{n,m}(f)$.

(i) If $a_{n,m}(f) \neq 0$ then

$$\begin{aligned} \frac{5-\sqrt{5}}{2} n < \quad & m \quad < \frac{5+\sqrt{5}}{2} n, \\ \frac{5-\sqrt{5}}{10} m < \quad & n \quad < \frac{5+\sqrt{5}}{10} m. \end{aligned}$$

(ii) $a_{\overline{\mu}/\delta}(f) = a_{m-n,m}(f)$.

(iii) $a_{\overline{\mu/\delta}}(f) = a_{n,5n-m}(f)$.

(iv) The Galois-conjugated Hilbert modular form $f(z_2, z_1)$ has the q-expansion

$$\sum_{n,m} a_{n,m}(f) q_1^n q_2^{5n-m}.$$

Proof. (i). Form the definitions of n and m follows that $\mu = \frac{m}{2} + (\frac{m}{2} - n)\sqrt{5}$. Since $\mu \gg 0$ there is the inequality

$$|\frac{m}{2} - n| < \frac{\sqrt{5}}{5} \frac{m}{2}$$

and (i) follows immediately. (ii). Following the definition of the coefficients $a_{n',m'}(f)$ we find

$$a_{n',m'}(f) = a_{\overline{\mu}/\delta}(f) \Leftrightarrow n' = \mathrm{tr}(\overline{\mu}/\delta) = m - n \text{ and } m' = \mathrm{tr}(\overline{\mu}) = m.$$

Because of (i), $m-n$ is always positive. (iii). Analogously

$$a_{n',m'}(f) = a_{\overline{\mu/\delta}}(f) \Leftrightarrow n' = \mathrm{tr}(\overline{\mu/\delta}) = n \text{ and } m' = \mathrm{tr}(\delta\overline{\mu/\delta}) = 5n-m.$$

For (iv) use

$$f(z_2,z_1) = \sum_{\nu\in\mathbb{Z}^{\#}_{F\gg 0}} a_\nu(f)\exp(2\pi i\mathrm{Tr}(\nu(z_2,z_1))) = \sum_{\nu\in\mathbb{Z}^{\#}_{F\gg 0}} a_{\overline{\nu}}(f)\exp(2\pi i\mathrm{Tr}(\nu(z_1,z_2))$$

and apply (iii). □

Lemma 3.31 **($F=\mathbb{Q}[\sqrt{2}]$)** *Let*

$$f = \sum_{\mu\in\mathbb{Z}_{F\gg 0}} a_{\mu/\delta}(f)q_1^{\mathrm{tr}(\mu/\delta)}q_2^{\mathrm{tr}(\mu/2)} \in M_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Let $\mu\in\mathbb{Z}_F$ be totally positive and set $n := \mathrm{tr}(\beta_2^{-1}\mu)$ and $m := \mathrm{tr}(\mu)$, i.e. $a_{\beta_1^{-1}\mu}(f) = a_{n,m}(f)$.

(i) If $a_{n,m}(f)\neq 0$ then

$$\begin{array}{rcl} (2-\sqrt{2})n < & m & < (2+\sqrt{2})n, \\ \dfrac{2-\sqrt{2}}{2}m < & n & < \dfrac{2+\sqrt{2}}{2}m. \end{array}$$

(ii) $a_{\overline{\mu}/2\delta}(f) = a_{2m-n,n}(f)$.

(iii) $a_{\overline{\mu/2\delta}}(f) = a_{n,4n-m}(f)$.

(iv) The Galois-conjugated Hilbert modular form f has the q-expansion

$$f(z_2,z_1) = \sum_{n,m} a_{n,m}(f)q_1^n q_2^{4n+2m}.$$

Proof. (i). If $a_{\mu/\delta}(f)\neq 0$ then μ must be totally positive. Since $\mu = m+(m-n)\sqrt{2}$ it follows that

$$|m-n| < \frac{\sqrt{2}}{2}m \quad -\frac{\sqrt{2}}{2}m \le m-n \le \frac{\sqrt{2}}{2}m$$

and (i) follows immediately. (ii). Following the definition of the coefficients $a_{n',m'}(f)$ we find

$$a_{n',m'}(f) = a_{\overline{\mu}/\delta}(f) \Leftrightarrow n' = \mathrm{tr}(\overline{\mu}/\delta) = 2m-n \text{ and } m' = \mathrm{tr}(\overline{\mu}/2) = m.$$

Because of (i), $2m - n$ is always positive. (iii). Analogously

$$a_{n',m'}(f) = a_{\overline{\mu/\delta}}(f) \Leftrightarrow n' = \mathrm{tr}(\overline{\mu/\delta}) = n \text{ and } m' = \mathrm{tr}\left(\frac{\delta\,\overline{\mu}}{\overline{\delta}\,2}\right) = 4n - m.$$

For (iv) use

$$f(z_2, z_1) = \sum_{\nu \in \mathbb{Z}^{\#}_{F \gg 0}} a_\nu(f) \exp(2\pi i \mathrm{Tr}(\nu(z_2, z_1))) = \sum_{\nu \in \mathbb{Z}^{\#}_{F \gg 0}} a_{\overline{\nu}}(f) \exp(2\pi i \mathrm{Tr}(\nu(z_1, z_2))$$

and apply (iii). $\square$

Lemma 3.32 ($F = \mathbb{Q}[\sqrt{3}]$)

(i) Let

$$f = \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\mu/\delta}(f) q_1^{\mathrm{tr}(\mu/2)} q_2^{\mathrm{tr}(\mu\beta_2/\delta)} \in \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Let $\mu \in \mathbb{Z}_F$ be totally positive and set $n := \mathrm{tr}(\mu/\delta)$ and $m := \mathrm{tr}(\mu)$, i.e. $a_{\mu/\delta}(f) = a_{n,m}(f)$.

(a) If $a_{n,m}(f) \neq 0$ then

$$\begin{aligned} \frac{-1+\sqrt{3}}{2} n < \quad m \quad &< \frac{1+\sqrt{3}}{2} n, \\ (-1+\sqrt{3}) m < \quad n \quad &< (1+\sqrt{3}) m. \end{aligned}$$

(b) $a_{\overline{\mu}/\delta}(f) = a_{n,2n-m}(f)$.

(c) $a_{\overline{\mu/\delta}}(f) = a_{n,5n-m}(f)$.

(d) The Galois-conjugated Hilbert modular form $f(-z_2, -z_1)$ has the q-expansion

$$\sum_{n,m} a_{n,m}(f) q_1^n q_2^{5n-m}.$$

(ii) Let

$$f = \sum_{\mu \in \mathbb{Z}^{\#}_{F \gg 0}} a_\mu(f) q_1^{\mathrm{tr}(\mu)} q_2^{\mathrm{tr}(\mu\beta_2)} \in M_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Let $\mu \in \mathbb{Z}_F$ be totally positive and set $n := \mathrm{tr}(\mu)$ and $m := \mathrm{tr}(\mu\beta_2)$, i.e. $a_\mu(f) = a_{n,m}(f)$.

(a) If $a_{n,m}(f) \neq 0$ then

$$\begin{aligned} (3-\sqrt{3}) n < \quad m \quad &< (3+\sqrt{3}) n, \\ \frac{3-\sqrt{3}}{6} m < \quad n \quad &< \frac{3+\sqrt{3}}{6} m. \end{aligned}$$

(b) $a_{\overline{\mu}}(f) = a_{n,6n-m}(f)$.

(c) The Galois-conjugated Hilbert modular form $f(z_2, z_1)$ *has the q-expansion*

$$\sum_{n,m} a_{n,m}(f) q_1^n q_2^{6n-m}.$$

Proof. (i). Form the definitions of n and m follows that $\mu = n + (n-m)\sqrt{3}$. Since $\mu \gg 0$ there is the inequality

$$|m - n| < \frac{\sqrt{3}}{3} n$$

and (a) follows immediately. (b). Following the definition of the coefficients $a_{n',m'}(f)$ we find

$$a_{n',m'}(f) = a_{\overline{\mu}/\delta}(f) \Leftrightarrow n' = \mathrm{tr}(\overline{\mu}/2) = n \text{ and } m' = \mathrm{tr}(\overline{\mu}\frac{3-\sqrt{3}}{6}) = 2n - m.$$

Because of (a), $2n - m$ is always positive. For (c) use

$$\begin{aligned} f(-z_2, -z_1) &= \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\mu/2\sqrt{3}}(f) \exp(2\pi i \mathrm{Tr}(\mu/2\sqrt{3}(-z_2, -z_1))) \\ &= \sum_{\mu \in \mathbb{Z}_{F \gg 0}} a_{\overline{\mu}/2\sqrt{3}}(f) \exp(2\pi i \mathrm{Tr}(\mu/2\sqrt{3}(z_1, z_2)) \end{aligned}$$

and apply (b).

(ii). Since $\mu = \frac{n}{2} + \frac{m-3n}{6}\sqrt{3} \gg 0$ there is the inequality

$$|\frac{m-3n}{6}| < \frac{\sqrt{3}}{6} n$$

and (a) follows immediately. (b) and (c) are clear. □

Chapter 4

Lattices and Modular Forms

Let $F = \mathbb{Q}[\sqrt{D}]$ be a real quadratic number field. Let ε_0 be the fundamental unit with $\varepsilon_0^{(1)} > 1$.

4.1 The Theta Series of a Lattice

Kloosterman introduced theta series over number fields in 1930, see [Klo30]. This section follows also [Maa40], [Ebe02, 5.7] and [Neb13]. The definition of the theta series differ slightly in the references.

The generating function of the cardinality of the layers

$$\{\Lambda(\alpha) \mid \alpha \gg 0\}$$

is called the theta series of Λ. It has an interesting structure because it is a Hilbert modular form. In fact, we define three types of theta series. The theta series of Type (i), (ii), or (iii) of a lattice of Type (i), (ii), or (iii), respectively, is a Hilbert modular form. Type (i) and (ii) theta series were used by Maaß in [Maa40], and Type (iii) theta series were used by Kloosterman in [Klo30] and by Skorrupa in [Ebe02, 5.7].

Definition 4.1 (Theta Series)
Let (Λ, Q) be an even lattice. In the following we define theta series $\Theta^{(i)}_{(\Lambda,Q)}$, $\Theta^{(ii)}_{(\Lambda,Q)}$, and $\Theta^{(iii)}_{(\Lambda,Q)}$ of (Λ, Q). Often we omit (i), (ii), or (iii), and we write Θ_Λ instead of $\Theta_{(\Lambda,Q)}$.

Type (i): *Assume that $\mathcal{N}(\varepsilon_0) = -1$. Then $\delta := \varepsilon_0\sqrt{d_F}$ generates the different ideal. The* theta series *of Type (i) of (Λ, Q) is* $\Theta^{(i)}_{(\Lambda,Q)} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$,

$$\Theta^{(i)}_{(\Lambda,Q)}(z) = \sum_{\lambda\in\Lambda} e^{2\pi i \operatorname{Tr}(z\, Q(\lambda)/\delta)} = 1 + \sum_{\mu\in\mathbb{Z}_{F\gg 0}} |\Lambda(\mu)| e^{2\pi i \operatorname{Tr}(z\,\mu/\delta)}$$

Type (ii): *The* theta series *of Type (ii) of* (Λ, Q) *is* $\Theta^{(ii)}_{(\Lambda,Q)} : \mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{C}$,

$$\Theta^{(ii)}_{(\Lambda,Q)}(z) = \sum_{\lambda\in\Lambda} e^{2\pi i\,\mathrm{Tr}(z\,Q(\lambda)/\sqrt{d_F})} = 1 + \sum_{\mu\in\mathbb{Z}_{F\gg 0}} |\Lambda(\mu)| e^{2\pi i\,\mathrm{Tr}(z\,\mu/\sqrt{d_F})}.$$

Type (iii): *The theta series of Type (iii) of* (Λ, Q) *is* $\Theta^{(iii)}_{(\Lambda,Q)} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$,

$$\Theta^{(iii)}_{(\Lambda,Q)}(z) = \sum_{\lambda\in\Lambda} e^{2\pi i\,\mathrm{Tr}(z\,Q(\lambda))} = 1 + \sum_{\mu\in\mathbb{Z}_{F\gg 0}} |\Lambda(\mu)| e^{2\pi i\,\mathrm{Tr}(z\,\mu)}.$$

If $\mathcal{N}(\varepsilon_0) = -1$, then all three types are equivalent. By applying $\left(\begin{smallmatrix}\varepsilon_0 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\varepsilon_0\sqrt{d_F} & 0\\ 0 & 1\end{smallmatrix}\right)$ to $\Theta^{(i)}_{(\Lambda,Q)}(z)$, one gets theta series of Type (ii) or (iii), respectively.

If $\mathcal{N}(\varepsilon_0) = 1$, i.e. $h_F^+ = 2h_F$, then there is no totally positive generator of the different. Type-(i) theta series cannot be defined. The Types-(ii) and -(iii) theta series live in different, non-equivalent spaces of modular forms.

In Section 3.1 we discussed the action of the Hilbert modular group $\mathrm{SL}_2(\mathbb{Z}_F)$ on the holomorphic functions. Theta series have nice properties under this action. For example and most importantly, there is the formula

$$\Theta_\Lambda\left(\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} z\right) = (-z)^{n/2} c(\Lambda, Q)\ \Theta_{\Lambda^\#}(z), \tag{I}$$

where $c(\Lambda, Q) \in \mathbb{C}$ is a constant depending on (Λ, Q).

If we are looking at theta series of Type (iii), then $c(\Lambda, Q) = \det(L_1)$. This was for example proved by Skorrupa in [Ebe02, Proposition 5.7]. We follow Skoruppa's proof of the Formula (I).

Lemma 4.2 *Let* $Z = X + iY \in \mathbb{C}^{m\times m}$ *be a symmetric complex matrix and assume that* Y *is positive definite, where* $X, Y \in \mathbb{R}^{n\times n}$.

Then

$$\int_{\mathbb{R}^m} e^{-\pi i\, xZ^{-1}x^{tr}} e^{-2\pi i\sum_{i=1}^m x_i y_i} dx \ = \ \sqrt{\det(\tfrac{Z}{i})}\ e^{\pi i\, yZy^{tr}},$$

where the square root should be positive.

Proof. [Ebe02, Lemma 5.6]. □

With this lemma we can prove versions of Formula (I). It follows directly that the theta series of an unimodular lattice is a Hilbert modular form.

Theorem 4.3

Type (i): *Let* (Λ, Q) *be a lattice of Type (i) (i.e. even and unimodular) and rank* n*. Then its theta series of Type (i) is holomorphic on* $\mathbb{H} \times \mathbb{H}$ *and a Hilbert modular form of level one and parallel weight* $k = n/2$*, i.e.*

$$\Theta^{(i)}_{(\Lambda,Q)}(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

If (Λ, Q) *is Galois-invariant, then* $\Theta^{(i)}_{(\Lambda,Q)}$ *is symmetric.*

Type (ii): *Let* (Λ, Q) *be an even unimodular lattice of rank* n*. Then its theta series of Type (ii) is holomorphic on* $\mathbb{H} \times \overline{\mathbb{H}}$ *and a Hilbert modular form of level one and parallel weight* $k = n/2$*, i.e.*

$$\Theta^{(ii)}_{(\Lambda,Q)}(z) \in \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

If (Λ, Q) *is a Type (ii) lattice (i.e. Galois-invariant), then* $\Theta^{(i)}_{(\Lambda,Q)}$ *is symmetric.*

Type (iii): *If* (Λ, Q) *is an even trace unimodular lattice of rank* n*, then the theta series of Type (iii) is holomorphic on* $\mathbb{H} \times \mathbb{H}$ *and a Hilbert modular form of level one and weight* $k = n/2$*, i.e.*

$$\Theta^{(iii)}_{(\Lambda,Q)} \in M_k(\mathrm{SL}_2(\mathbb{Z}_F)).$$

If (Λ, Q) *is a Type (iii) lattice (i.e. Galois-invariant), then* $\Theta^{(i)}_{(\Lambda,Q)}$ *is symmetric.*

For Types (ii) and (iii), if Λ *is fundamentally invariant, then* $\Theta_{(\Lambda,Q)}$ *is fundamentally symmetric.*

Proof. (i). As mentioned, $\Theta_\Lambda(\left(\begin{smallmatrix}\varepsilon_0 & 0\\ 0 & 1\end{smallmatrix}\right) z)$ is a theta series of Type (ii). Hence (i) follows from (ii).

(ii) The group $\mathrm{SL}_2(\mathbb{Z}_F)$ is generated by the matrices

$$\begin{pmatrix}1 & \mu\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\varepsilon_0 & 0\\ 0 & \varepsilon_0^{-1}\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

where $\mu \in \mathbb{Z}_F$. It is straightforward to check that the theta series is invariant under transformations of matrices of the first and second kind. We show the invariance under transformation with $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ by verifying a variation of the Formula (I).

For $j = 1, 2$ the field $\mathbb{R}$ is an $F-$module via the action $(\alpha, r) \mapsto \alpha^{(j)} \cdot r$. Denote it by $F^{(j)}$ and define $V^{(j)} := F^n \otimes_F F^{(j)}$ $(\cong \mathbb{R}^n)$. We extend

$$F^n \times F^n \to F^{(j)}, (\lambda, \mu) \mapsto B(\lambda, \mu)^{(j)}$$

$\mathbb{R}$-bilinearly to a scalar product $(\ ,\ )_j$ of $V^{(j)}$. So $(V^{(j)}, (\ ,\ )_j)$ is an Euclidean space and there is an isometry φ_j of $V^{(j)}$ and the standard Euclidean space on $\mathbb{R}^n$. Especially then $B(\lambda,\mu)^{(j)} = \varphi_j(\lambda)^{\mathrm{tr}}\varphi_j(\mu)$. By combining all real places, $V^{(j)} \times V^{(j)}$ is isometric to $\mathbb{R}^{2n}$ with the usual scalar product $x \cdot y := x^{\mathrm{tr}} y$. We get an isometric embedding

$$\varphi : F^n \to \mathbb{R}^{2n},\ \lambda \mapsto (\varphi_1(\lambda), \varphi_2(\lambda)).$$

Then $L := \varphi(\Lambda) \subseteq \mathbb{R}^{2n}$ is an even $\mathbb{Z}$-lattice. Since $\varphi(\lambda)\cdot\varphi(\mu) = \operatorname{tr}(B(\lambda,\mu))$ for all $\lambda, \mu \in F^n$, especially for all lattice points, L is isometric to the trace lattice L_1. Hence $\det L = \det L_1 = \mathcal{N}(\mathbb{Z}_F^{\#})^{-n} = d_F^n$, see Lemma 1.8. And there is a similarity $f \in \mathrm{GL}_{2n}(\mathbb{R})$ of norm d_F such that $f(L^{\#}) = L$, see Proposition 2.3.

Let $z \in \mathbb{H}^2$ and set $Z := \operatorname{diag}(\frac{z_1}{\sqrt{d_F}} I_n,\ \frac{-z_2}{\sqrt{d_F}} I_n) \in \mathbb{C}^{2n\times 2n}$. The theta series of Λ is

$$\Theta_{(\Lambda,Q)}(z_1,z_2) = \sum_{y\in L} e^{\pi i\ y^{\mathrm{tr}} Z y}.$$

One sees easily that $\Theta_{(\Lambda,Q)}$ is holomorphic, because $|e^{\pi i\ yZy^{tr}}|$ is bounded and exponentially decreasing for $\mathrm{Im}(z_1), \mathrm{Im}(z_2) \to \infty$

For $y \in \mathbb{R}^{2n}$ let $y^{(1)}$ denote the first n entries of y and $y^{(2)}$ the last. So

$$\Theta_{(\Lambda,Q)}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) z\right) = \sum_{y\in L^{\#}} \exp\left(\pi i \left(f(y^{(1)}) \cdot f(y^{(1)}) \tfrac{-1}{z_1\sqrt{d_F}} + f(y^{(2)}) \cdot f(y^{(2)}) \tfrac{1}{z_2\sqrt{d_F}}\right)\right).$$

Since $f(x)\cdot f(x) = d_F \cdot x \cdot x$ for all $x \in \mathbb{R}^{2n}$, this is equal to the sum

$$\sum_{y\in L^{\#}} \exp\left(\pi i \left(y^{(1)} \cdot y^{(1)} \tfrac{\sqrt{d_F}}{-z_1} + y^{(2)} \cdot y^{(2)} \tfrac{\sqrt{d_F}}{z_2}\right)\right) = \sum_{y\in L^{\#}} e^{\pi i\, y^{\mathrm{tr}} Z^{-1} y}.$$

Using Poisson's summation formula (see for instance [Ebe02, Theorem 2.3]) this may be written as

$$\det(L^{\#})^{-1/2} \sum_{y\in L} \int_{\mathbb{R}^{2n}} e^{-\pi i\, x Z^{-1} x^{tr}} e^{-2\pi i\, x y^{tr}}\, dx.$$

The factor is $\det(L^{\#})^{-1/2} = \det(L)^{1/2} = d_F^{n/2}$. The matrix Z is symmetric and the imaginary part is positive definite. Hence we may apply Lemma 4.2:

$$\Theta_{(\Lambda,Q)}\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} z\right) \;=\; d_F^{n/2} \det\left(\frac{Z}{i}\right)^{1/2} \underbrace{\sum_{y\in L} e^{2\pi i\, y^{\mathrm{tr}} Z y}}_{=\Theta_{(\Lambda,Q)}(z)}.$$

We have

$$d_F^{n/2} \det\left(\frac{Z}{i}\right)^{1/2} = d_F^{n/2} \mathcal{N}(\sqrt{d_F})^{n/2}\, i^n (-z)^{n/2} = (-z)^{n/2}.$$

This proves the first part of (ii).

Secondly, if Λ is Galois-invariant, there is a semi-linear endomorphism $\sigma : \Lambda \to \Lambda$ such that $Q(\sigma(\lambda)) = \overline{Q(\lambda)}$ for all $\lambda \in \Lambda$. Hence

$$\begin{aligned}\Theta_\Lambda(-z_2, -z_1) &= \sum_{\lambda\in\Lambda} \exp\left(2\pi i \operatorname{Tr}\left(\tfrac{\overline{Q(\lambda)}}{\sqrt{d_F}} z\right)\right)\\ &= \sum_{\lambda\in\Lambda} \exp\left(2\pi i \operatorname{Tr}\left(\tfrac{Q(\sigma(\lambda))}{\sqrt{d_F}} z\right)\right) = \Theta_\Lambda(z_1, z_2)\end{aligned}$$

and therefore Θ_Λ is symmetric.

Thirdly, if Λ is fundamentally invariant, then there is a $\tau : \Lambda \to \Lambda$ such that $Q(\tau(\lambda)) = \varepsilon_0 Q(\lambda)$ for all $\lambda \in \Lambda$. Hence

$$\Theta_\Lambda\left(\left(\begin{smallmatrix}\varepsilon_0 & \\ & 1\end{smallmatrix}\right) z\right) = \sum_{\lambda\in\Lambda} \exp\left(2\pi i \operatorname{Tr} \tfrac{Q(\lambda)}{\sqrt{d_F}} (\varepsilon z)\right) = \sum_{\lambda\in\Lambda} \exp\left(2\pi i \operatorname{Tr} \tfrac{Q(\tau(\lambda))}{\sqrt{d_F}} z\right) = \Theta_\Lambda(z)$$

s and Θ_Λ is fundamentally symmetric.

(iii) Analogously. See also Skoruppa's notes in [Ebe02, 5.7]. □

The proof of the symmetry shows that, in general, the series $\Theta_{(\Lambda,Q)}(z_2, z_1)$ is the theta series of the Galois-conjugate lattice $(\overline{\Lambda}, \overline{Q})$.

4.2 q-Expansions of Theta Series and Extremal Lattices

The number of vectors of a given length in a lattice is given by the coefficients of the theta series. The norm of a vector may be calculated by using the trace norms. So the theta series of the trace lattices play an important role. In general, let $L_1 = (\Lambda, \mathfrak{q}_1)$ and $L_2 = (\Lambda, \mathfrak{q}_2)$ be $\mathbb{Z}$-lattices. We call

$$\Theta_{(L_1,L_2)} = \sum_{\lambda\in\Lambda} q_1^{\mathfrak{q}_1(\lambda)} \cdot q_2^{\mathfrak{q}_2(\lambda)} \in \mathbb{C}[[q_1, q_2]]$$

the *merged theta series* of L_1 and L_2. The merged theta series is the generating function of the cardinalities of the sets $L_1(a_1) \cap L_2(a_2)$, where $a \in \mathbb{Z}^2$. The restrictions of the merged theta series to $q_2 = 1$ or $q_1 = 1$ give the single theta series of L_1 or L_2, respectively. But the other way is not true; the knowledge of the single theta series is not sufficient to determine the merged theta series.

The following theorem shows that the merged theta series of the trace lattices Λ_1 and Λ_2 of a lattice Λ of Type (i), (ii), or (iii) (see Definition 2.4) is the standard q-expansion (see Definition 3.26) of the theta series of Λ.

Theorem 4.4
Let (Λ, Q) be a lattice of Type (i), (ii), or (iii). So (Λ_1, Q_1), (Λ_2, Q_2) are trace lattices of (Λ, Q) with respect to α_1 or α_2, respectively (see Definition 2.4). Define

$$\beta := \begin{cases} \sqrt{d_F}\varepsilon_0 & \text{for Type (i),} \\ \sqrt{d_F} & \text{for Type (ii),} \\ 1 & \text{for Type (iii),} \end{cases}$$

and $B := (\beta\alpha_1, \beta\alpha_2)$.

Then the q*-expansion of the theta series of Type (i), (ii), or (iii), respectively, of* Λ *with respect to* B *is*

$$\Theta_{(\Lambda,Q)}(q_1, q_2) = \sum_{\lambda\in\Lambda} q_1^{Q_1(\lambda)}\, q_2^{Q_2(\lambda)} = \sum_{n,m\geq 0} |\Lambda_1(n) \cap \Lambda_2(m)|\; q_1^n\, q_2^m.$$

Proof. The q-expansion of Θ_Λ with respect to B is

$$\Theta_\Lambda(q_1, q_2) = \sum_{\mu\in(\mathbb{Z}_F)_{\gg 0}} |\Lambda(\mu)|\, q_1^{\mathrm{tr}(\alpha_1\nu)}\, q_2^{\mathrm{tr}(\alpha_2\nu)}.$$

Let $n, m \geq 0$. Then $\lambda \in \Lambda_1(n) \cap \Lambda_2(m) \subseteq \Lambda$ if and only if $Q_1(\lambda) = \mathrm{tr}(\alpha_1 Q(\lambda)) = n$ and $q_2(\lambda) = \mathrm{tr}(\alpha_2 Q(\lambda)) = m$. Hence $c_{n,m} = |\Lambda(Q(\lambda))|$, and the claimed identity is true. $\square$

The q-expansion of the theta series of Λ is a power series in (q_1, q_2),

$$\Theta_\Lambda \in \mathbb{C}[[q_1, q_2]].$$

The theta series of Λ_1 and Λ_2 are obviously given by $\Theta_\Lambda(q_1, 1)$ and $\Theta_\Lambda(1, q_2)$, but the theta series of other trace lattices can also be easily computed with the q-expansion of Θ_Λ.

Corollary 4.5 *Let* Λ *be a a lattice of Type (i), (ii), or (iii). Let* $\gamma \in \mathbb{Z}_F^\#$ *(for Types (i) or (ii)) or* $\gamma \in \mathbb{Z}_F$ *(for Type (iii)) be totally positive. Then* $\gamma = c_1\alpha_1 + c_2\alpha_2$ *for some* $c_1, c_2 \in \mathbb{Q}$*. The theta series of the trace lattice* L_γ *is*

$$\Theta_{L_\gamma}(q) = \Theta_\Lambda\left(q^{c_1}, q^{c_2}\right).$$

Proof. Let $\lambda \in \Lambda$. Then $\mathfrak{q}_\gamma(\lambda) = \mathrm{tr}(\gamma Q(\lambda)) = c_1\mathrm{tr}(\alpha_1 Q(\lambda)) + c_2\mathrm{tr}(\alpha_2 Q(\lambda)) = c_1 Q_1(\lambda) + c_2 Q_2(\lambda)$. So

$$\theta_{L_\gamma}(q) = \sum_{\ell\geq 0} \sum_{\substack{n,m\geq 0,\\ c_1 n + c_2 m = \ell}} |\Lambda_1(n) \cap \Lambda_2(m)|\, q^\ell = \Theta_\Lambda\left(q^{c_1}, q^{c_2}\right).$$

$\square$

In Definition 3.24 we defined an ordering $\leq_B$ of the monomials of $\mathbb{C}[[q_1, q_2]]$ and an valuation ν_B of Hilbert modular forms. Especially, we defined extremal Hilbert modular forms. We use that to define extremal lattices.

Definition 4.6 (Extremal lattices)
Let Λ be a lattice of Type (i), (ii), or (iii). Let n be the rank of Λ and $k = \frac{n}{2}$.

Λ is called extremal *if its theta series of Type (i), (ii), or (iii) in the standard q-expansion is an extremal Hilbert modular form of weight k in*

$$\begin{array}{ll} M_k(\mathrm{SL}_2(\mathbb{Z}_F)) & \textit{for Type (i),} \\ \overline{M}_k^{+}(\mathrm{SL}_2(\mathbb{Z}_F)) & \textit{for Type (ii), or} \\ M_k^{+}(\mathrm{SL}_2(\mathbb{Z}_F)) & \textit{for Type (iii).} \end{array}$$

Remark 4.7 *If there is an extremal lattice over $\mathbb{Z}$, it is not only the lattice with an extremal theta series but also with the maximal minimum among all even unimodular lattices. For extremal lattices over number fields, this is somehow also true, if we take the total ordering $\leq_A$, where $A = (\alpha_1, \alpha_2)$ depends on the type.*

More explicit, if Λ is extremal of Type (i), (ii), or (iii), then for all $\mathbb{Z}_F$-lattices M of Type (i), (ii), or (iii) of the same rank as Λ we have

$$\min_A M \ \leq_A \ \min_A \Lambda.$$

Remark 4.8 *Using the q-expansion of the theta series, we get estimates for the minima of the trace lattices.*

(i) For $F = \mathbb{Q}[\sqrt{5}]$ (Type (i)) we get by Lemma 3.30

$$\begin{aligned} \frac{5-\sqrt{5}}{2} \min \Lambda_1 < \quad \min \Lambda_2 \quad &< \frac{5+\sqrt{5}}{2} \min \Lambda_1, \\ \frac{5-\sqrt{5}}{10} \min \Lambda_2 < \quad \min \Lambda_1 \quad &< \frac{5+\sqrt{5}}{10} \min \Lambda_2. \end{aligned}$$

More general, if $\lambda \in \Lambda$ then

$$\begin{aligned} \frac{5-\sqrt{5}}{2} Q_1(\lambda) < \quad Q_2(\lambda) \quad &< \frac{5+\sqrt{5}}{2} Q_1(\lambda), \\ \frac{5-\sqrt{5}}{10} Q_2(\lambda) < \quad Q_1(\lambda) \quad &< \frac{5+\sqrt{5}}{10} Q_2(\lambda). \end{aligned}$$

In Proposition 2.7 we already formulated estimates for the minima of the trace lattices. For the $\sqrt{5}$-case, recall that

$$2 \min \Lambda_1 \leq \min \Lambda_2 \leq \tfrac{5}{2} \min \Lambda_1.$$

These estimates are better, because $\frac{5-\sqrt{5}}{2} \approx 1.38 < 2$ and $\frac{5}{2} < \frac{5+\sqrt{5}}{2} \approx 3.69$.

(ii) For $F = \mathbb{Q}[\sqrt{2}]$ (Type (i)) we get by Lemma 3.31

$$\begin{aligned} (2-\sqrt{2}) \min \Lambda_1 < \quad \min \Lambda_2 \quad &< (2+\sqrt{2}) \min \Lambda_1, \\ \frac{2-\sqrt{2}}{2} \min \Lambda_2 < \quad \min \Lambda_1 \quad &< \frac{2+\sqrt{2}}{2} \min \Lambda_2. \end{aligned}$$

The same estimates are true if one replaces $\min \Lambda_1$ *and* $\min \Lambda_2$ *by* $Q_1(\lambda)$ *and* $Q_2(\lambda)$*, respectively, where* $\lambda \in \Lambda$*. The estimates given in Proposition 2.7 are*

$$\min \Lambda_1 \leq \min \Lambda_2 \leq 2 \min \Lambda_1.$$

These estimates are better, because $2 - \sqrt{2} \approx 0.59 < 1$ *and* $2 < 2 + \sqrt{2} \approx 3.41$.

(iii) For $F = \mathbb{Q}[\sqrt{3}]$ *and* Λ *is even unimodular (Type (ii)) we get by Lemma 3.32(i):*

$$\begin{aligned} \frac{-1+\sqrt{3}}{2} \min \Lambda_1 < \quad \min \Lambda_2 \quad &< \frac{1+\sqrt{3}}{2} \min \Lambda_1, \\ (-1+\sqrt{3}) \min \Lambda_2 < \quad \min \Lambda_1 \quad &< (1+\sqrt{3}) \min \Lambda_2. \end{aligned}$$

The same estimates are true if one replaces $\min \Lambda_1$ *and* $\min \Lambda_2$ *by* $Q_1(\lambda)$ *and* $Q_2(\lambda)$*, respectively, where* $\lambda \in \Lambda$*. By Proposition 2.7,*

$$\tfrac{1}{2} \min \Lambda_1 \leq \min \Lambda_2 \leq \tfrac{4}{3} \min \Lambda_1.$$

These estimates are better, because $\frac{-1+\sqrt{3}}{2} \approx 0.37 < \frac{1}{2}$ *and* $\frac{4}{3} < \frac{1+\sqrt{3}}{2} \approx 1.37$.

(iv) For $F = \mathbb{Q}[\sqrt{3}]$ *and* Λ *is trace even unimodular (Type (iii)) we get by Lemma 3.32(ii):1*

$$\begin{aligned} (3-\sqrt{3}) \min \Lambda_1 < \quad \min \Lambda_2 \quad &< (3+\sqrt{3}) \min \Lambda_1, \\ \frac{3-\sqrt{3}}{6} \min \Lambda_2 < \quad \min \Lambda_1 \quad &< \frac{3+\sqrt{3}}{6} \min \Lambda_2. \end{aligned}$$

The same estimates are true if one replaces $\min \Lambda_1$ *and* $\min \Lambda_2$ *by* $Q_1(\lambda)$ *and* $Q_2(\lambda)$*, respectively, where* $\lambda \in \Lambda$*. The estimates in in Proposition 2.7 are*

$$\frac{3}{2} \min \Lambda_1 \ \leq \ \min \Lambda_2 \ \leq \ 4 \min \Lambda_1.$$

These estimates are better, because $(3-\sqrt{3}) \approx 1.26 < \frac{3}{2}$ *and* $4 < 3+\sqrt{3} \approx 4.73$.

Chapter 5

Spherical Theta Series

Let $(L, \mathfrak{q})$ be a $\mathbb{Z}$-lattice. To L we may define not only the usual theta series $\Theta_L = \sum_{\lambda\in\Lambda} e^{2\pi i\,\mathfrak{q}(\lambda)}$, but we may also plug in some coefficients. We get so-called spherical theta series $\Theta_{L,P} = \sum_{\lambda\in\Lambda} P(\lambda)e^{2\pi i\,q(\lambda)}$, where P is a harmonic polynomial. These theta series are modular forms for some level and character.

Firstly, this gives a connection between lattices and spherical t-designs, which were introduced by Delsarte, Goethals, and Seidel in [DelGoeSei77].

Secondly, theta series with spherical coefficients provide very useful techniques to classify (extremal) unimodular or p-modular lattices. Venkov used them to find all 24 Niemeier lattices (unimodular lattices in dimension 24). His proof is much simpler than Niemeier's original proof, see [ConSlo99, Chapter 16]. Bachoc and Venkov developed a method to (partly) classify lattices by using spherical theta series, see [BacVen01].

In this chapter I extend the Bachoc-Venkov method to lattices over number fields. In the next chapters this method is used to classify lattices over number fields.

5.1 Harmonic Polynomials

This section follows [Hel00], [Ogg69], and [Ven01].

Let $n \in \mathbb{N}$. We fix a Euclidean space $(V, (\ ,\ ))$, i.e. a real vector space V of dimension n and a positive definite inner product $(\ ,\ )$. Also let $E = (e_1, \dots, e_n)$ be a fixed basis and let $G = ((e_i, e_j))_{i,j}$ be its Gram matrix. We identify V with $\mathbb{R}^n$ with respect to this basis.

The orthogonal group of the Euclidean space is

$$\begin{aligned} O(V, (\ ,\ )) &:= \{g \in \mathrm{GL}(V) \mid (gv, gv) = (v, v) \text{ for all } v \in V\} \\ &\cong \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g^{\mathrm{tr}}Gg = G\}. \end{aligned}$$

Let V^* be the dual space of V, and let $E^* = (e_1^*, \dots, e_n^*)$ be the dual basis of E, i.e. $e_i^*(e_j) = \delta_{i,j}$. The symmetric algebra of V^* is the graduated algebra

$$S(V^*) = \sum_{m\geq 0} S^m(V^*).$$

We identify the polynomial ring $\mathcal{P} := \mathbb{R}[x_1, \ldots, x_n]$ with $S(V^*)$ via

$$x_j \mapsto e_j^*.$$

We write $x := (x_1, \ldots, x_n)^{\mathrm{tr}}$ for short.

The orthogonal group $O(V, (\ ,\ ))$ acts on $\mathcal{P}$ via $g \cdot p := p \circ g^{-1}$.

Let $\mathcal{P}_d$ be the space of the homogeneous polynomials of degree $d \in \mathbb{Z}_{\geq 0}$. Let $i \in \mathbb{Z}^n_{\geq 0}$ be a multi-index. We set $|i| := i_1 + \cdots + i_n$. We use the multi-index notation $x^i := x_1^{i_1} \ldots x_n^{i_n}$. A basis of the the space $\mathcal{P}_m$ are the monomials x^i with $|i| = m$.

Since G is real symmetric, there exists $A \in \mathrm{GL}_n(\mathbb{R})$ such that $A^{\mathrm{tr}}A = G$. We set $y := Ax$. Let $\cdot$ be the usual dot product on $\mathbb{R}^n$, i.e. $u \cdot w = u^{\mathrm{tr}}w$. We have an isometry

$$\varphi : (V, (\ ,\ )) \to (\mathbb{R}^n,\ \cdot), v \mapsto Av$$

and an algebra isomorphism $\varphi^* : \mathbb{R}[y] \to \mathbb{R}[x], p \mapsto p \circ A$. So for $p \in \mathbb{R}[y]$ we have $\varphi^*(p)(v) = p(\varphi(v))$ for all $v \in V$, i.e. $\varphi^*(p)(x) = p(y)$.

The usual orthogonal group is

$$O_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g^{\mathrm{tr}}g = 1\} = AO(V, (\ ,\ ))A^{-1}.$$

It acts on $\mathbb{R}^n$ and $\mathbb{R}[y]$, and for $g \in O(V, (\ ,\ ))$ we have

$$\varphi(gv) = AgA^{-1}\varphi(v) \text{ and } \varphi^*(AgA^{-1}p(y)) = g\varphi^*(p(y)).$$

We introduce the formal vector $\nabla = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})^{\mathrm{tr}}$, called *nabla*. For $i \in \mathbb{Z}^n_{\geq 0}$ let $\nabla^i := \frac{\partial^{i_1}}{\partial y_1^{i_1}} \cdot \cdots \cdot \frac{\partial^{i_n}}{y_n^{i_n}}$. So if $i, j \in \mathbb{Z}^n_{\geq 0}$ with $|i| = |j|$, then

$$\nabla^i y^j = \begin{cases} 0 & \text{if } i \neq j, \\ i_1! \cdot \cdots \cdot i_n! & \text{if } i = j. \end{cases}$$

This defines a scalar product on $\mathcal{P}_d$:

$$[\varphi^*(p), \varphi^*(q)] := \tfrac{1}{d!}\ p(\nabla)\, q(y).$$

Lemma 5.1 $[\ ,\ ]$ *is a scalar product.*

For $v \in V$ let $\rho_v(x) := (x, v) \in \mathcal{P}_1$ be the linear form to v. Then for all $p(x) \in \mathcal{P}_d$:

$$[p, \rho_v^d] = \ p(v).$$

Proof. [,] is clearly bilinear. Let $p(y), q(y) \in \mathbb{R}[y]$ homogeneous of degree d. We write $p(y) = \sum_{|i|=d} p_i y^i$, $q(y) = \sum_{|i|=d} q_i y^i$, where $p_i, q_i \in \mathbb{R}$. Then

$$\begin{aligned} d![\varphi^*(p), \varphi^*(q)] = p(\nabla)q(y) &= \sum_{|i|=|j|=d} p_i\, q_j\, \nabla^i y^j = \sum_{|i|=d} p_i\, q_i\; i_1! \cdots i_n! \\ &= d![\varphi^*(q), \varphi^*(p)], \end{aligned}$$

$d![\varphi^*(p), \varphi^*(p)] = \sum_{|i|=d} p_i^2\, i_1! \cdots i_n! \geq 0$, and $[\varphi^*(p), \varphi^*(p)] = 0$ if and only if $p = 0$. So [,] is a scalar product.

Clearly $\rho_v(x)^d = \varphi^*((y \cdot \varphi(v))^d)$. For $i \in \mathbb{Z}^d_{\geq 0}$ and $w \in \mathbb{R}^n$ write $w^i := w_1^{i_1} \cdot \cdots \cdot w_n^{i_n}$. We calculate

$$(y \cdot \varphi(v))^d = \sum_{|i|=d} \frac{d!}{i_1! \cdots i_n!}\; \varphi(v)^i y^i.$$

Let $p(x) \in \mathcal{P}_d$ and $q(y) = \sum_{|i|=d} q_i y^i \in \mathbb{R}[y]$ such that $\varphi^*(q(y)) = p(x)$. Then

$$[p, \rho_v^d] = \tfrac{1}{d!}\, q(\nabla)(y \cdot \varphi(v))^d = \sum_{|i|=d} q_i\; \varphi(v)^i \;=\; q(\varphi(v)) = p(v).$$

$\square$

The Euclidean inner product defines a quadratic polynomial $\omega(y) = y \cdot y$, so $\omega^*(x) := \varphi^*(\omega(y)) = (x, x)$. Since $\omega(y)$ is $O_n(\mathbb{R})$-invariant, $\omega^*(x)$ is $O(V, (\ ,\ ))$-invariant.

Definition 5.2 (Laplace Operator) *The Laplace operator is*

$$\Delta := \omega(\nabla) = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}.$$

Lemma 5.3 *Let $p(x) \in \mathcal{P}$.*

$$\Delta p(x) = \sum_{i,j,k=1}^{n} \frac{\partial^2 p(x)}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial y_i} \frac{\partial x_k}{\partial y_i} = \sum_{j,k=1}^{n} G^{-1}_{j,k} \frac{\partial^2 p(x)}{\partial x_j \partial x_k},$$

where G is the fixed Gram matrix of $(V, (\ ,\ ))$ and $G^{-1} = \left(G^{-1}_{j,k}\right)_{j,k}$. So

$$\Delta = \nabla_x^{\mathrm{tr}}\, G^{-1}\, \nabla_x, \;\; \textit{where } \nabla_x = \left(\tfrac{\delta}{\delta x_1}, \ldots, \tfrac{\delta}{\delta x_n}\right)^{\mathrm{tr}}.$$

Proof. By the chain rule,

$$\Delta p(x) = \sum_{i,j,k=1}^{n} \frac{\partial^2 p(x)}{\partial x_j \partial x_k} \frac{\partial x_j}{\partial y_i} \frac{\partial x_k}{\partial y_i}.$$

Since $x = A^{-1}y$, we have $\frac{\partial x_j}{\partial y_i} = A^{-1}_{j,i}$. So

$$\sum_{i=1}^{n} \frac{\partial x_j}{\partial y_i}\frac{\partial x_k}{\partial y_i} = \sum_{i=1}^{n} A^{-1}_{j,i} A^{-\mathrm{tr}}_{i,k} = (A^{-1}A^{-\mathrm{tr}})_{j,k} = G^{-\mathrm{tr}}_{j,k} = G^{-1}_{j,k}.$$

□

Lemma 5.4 *Δ is $O(V, (\ ,\ ))$-invariant and maps $\mathcal{P}_d$ surjectively onto $\mathcal{P}_{d-2}$.*

Proof. Let $g \in O(V, (\ ,\ ))$ and $p \in \mathcal{P}_d$. Then by the chain rule,

$$\nabla_x\, p \circ g^{-1}(x) = g^{-\mathrm{tr}} \nabla_x p(g^{-1}x).$$

So

$$\Delta(gp)(x) = \nabla_x^{\mathrm{tr}}\, G^{-1}\, \nabla_x (p \circ g^{-1})(x) = \nabla_x^{\mathrm{tr}}\, g^{-1} G^{-1} g^{-\mathrm{tr}}\, \nabla_x(p)(g^{-1}x)).$$

Since $g \in O_n$ we have $g^{-1}G^{-1}g^{-\mathrm{tr}} = G^{-1}$ and hence $\Delta(gp) = g\Delta(p)$.

The proof of the surjectivity can also be found in [Ven01, Prop. 1.3]. Let $\varphi^*(p) \in \mathcal{P}_{d-2}$ be orthogonal to $\Delta(\mathcal{P}_d)$ with respect to $[\ ,\ ]$. Since $\omega^*\varphi^*(p) \in \mathcal{P}_d$ we have for all $\varphi^*(q) \in \mathcal{P}_d$:

$$\begin{aligned} d![\omega^*\varphi^*(p), \varphi^*(q)] &= (\omega p)(\nabla)\, q(y) = \Delta p(\nabla) q(y) = p(\nabla)\Delta q(y) \\ &= [\varphi^*(p), \Delta(\varphi^*(q))] = 0. \end{aligned}$$

Hence $\omega^*\varphi^*(p) = 0$ and so $\varphi^*(p) = 0$. □

Remark 5.5 ([Ven01, p. 15]) *Let $\lambda \in V$, $m \geq 1$, and $d \geq 2$.*

$$\begin{aligned} \Delta((x,\lambda)) &= 0 \textit{ and } \Delta((x,\lambda)^d) = d(d-1)(\lambda,\lambda)(x,\lambda)^{d-2}. \\ \Delta(\omega^*(x)^m) &= 2m(n+2m-2)\omega^*(x)^{m-1}. \\ \Delta(\omega^*(x)^m(x,\lambda)^d) &= 2m(n+2m+2d-2)(x,\lambda)^d\omega^*(x)^{m-1} \\ &\quad + d(d-1)(\lambda,\lambda)(x,\lambda)^{d-2}\omega^*(x)^m. \end{aligned}$$

Definition 5.6 (Harmonic Polynomials)
A polynomial $p \in \mathcal{P}_d$ is called harmonic *or* spherical *if $\Delta(p) = 0$, i.e. if*

$$\sum_{j,k=1}^{n} G^{-1}_{j,k} \frac{\partial^2 p(x)}{\partial x_j \partial x_k} = 0.$$

The space of the homogeneous harmonic polynomials is $\mathcal{H}_d := \ker(\Delta) \leq \mathcal{P}_d$.

Theorem 5.7 *(i) $\mathcal{H}_d$ is an irreducible $\mathbb{R}O(V, (\ ,\ ))$-module.*

(ii) A harmonic polynomial divisible by $\omega^*(x)$ is zero.

(iii) There is an orthogonal decomposition

$$\mathcal{P}_d = \bigoplus_{m=0}^{\lfloor d/2 \rfloor} \omega^*(x)^m \mathcal{H}_{d-2m}.$$

Proof. This is well known for $\mathbb{R}[y]$, see for example [Ven01, Théorème 2.1] or [Hel00, pp. 345]. By applying φ^*, this is also true for $\mathbb{R}[x]$. □

Extension to the complex numbers We extend the Euclidean space to the complex space $V(\mathbb{C}) := \mathbb{C} \otimes V$ and the $\mathbb{C}$-linear extension of $(\ ,\ )$. So we have the polynomial ring $\mathcal{P}(\mathbb{C}) := \mathbb{C}[x_1, \dots, x_n] \cong S(V(\mathbb{C})^*)$. We write $\mathcal{P}_d(\mathbb{C})$ for the homogeneous polynomials and $\mathcal{H}_d(\mathbb{C})$ for the homogeneous harmonic polynomials of degree d.

Also we extend $\varphi : V(\mathbb{C}) \to \mathbb{C}^n$ and $\varphi^* : \mathbb{C}[y] \to \mathcal{P}(\mathbb{C})$.

For $p = \sum_{i \in \mathbb{Z}^n_{\geq 0}} p_i y^i \in \mathbb{C}[y]$ we write $\overline{p} := \sum_{i \in \mathbb{Z}^n_{\geq 0}} \overline{p_i} y^i$. We extend $[\ ,\ ]$ to a sesquilinear form on $\mathcal{P}_d(\mathbb{C})$:

$$[\varphi^*(p), \varphi^*(q)] = \overline{p}(\nabla) q(y).$$

Analogously to Lemma 5.1, we have

$$[q, (x, \lambda)^d] = \overline{q}(\lambda)$$

for all $q \in \mathcal{P}_d(\mathbb{C})$ and $\lambda \in V(\mathbb{C})$.

The harmonic polynomials can be written in the following way.

Lemma 5.8 *Let $d \geq 2$. $\mathcal{H}_d(\mathbb{C})$ is generated by*

$$\left\{ (x, \lambda)^d \mid \lambda \in V(\mathbb{C}),\ (\lambda, \lambda) = 0 \right\}.$$

Proof. The proof can also be found is [Hel00, pp.345], [Ogg69, Chapter VI], or [Ebe02, Theorem 3.1].

Let $p(x) := (x, \lambda)^d$ with $(\lambda, \lambda) = 0$. By Remark 5.5,

$$\Delta^*(p(x)) = d(d-1) \underbrace{(\lambda, \lambda)}_{=0} (x, \lambda)^{d-2} = 0,$$

so $p \in \mathcal{H}_d(\mathbb{C})$.

To see the other direction, let $p \in \mathcal{H}_d(\mathbb{C})$ be orthogonal to all $(x, v)^d$ with $(v, v) = 0$. Hence $0 = [p, (x, v)^d] = \overline{p}(v)$ for all zeros of $\omega^*(x)$. Therefore, by Hilbert's Nullstellensatz, $\omega^*(x)$ divides $\overline{p}$ and also $p \in \mathcal{H}_d(\mathbb{C})$. But this is only possible for $p = 0$, see Theorem 5.7(ii). □

5.2 Gegenbauer Polynomials

We want to define some interesting harmonic polynomials. We introduce Gegenbauer polynomials, which were used in this context in [Vil68] and [DelGoeSei77]. And we define zonal polynomials, see also [Ven01], [BacVen01], and [Jür15].

Definition 5.9 (Gegenbauer Polynomials, cf. [Vil68])
Fix a parameter $p \in \mathbb{R}_{\geq 0}$. On $\mathbb{R}[t]$ we define the scalar product

$$(f, g) = \int_{-1}^{1} f(t)g(t)(1-t^2)^{p-\frac{1}{2}} dt.$$

The Gegenbauer polynomials G_0^p, G_1^p, G_2^p, etc. are the orthonormal polynomials with respect to the scalar product, i.e. G_d^p is homogeneous of degree d and $(G_d^p, G_\ell^p) = \delta_{d,\ell}$.

The first Gegenbauer polynomials are

$$\begin{aligned} G_0^p(t) &= 1, \\ G_1^p(t) &= 2pt, \\ G_2^p(t) &= 2p(p+1)t^2 - p, \\ G_3^p(t) &= \tfrac{4}{3}p(p+1)(p+2)t^3 - 2p(p+1)t. \end{aligned}$$

Let $G_d^p(t) = \sum_{m=0}^{\lfloor \frac{d}{2} \rfloor} p_m \, t^{d-2m}$, then

$$p_m = (-1)^m \frac{\Gamma(p+d-m)2^{d-2m}}{\Gamma(p)m!(d-2m)!}.$$

The Gegenbauer polynomials also fulfill the recurrence relations

$$G_d^p(t) = \tfrac{2(p+1)+d}{2(p+1)} G_d^{p-1}(t) + tG_{d-1}^p(t)$$

and

$$G_d^p(t) = \tfrac{2(d+p-1)}{d} \, t \, G_{d-1}^p(t) - \tfrac{d+2p-2}{d} G_{d-2}^p(t).$$

Vilenkin gives a basis of $\mathcal{P}_d$ with Gegenbauer polynomials in [Vil68, Chapter IX]. It is complicated to describe the basis, and for us not necessary. So we just define very useful zonal polynomials.

Definition 5.10 (The Polynomials P_d)
Let $n \geq 2$ be even and $d \in \mathbb{Z}_{\geq 0}$. Set $p := (n-2)/2$. We homogenize the Gegenbauer polynomial $G_d^p(t)$:

$$G_d^p(t, s) = \sum_{m=0}^{\lfloor d/2 \rfloor} p_m \, t^{d-2m} \, s^{2m} \in \mathbb{R}[t, s].$$

We define

$$P_d^\lambda(x) := \sum_{m=0}^{\lfloor d/2 \rfloor} p_m \, (x,\lambda)^{d-2m} \, (x,x)^m \, (\lambda,\lambda)^m \in \mathcal{P}_d.$$

If λ is fixed, we often write P_d instead of P_d^λ.

The polynomial P_d^λ defines a map $V \to \mathbb{R}$, $\mu \mapsto P_d^\lambda(\mu)$, and

$$P_d^\lambda(\mu) = G_d^p\left((\lambda,\mu), \sqrt{(\lambda,\lambda)(\mu,\mu)}\right) = \sum_{m=0}^{\lfloor d/2 \rfloor} p_m \, (\lambda,\mu)^{d-2m} \, (\lambda,\lambda)^m (\mu,\mu)^m.$$

So we will sometimes write $G_d^p((x,\lambda), \sqrt{(x,x)(\lambda,\lambda)})$ instead of $P_d^\lambda(x)$. The square root is just a convenient notation because it does not appear in the polynomial.

Definition 5.11 (Zonal Polynomials)
Let $\lambda \in V$. A polynomial $p \in \mathcal{P}_d$ is called zonal *to λ if $gp = p$ for all $g \in O(V, (\ ,\ ))$ with $g\lambda = \lambda$.*

A polynomial $p \in \mathcal{P}$ is zonal to λ if and only if for all $\mu, \mu' \in V$:

$$(\mu,\mu) = (\mu',\mu') \text{ and } (\lambda,\mu) = (\lambda,\mu') \implies P(\mu) = P(\mu').$$

Theorem 5.12 *Let $\lambda \in V$ and $d \in \mathbb{Z}_{\geq 0}$.*

The polynomial P_d^λ is harmonic and homogeneous of degree d and zonal with respect to λ.

The space of zonal harmonic polynomials is 1-dimensional, i.e.

$$\{p \in \mathcal{H}_d \mid p \text{ is zonal to } \lambda\} = \langle P_d^\lambda \rangle.$$

Proof. See [Vil68, Chapter IX], [Ven01], or [Jür15, Kapitel 2]. □

5.3 Spherical Designs

Spherical designs are interesting geometric objects. Many interesting designs can be constructed with lattices, and layers of extremal lattices are often good designs. This section follows [Ven01] and [DelGoeSei77].

Definition 5.13 (Spherical t-Designs)
Let $X \subset V$ be non-empty and finite, and all elements of X should have the same square length. Let $t \in \mathbb{N}$. Then X is called a spherical t-design *if for all $P \in \mathcal{H}_d$ with $1 \leq d \leq t$:*

$$\sum_{x \in X} P(x) \ = \ 0.$$

Usually one considers $V = \mathbb{R}^n$. Let $S^{n-1} := \{x \in \mathbb{R}^n \mid (x,x) = 1\}$ be the unit sphere. A finite set $X \subseteq S^{n-1}$ is a t-design if and only if for all polynomials $p(x)$ of degree less or equal t:

$$\int_{S^{n-1}} p(x)\,dx \;=\; \frac{1}{|X|}\sum_{x\in X} p(x).$$

The measure dx is normalized such that S^{n-1} has measure 1. So t-designs can be used for polynomial approximations. For us, other equivalent characterizations are more useful.

Proposition 5.14 (Venkov)
Let X be as before, and additionally assume that X is symmetric, i.e. $-X = X$. Let $a > 0$ be the square length of any element of X, and let $t \in \mathbb{N}$ be even.

Then the following are equivalent:

(i) X is a spherical t-design.

(ii) X is a spherical $(t+1)$-design.

(iii) For all $p \in \mathcal{P}_d$ and $g \in O(V, (\ ,\ ))$:

$$\sum_{x\in X} p(x) = \sum_{x\in X} (gp)(x).$$

(iv) There is a constant $c_t \in \mathbb{R}$ such that for all $y \in V$:

$$\sum_{x\in X} (x,y)^t \;=\; c_t\ a^{t/2}\ (y,y)^{t/2}\ |X|.$$

Proof. Since $-X = X$, the condition $\sum_{x\in X} p(x) = 0$, $p \in \mathcal{H}_t$, is trivial for t odd. So (i) is equivalent to (ii). For the proofs of the other equivalences see [Ven01, Théorème 3.2]. □

The constant in (iv) is always

$$c_t := \prod_{\ell=1}^{t/2} \frac{2\ell-1}{n+2\ell-2}.$$

One gets the following additional equivalences.

Proposition 5.15 *Let X, a, and t be as before. X is a t-design if and only if*

$$\sum_{x,y\in X} (x,y)^t \;=\; c_t\ a^t\ |X|^2.$$

Proof. [Ven01, Théorème 8.1] □

Theorem 5.16 (Delsarte-Goethals-Seidel)
Let $X \subseteq V$ be non-empty, finite, and symmetric. Assume that all elements of X have the same square length. Let $t \in \mathbb{N}$.

X is a spherical t-design if and only if for all $\lambda \in X$ and all $d = 1, \ldots, t$:

$$\sum_{x \in X} P_d^\lambda(x) \ = 0.$$

Proof. [DelGoeSei77, Theorem 5.5] □

Example 5.17 *The minimal vectors of the $\mathbb{Z}$-lattice $\mathbb{E}_8$ form a 7-design, the minimal vectors of $\mathbb{E}_8 \perp \mathbb{E}_8$ form a 3-design, and the minimal vectors of the Leech lattice form an 11-design. In general, one can show that the layers $L(m)$ of an extremal even unimodular lattice L are spherical 7, 3, or 11-designs, if $\dim L$ is modulo 24 equal to 8, 16, or 0, respectively. This was observed by Venkov in [Ven84].*

Bachoc and Venkov also considered modular lattices, see [BacVen01]. They observed the following.

If L is extremal even 5-modular, then for all $m \in \mathbb{N}$ with $L(m) \neq 0$:

- *If $m \equiv 0 \pmod 8$, then $L(m)$ is a spherical 3-designs.*
- *If $m \equiv 4 \pmod 8$, then $L(m) \cup \sqrt{5}L^\#(m)$ is a spherical 3-design.*

If L is extremal even 2-modular, then for all $m \in \mathbb{N}$ with $L(m) \neq 0$:

- *If $m \equiv 0 \pmod{16}$, then $L(m)$ is a spherical 7-designs.*
- *If $m \equiv 4 \pmod{16}$, then $L(m)$ is a spherical 5-designs, and $L(m) \cup \sqrt{2}L^\#(m)$ is a spherical 7-design.*
- *If $m \equiv 8 \pmod{16}$, then $L(m)$ is a spherical 3-designs.*
- *If $m \equiv 12 \pmod{16}$, then $L(m) \cup \sqrt{2}L^\#(m)$ is a spherical 3-design.*

If L is extremal even 3-modular, then for all $m \in \mathbb{N}$ with $L(m) \neq 0$:

- *If $m \equiv 0, 2 \pmod{12}$, then $L(m)$ is a spherical 5-designs, and $L(m) \cup \sqrt{3}L^\#(m)$ is a spherical 7-design.*
- *If $m \equiv 4, 6 \pmod{12}$, then $L(m)$ is a spherical 3-designs.*
- *If $m \equiv 8, 10 \pmod{12}$, then $L(m) \cup \sqrt{3}L^\#(m)$ is a spherical 3-design.*

5.4 Spherical Theta Series

The following theorem gives a connection of harmonic polynomials to lattices and modular forms. For classical spherical theta series we refer to [KoeKri07], [Ogg69], and [Ebe02].

Theorem 5.18 (Classical Spherical Theta Series)
Let n be even, and again let $(V, (\ ,\ ))$ be a Euclidean space of dimension n. Let $L \subseteq V$ be a $\mathbb{Z}-$lattice and let $P \in \mathcal{H}_d$ be a homogeneous harmonic polynomial of degree d. Define the spherical *or* harmonic theta series

$$\Theta_{L,P}(z) := \sum_{\lambda \in L} P(\lambda)\, q^{(\lambda,\lambda)/2}, \text{ where } q = \exp(2\pi i\, z) \text{ and } z \in \mathbb{H}.$$

We have the identity

$$\Theta_{L,P}(-\tfrac{1}{z}) = (-z)^{n/2+d}\, i^{n/2}\, \det(L)^{-1/2}\, \Theta_{L^\#,P}.$$

If L is even unimodular, then $\Theta_{L,P}$ is a modular form of weight $n/2 + d$ and is a cusp form if $d > 0$.

Proof. [Ebe02, Proposition 3.1 and Theorem 3.3] □

One finds many spherical t-designs when calculating spherical theta series. For example, if $\Theta_{L,P} = 0$ for all harmonic spherical polynomials P of degree 1 to t, then all layers of L are spherical t-designs. This was discussed in [BacVen01]. Especially, one may prove the claims of Example 5.17.

Harmonic Polynomials over Real Quadratic Number Fields
We want to extend the theory of harmonic polynomials to a real quadratic number field F, so that the polynomials have coefficients in F rather than in $\mathbb{R}$ (or in $\mathbb{C}$).

For our purposes it is sufficient to use an embedding of F into $\mathbb{R}$. So let V be a F-vector space of dimension n and let $Q : V \to F$ be a totally positive definite form with bilinear form B. Let $E = (e_1, \ldots, e_n)$ be a basis of V and $G = (B(e_i, e_j))_{i,j}$ be the Gram matrix of E.

Let $j = 1, 2$. The field $\mathbb{R}$ becomes an F-algebra via $(\alpha, r) \mapsto \alpha^{(j)} \cdot r$. We write $F^{(j)}$ for the copy of $\mathbb{R}$. Also, $V^{(j)} := F^{(j)} \otimes F^n$ ($\cong \mathbb{R}^n$) and let $(\ ,\ )_j$ be the extension of $\sigma_j \circ B(\ ,\ )$ to $V^{(j)}$. So $(V^{(j)}, (\ ,\ )_j)$ is a Euclidean space, $E^{(j)} = (1 \otimes e_1, \ldots, 1 \otimes e_n) \in {V^{(j)}}^n$ is a basis, and $G^{(j)}$ is its Gram matrix.

We identify $\mathcal{P}^{(j)} := \mathbb{R}[x_1^{(j)}, \ldots, x_n^{(j)}] = S({V^{(j)}}^*)$ and define harmonic polynomials, Gegenbauer polynomials, etc. like in Section 5.1. The map $P \mapsto \sigma_j \circ P$ embeds $F[x_1, \ldots, x_n]$ in $\mathcal{P}^{(j)}$.

By extension to the complex numbers, we get polynomial rings

$$\mathcal{P}^{(1)}(\mathbb{C}) = \mathbb{C}[x_1^{(1)}, \ldots, x_n^{(1)}] \text{ and } \mathcal{P}^{(2)}(\mathbb{C}) = \mathbb{C}[x_1^{(2)}, \ldots, x_n^{(n)}].$$

Definition 5.19 (Harmonic Polynomials)
A polynomial $P \in F[x_1, \ldots, x_n]$ *is called* harmonic *or* spherical *if* $\sigma_1 \circ P$ *is harmonic.*

Remark 5.20 $\sigma_1 \circ P$ *is harmonic if and only if* $\sigma_2 \circ P$ *is harmonic.*

We may also define the Laplace operator Δ *over* F *similar to Section 5.1, i.e.* $\Delta = \sum_{i,j=1}^{n} G_{i,j}^{-1} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$. *Here the derivations* $\frac{\partial}{\partial x_i}$ *are*

$$\tfrac{\partial}{\partial x_i}\Big(\sum_{m=0}^{d} P_m x_i^m\Big) := \sum_{m=1}^{d} m P_m x_i^{m-1}$$

for arbitrary $\sum_{m=0}^{d} P_m x_i^m \in F[x_1, \ldots, x_n]$ *with* $P_m \in F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m]$.

A polynomial $P \in F[x_1, \ldots, x_n]$ *is harmonic if and only if* $\Delta P = 0$.

For instance, we may also find spherical t-designs as subsets of F^n. Layers of lattices over F are sometimes as good t-designs as the layers of their trace lattices. Sometimes they are better and sometimes they are worse.

The relation to spherical theta series and Hilbert modular forms is not so easy as for the classical case. A theta series with values of harmonic polynomials as Fourier coefficients is in general not a Hilbert modular form. Instead one has to consider norms of harmonic polynomials.

Definition 5.21 (Spherical Theta Series)
Let (Λ, Q) *be a* $\mathbb{Z}_F$*-lattice of rank* n. *Assume that* (Λ, Q) *is of Type (i), (ii), or (iii). Let* $P \in F[x_1, \ldots, x_n]$ *be harmonic and homogeneous of degree* d.

We define spherical theta series to (Λ, Q) *and* P *depending on the type of* (Λ, Q). *For* $\mu \in F$ *and* $z \in \mathbb{H} \times \mathbb{H}$ *or* $z \in \mathbb{H} \times \overline{\mathbb{H}}$ *let*

$$q^{\mu} := \begin{cases} \exp(2\pi i \ \mathrm{Tr}(z\mu/\varepsilon_0\sqrt{d_F})) & \text{for Type (i) theta series, i.e.} \\ & \mathcal{N}(\varepsilon_0) = 1 \text{ and } z \in \mathbb{H} \times \mathbb{H}, \\ \exp(2\pi i \ \mathrm{Tr}(z\mu/\sqrt{d_F})) & \text{for Type (ii) theta series, i.e.} \\ & z \in \mathbb{H} \times \overline{\mathbb{H}}, \\ \exp(2\pi i \ \mathrm{Tr}(z\mu)) & \text{for Type (iii) theta series, i.e.} \\ & z \in \mathbb{H} \times \mathbb{H}. \end{cases}$$

Then

$$\Theta_{(\Lambda,Q),\mathcal{N}P} := \sum_{\lambda \in \Lambda} \mathcal{N}(P(\lambda))\, q^{Q(\lambda)}$$

is called the spherical theta series *or* harmonic theta series *of* (Λ, Q) *and* P.

Theorem 5.22 (Eichler)
$\Theta_{(\Lambda,Q),\mathcal{N}P}$ *is a Hilbert theta series of weight* $k = \frac{n}{2} + d$ *and level one, i.e.*

$$\Theta_{(\Lambda,Q),\mathcal{N}P} \in \begin{cases} M_k(\mathrm{SL}_2(\mathbb{Z}_F)) & \text{for Types (i) and (iii),} \\ \overline{M}_k(\mathrm{SL}_2(\mathbb{Z}_F)) & \text{for Type (ii).} \end{cases}$$

This was first proved by Eichler [Eic77]. Extensions were proved by Richter [Ric02, Theorem 2] and Walling [Wal93] (the last two proved different results with coincide in our special case).

As shown by Richter or Walling, the transformation formula of the previous proposition is also true if the lattice is not totally positive or if one considers translations of a lattice, respectively.

Proof. We follow the same lines as the proof of Theorem 4.3. Again, we do this only for Type (ii) theta series. One proves this for Types (i) and (iii) analogously.

And we use the fact that a homogeneous polynomial of degree d is harmonic if and only if it is the linear combination of polynomials $(x,w)^d$ with $(w,w)=0$, see Lemma 5.8. We have $\mathcal{N}(P(\lambda)) = P^{(1)}(\lambda^{(1)})P^{(2)}(\lambda^{(2)})$, where $P^{(j)} = \sigma_j \circ P^{(j)} \in \mathcal{P}^{(j)}$ is harmonic. Hence it is a product of polynomials $(x^{(j)}, w^{(j)}))_j^d$ with $(w^{(j)}, w^{(j)})_j = 0$.

So let $w^{(j)} \in \mathbb{C} \otimes V^{(j)}$ with $(w^{(j)}, w^{(j)}) = 0$ and

$$h(x^{(1)}, x^{(2)}) := (x^{(1)}, w^{(1)})_1^d (x^{(2)}, w^{(2)})_2^d.$$

It is sufficient to show that

$$\Theta_{(\Lambda,Q),h}(z) := \sum_{\lambda\in\Lambda} h(\lambda^{(1)}, \lambda^{(2)}) \exp(2\pi i \ \mathrm{Tr}(zQ(\lambda)/\sqrt{d_F}))$$

is a modular form.

The invariance for the matrices $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$, where $\mu \in \mathbb{Z}_F$ and $\varepsilon \in \mathbb{Z}_F^*$, is straight forward to show. We show the invariance under the transformation with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

There is an isometry φ_j of the Euclidean spaces $V^{(j)}$ with $(\ ,\ )_j$ and $\mathbb{R}^n$ with the standard Euclidean space on $\mathbb{R}^n$, where $j = 1, 2$. By combining the two real embeddings, there is an isometric embedding $\varphi : V \to \mathbb{R}^{2n}, \lambda \mapsto (\varphi_1(\lambda), \varphi_2(\lambda))$, where $\mathbb{R}^{2n}$ has the dot scalar product $(x,y) \mapsto x \cdot y := x^{\mathrm{tr}} y$. We know that $L := \varphi(\Lambda) \subseteq \mathbb{R}^{2n}$ is an even $\mathbb{Z}$-lattice and that $\det L = \det L_1 = \mathcal{N}(\mathbb{Z}_F^{\#})^{-n} = d_F^n$. Actually $L^{\#} = \varphi(\Lambda^*) = \varphi(\frac{1}{\sqrt{d_F}}\Lambda)$, and hence

$$\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, (y^{(1)}, y^{(2)})^{\mathrm{tr}} \mapsto \left(\sqrt{d_F} y^{(1)}, -\sqrt{d_F} y^{(2)}\right)^{\mathrm{tr}}$$

is a similarity of norm d_F which maps $L^{\#}$ onto L. For $y \in \mathbb{R}^2 n$ we let $y^{(1)}$ are the first n entries of y and $y^{(2)}$ are the last.

We may write $h(\lambda^{(1)}, \lambda^{(2)}) = (\varphi_1(\lambda) \cdot \widetilde{w}^{(1)})^d \cdot (\varphi_2(\lambda) \cdot \widetilde{w}^{(2)})^d$ for some $(\widetilde{w}^{(1)}, \widetilde{w}^{(2)})^{\mathrm{tr}} \in \mathbb{C}^{2n}$ with $\widetilde{w}^{(j)} \cdot \widetilde{w}^{(j)} = 0$. For short we set $h'(y) := (y^{(1)} \cdot \widetilde{w}^{(1)})^d \cdot (y^{(2)} \cdot \widetilde{w}^{(2)})^d$.

So

$$\Theta_{(\Lambda,Q),h}(z) = \sum_{y\in L} h'(y) e^{\pi i \ (y^{(1)} \cdot y^{(1)} z_1 - y^{(2)} \cdot y^{(2)} z_2)/\sqrt{d_F}},$$

and hence the theta series is holomorphic on $\mathbb{H} \times \overline{\mathbb{H}}$. Let $z \in \mathbb{H} \times \overline{\mathbb{H}}$.

Since $\psi(L^{\#}) = L$, the transformation of the theta series may be written as

$$\Theta_{(\Lambda,Q),h}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) z\right) = \sum_{y\in L^{\#}} h'(\psi(y))\ \exp\left(\pi i \left(\psi(y^{(1)})\cdot\psi(y^{(1)})\tfrac{-1}{z_1\sqrt{d_F}} + \psi(y^{(2)})\cdot\psi(y^{(2)})\tfrac{1}{z_2\sqrt{d_F}}\right)\right).$$

Since $\psi(x)\cdot\psi(x) = d_F\, x\cdot x$ for all $x\in\mathbb{R}^{2n}$ and $\psi(y)^{(j)} = \sqrt{d_F}^{(j)} y^{(j)}$, this is equal to

$$\mathcal{N}(\sqrt{d_F})^d \sum_{y\in L^{\#}} f_1(y^{(1)}) f_2(y^{(2)}),$$

where $f_j(y^{(j)}) = (y^{(j)}\cdot\widetilde{w}^{(j)})^d \exp\left(\pi i\, y^{(j)}\cdot y^{(j)}\,\frac{\sqrt{d_F}^{(j)}}{-z_j}\right)$.

Using Poisson's summation formula (see for instance [Ebe02, Theorem 2.3]), this is equal to

$$\mathcal{N}(\sqrt{d_F})^d \det(L^{\#})^{-1/2} \sum_{y\in L}\int_{\mathbb{R}^{2n}} f_1(x^{(1)}) e^{-2\pi i\, x^{(1)}\cdot y^{(1)}} f_2(x^{(2)}) e^{-2\pi i\, x^{(2)}\cdot y^{(2)}}\, dx,$$

where $x = (x^{(1)}, x^{(2)})^{\mathrm{tr}} \in \mathbb{R}^{2n}$. The prefactor is

$$\mathcal{N}(\sqrt{d_F})^d \det(L^{\#})^{-1/2} = (-d_F)^d \det(L)^{1/2} = (-1)^d d_F^{d+n/2}.$$

$$\int_{\mathbb{R}^n} f_1(x^{(1)}) e^{-2\pi i\, x^{(1)}\cdot y^{(1)}}\, dx^{(1)} = \left(\frac{z_1}{\sqrt{d_F}\, i}\right)^{n/2+d} \left(\frac{y^{(1)}\cdot\widetilde{w}_1}{i}\right)^d e^{\pi i\, y^{(1)}\cdot y^{(1)} z_1/\sqrt{d_F}}$$

and

$$\int_{\mathbb{R}^n} f_2(x^{(2)}) e^{-2\pi i\, x^{(2)}\cdot y^{(2)}}\, dx^{(2)} = \left(\frac{-z_2}{\sqrt{d_F}\, i}\right)^{n/2+d} \left(\frac{y^{(2)}\cdot\widetilde{w}_2}{i}\right)^d e^{\pi i\, y^{(2)}\cdot y^{(2)} (-z_2)/\sqrt{d_F}}.$$

The exponential factors multiply to $\exp(2\pi i\ \mathrm{Tr}(zQ(\lambda)/\sqrt{d_F}))$, where $\lambda\in\Lambda$ with $\varphi(\lambda) = y = (y^{(1)}, y^{(2)})$.

The polynomial factors multiply to $h'(y)$, which is equal to $h(\lambda^{(1)},\lambda^{(2)})$. In conclusion, the transformation of the theta series is equal to

$$\begin{aligned}(-1)^d d_F^{d+n/2} \left(\frac{z_1}{\sqrt{d_F}\, i}\right)^{n/2+d} \left(\frac{-z_2}{\sqrt{d_F}\, i}\right)^{n/2+d} & i^{-2d} \sum_{\lambda\in\Lambda} h(\lambda^{(1)},\lambda^{(2)}) e^{2\pi i\,\mathrm{Tr}(zQ(\lambda)/\sqrt{d_F})} \\ &= (-z)^{n/2+d}\Theta_{(\Lambda,Q),h}(z).\end{aligned}$$

Therefore $\Theta_{(\Lambda,Q)}(z)$ and hence $\Theta_{(\Lambda,Q),\mathcal{N}P}(z)$ are Hilbert modular forms of weight $k = n/2 + d$.

$\square$

Galois Symmetry
The extension of the Galois automorphism $\alpha \mapsto \overline{\alpha}$ defines transformations

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H},\ (z_1, z_2) \mapsto (z_2, z_1)$$

and

$$\mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{H} \times \overline{\mathbb{H}},\ (z_1, z_2) \mapsto (-z_2, -z_1).$$

Hilbert modular forms invariant under one of these transformation are called (Galois) symmetric.

On the lattice side, an F-vector space V is also an F-vector space with the multiplication

$$F \times V \to V,\ (\alpha, \lambda) \mapsto \alpha * \lambda := \overline{\alpha} \cdot \lambda.$$

We denote this vector space with $\overline{V}$, compare also Definition 1.4. If we want to emphasis that we regard a $\lambda \in V$ in $\overline{V}$, we write $\lambda_{\overline{V}}$ instead of λ.

Let $n = \dim V$ and $E = (e_1, \dots, e_n)$ be a basis (of V and $\overline{V}$). The polynomial ring $F[x_1, \dots, x_n]$ is on one hand identified with $\mathcal{P} := S(V^*)$ and on the other hand with $\overline{\mathcal{P}} := S(\overline{V}^*)$ via

$$x_i \mapsto e_j^*.$$

Let $\lambda \in V$, then $\lambda = \sum_{i=1}^n \lambda_j \cdot e_i$ and $\lambda_{\overline{V}} = \sum_{i=1}^n \overline{\lambda_i} * e_i$ for some $\lambda_1, \dots, \lambda_n \in F$. If $P(x) \in \mathcal{P}$, then $P(\lambda) = P(\lambda_1, \dots, \lambda_n)$, and if $P \in \overline{\mathcal{P}}$, then $P(\lambda_{\overline{V}}) = P(\overline{\lambda}_1, \dots, \overline{\lambda}_n)$.

For $P(x) = \sum_{i \in \mathbb{Z}^n_{\geq 0}} p_i\, x^i \in F[x_1, \dots, x_n]$ let

$$\overline{P}(x) := \sum_{i \in \mathbb{Z}^n_{\geq 0}} \overline{p_i}\, x^i.$$

Then for $\lambda \in V$ we have

$$P(\lambda) = \overline{\overline{P}(\lambda_{\overline{V}})}$$

and especially

$$\mathcal{N}(P(\lambda)) = \mathcal{N}(\overline{P}(\lambda_{\overline{V}})).$$

With the last formula, we can state relations between the spherical theta series of (Λ, Q) and its Galois-conjugate $(\overline{\Lambda}, \overline{Q})$.

Theorem 5.23 *Let (Λ, Q) be a lattice of rank n. Let $P \in F[x_1, \dots, x_n]$ be homogeneous of degree d and harmonic.*

Then the following identities are true.

Type (i):

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2, z_1) = (-1)^d \Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1, z_2).$$

Type (ii):

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(-z_2, -z_1) = \Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1, z_2).$$

Type (iii):

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1)=\Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1,z_2).$$

Proof. We identify F with $F^{(1)}\subseteq\mathbb{R}$. (i).

$$\begin{aligned}\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1) &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\lambda))\exp\left(2\pi i\ \left(\frac{z_2Q(\lambda)}{\varepsilon_0\sqrt{d_F}}-\frac{z_1\overline{Q(\lambda)}}{\overline{\varepsilon}_0\sqrt{d_F}}\right)\right)\\ &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\lambda))\exp\left(2\pi i\ \left(\frac{z_1\overline{Q(\overline{\varepsilon}_0\lambda)}}{\varepsilon_0\sqrt{d_F}}-\frac{z_2Q(\overline{\varepsilon}_0\lambda)}{\overline{\varepsilon}_0\sqrt{d_F}}\right)\right)\\ &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\varepsilon_0\lambda))\exp\left(2\pi i\ \left(\frac{z_1\overline{Q(\lambda)}}{\varepsilon_0\sqrt{d_F}}-\frac{z_2Q(\lambda)}{\overline{\varepsilon}_0\sqrt{d_F}}\right)\right).\end{aligned}$$

Since P is homogeneous of degree d, we have

$$\mathcal{N}(P(\varepsilon_0\lambda))=\mathcal{N}(\varepsilon_0)^d\mathcal{N}(P(\lambda))=(-1)^d\mathcal{N}(P(\lambda)).$$

We will sum over $\overline{\Lambda}$ and use $\overline{Q}$ instead of Q. Then $\mathcal{N}(P(\lambda))=\mathcal{N}(\overline{P}(\lambda_{\overline{V}}))$. We get

$$\begin{aligned}\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1) &= (-1)^d\sum_{\lambda\in\overline{\Lambda}}\mathcal{N}(\overline{P}(\lambda_{\overline{V}}))\exp\left(2\pi i\ \left(\frac{z_1\overline{Q}(\lambda_{\overline{\Lambda}})}{\varepsilon_0\sqrt{d_F}}-\frac{z_2\overline{\overline{Q}(\lambda_{\overline{\Lambda}})}}{\overline{\varepsilon}_0\sqrt{d_F}}\right)\right)\\ &= (-1)^d\Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1,z_2).\end{aligned}$$

(ii).

$$\begin{aligned}\Theta_{(\Lambda,Q),\mathcal{N}P}(-z_2,-z_1) &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\lambda))\exp\left(2\pi i\ \left(\frac{-z_2Q(\lambda)}{\sqrt{d_F}}-\frac{-z_1\overline{Q(\lambda)}}{\sqrt{d_F}}\right)\right)\\ &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\lambda))\exp\left(2\pi i\ \left(\frac{z_1\overline{Q(\lambda)}}{\sqrt{d_F}}-\frac{z_2Q(\lambda)}{\sqrt{d_F}}\right)\right)\\ &= \sum_{\lambda\in\overline{\Lambda}}\mathcal{N}(\overline{P}(\lambda_{\overline{V}}))\exp\left(2\pi i\ \left(\frac{z_1\overline{Q}(\lambda_{\overline{\Lambda}})}{\sqrt{d_F}}-\frac{z_2\overline{\overline{Q}(\lambda_{\overline{\Lambda}})}}{\sqrt{d_F}}\right)\right)\\ &= \Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1,z_2).\end{aligned}$$

(iii).

$$\begin{aligned}\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1) &= \sum_{\lambda\in\Lambda}\mathcal{N}(P(\lambda))\exp\left(2\pi i\ \left(z_2Q(\lambda)+z_1\overline{Q(\lambda)}\right)\right)\\ &= \sum_{\lambda\in\overline{\Lambda}}\mathcal{N}(\overline{P}(\lambda_{\overline{\Lambda}}))\exp\left(2\pi i\ \left(z_1\overline{Q}(\lambda_{\overline{\Lambda}})+z_2\overline{\overline{Q}(\lambda_{\overline{\Lambda}})}\right)\right)\\ &= \Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z_1,z_2).\end{aligned}$$

□

Corollary 5.24 Type (i):

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1)=(-1)^d\Theta_{(\Lambda,Q),\mathcal{N}P}(z_1,z_2)$$

if and only if for all $\beta\in F$:

$$\sum_{\lambda\in\Lambda(\beta)}\mathcal{N}(P(\lambda))=\sum_{\lambda\in\Lambda(\overline{\beta})}\mathcal{N}(P(\lambda)).$$

If this is the case, then the spherical theta series $\Theta_{(\Lambda,Q),\mathcal{N}P}$ is symmetric if d is even and anti-symmetric if d is odd.

Type (ii): *The spherical theta series $\Theta_{(\Lambda,Q),\mathcal{N}P}$ is symmetric, i.e.*

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(-z_2,-z_1)=\Theta_{(\Lambda,Q),\mathcal{N}P}(z_1,z_2),$$

if and only if for all $\beta\in F$:

$$\sum_{\lambda\in\Lambda(\overline{\beta})}\mathcal{N}(P(\lambda))=\sum_{\lambda\in\Lambda(\beta)}\mathcal{N}(P(\lambda)).$$

Type (iii): *The spherical theta series $\Theta_{(\Lambda,Q),\mathcal{N}P}$ is symmetric, i.e.*

$$\Theta_{(\Lambda,Q),\mathcal{N}P}(z_2,z_1)=\Theta_{(\Lambda,Q),\mathcal{N}P}(z_1,z_2),$$

if and only if for all $\beta\in F$:

$$\sum_{\lambda\in\Lambda(\overline{\beta})}\mathcal{N}(P(\lambda))=\sum_{\lambda\in\Lambda(\beta)}\mathcal{N}(P(\lambda)).$$

Proof. We use the previous theorem. The identities of the theta series are true if and only if the coefficients of $\Theta_{(\overline{\Lambda},\overline{Q}),\mathcal{N}\overline{P}}(z)$ and of $\Theta_{(\Lambda,Q),\mathcal{N}P}(z)$ at β coincide, for all $\beta\in F$. So

$$\sum_{\lambda_{\overline{\Lambda}}\in\overline{\Lambda}(\beta)}\mathcal{N}(\overline{P}(\lambda_{\overline{V}}))=\sum_{\lambda\in\Lambda(\overline{\beta})}\mathcal{N}(P(\lambda))\overset{!}{=}\sum_{\lambda\in\Lambda(\beta)}\mathcal{N}(P(\lambda)).$$

□

Corollary 5.25 *Let (Λ,Q) be a lattice of rank n and Type (i), (ii), or (iii). Let $P\in F[x_1,\dots,x_n]$ be homogeneous of degree d and harmonic.*

Assume that (Λ,Q) is Galois invariant, i.e. there is a semi-endo-morphism $\sigma:\Lambda\to\Lambda$ with $Q(\sigma(\lambda))=\overline{Q(\lambda)}$ for all $\lambda\in\Lambda$. Assume that $\mathcal{N}P(\sigma(\lambda))=\mathcal{N}P(\lambda)$ for all $\lambda\in\Lambda$. Let $k:=\frac{n}{2}+d$. Then we have the following.

Type (i):

$$\Theta_{(\Lambda,Q),\mathcal{N}P} \in \begin{cases} M_k^+(\mathrm{SL}_2(\mathbb{Z}_F)) & \textit{if } d \textit{ is even,} \\ M_k^-(\mathrm{SL}_2(\mathbb{Z}_F)) & \textit{if } d \textit{ is odd.} \end{cases}$$

Type (ii):

$$\Theta_{(\Lambda,Q),\mathcal{N}P} \in \overline{M}_k^+(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Type (iii):

$$\Theta_{(\Lambda,Q),\mathcal{N}P} \in M_k^+(\mathrm{SL}_2(\mathbb{Z}_F)).$$

Proof. The condition of the last corollary is fulfilled in each case:

$$\sum_{\lambda \in \Lambda(\overline{\beta})} \mathcal{N}P(\lambda) = \sum_{\lambda \in \Lambda(\beta)} \mathcal{N}P(\sigma(\lambda)) = \sum_{\lambda \in \Lambda(\beta)} \mathcal{N}P(\lambda).$$

□

5.5 Configuration Numbers

The theta series of an extremal lattice is the unique extremal Hilbert modular form. Hence the number of vectors of given length is determined independently from the lattice. The theory of spherical theta series gives further constraints for the configuration of lattice vectors in the same length. We develop a method to find these constrains in this section.

Suppose that Λ is an extremal lattice of Type (i), (ii), or (iii). Let $\alpha \in F$ such that $\Lambda(\alpha/2) \neq \emptyset$, e.g. $\alpha = 2\min(\Lambda)$. Choose $\lambda \in \Lambda(\alpha/2)$, i.e. $\lambda \in \Lambda$ with square length $B(\lambda, \lambda) = \alpha$. The number of vectors of the same layer of Λ which have the same inner product with λ are often determined by modular forms.

Definition 5.26 (Configuration numbers)
Let $\lambda \in \Lambda$. For $\beta \gg 0$ and $\iota \in F$ define

$$N(\beta, \iota; \lambda) := \{\mu \in \Lambda(\beta/2) \mid B(\lambda, \mu) = \pm\iota\} \quad \textit{and} \quad n(\beta, \iota; \lambda) := |N(\beta, \iota; \lambda)|.$$

The configuration numbers of λ and β are the numbers

$$(\ n(\beta, \iota; \lambda) \mid \iota \in F)\,.$$

Lemma 5.27 *Let $\lambda \in \Lambda$. For $\beta \gg 0$ there are only finitely many $\iota \in F$ with $n(\beta, \iota; \lambda) \neq 0$.*

Proof. Let $\mu \in \Lambda(\beta/2)$ and $B(\lambda, \mu) = \iota = \iota_0 + \iota_1\sqrt{D}$ with $\iota_0, \iota_1 \in \mathbb{Q}$. Actually, we have more restrictions on ι_0 and ι_1, depending on the Type of the lattice. So we have $\iota_0, \iota_1 \in \frac{1}{2D}\mathbb{Z}$.

Also, μ is a lattice point of the embedded $\mathbb{Z}$-lattices $\Lambda^{(1)}$ and $\Lambda^{(2)}$. We identify F with $F^{(1)} \subseteq \mathbb{R}$, hence $B^{(1)}(\lambda, \mu) = \iota_0 + \iota_1\sqrt{D} \in \mathbb{R}$ and $B^{(2)}(\lambda, \mu) = \iota_0 - \iota_1\sqrt{D} \in \mathbb{R}$. The Cauchy-Schwartz inequality yields

$$\begin{aligned} |B(\lambda, \mu)^{(1)}| &= |\iota_0 + \iota_1\sqrt{D}| \leq \sqrt{\alpha^{(1)}\beta^{(1)}}, \\ |B(\lambda, \mu)^{(2)}| &= |\iota_0 - \iota_1\sqrt{D}| \leq \sqrt{\alpha^{(2)}\beta^{(2)}}. \end{aligned}$$

Only finitely many ι fulfill these conditions.

□

The proof gives a direct way to compute all possible ι's by solving the inequalities. One may also use the fact that $Q(\lambda \pm \mu) = \frac{\alpha+\beta}{2} \pm \iota$ must be 0 or totally positive.

So we have finitely many configuration numbers $n(\beta, \iota; \lambda)$. We want to find equations of these without using information about the concrete extremal lattice. So, if the lattice is not known, we treat the configuration numbers as unknowns, and may find solutions for them.

Often there are easy relations between these numbers. We call these equations between the numbers *trivial* equations.

Lemma 5.28 (Trivial Equations)
Let $\alpha, \beta \in \mathbb{Z}_F$ be totally positive and $\lambda \in \Lambda(\alpha/2)$. Let $\iota \in F$.

(i) $n(\alpha, \alpha; \lambda) = 2$.

(ii) $n(\varepsilon^2\beta, \varepsilon\iota, \lambda) = n(\beta, \iota; \lambda)$ *for* $\varepsilon \in \mathbb{Z}_F^*$.

(iii) If Λ is Galois invariant with semi-endomorphism σ, then $n(\overline{\beta}, \overline{\iota}; \sigma(\lambda)) = n(\beta, \iota; \lambda)$.

(iv) If $\Lambda(\gamma/2) = \emptyset$, then $n(\beta, (\alpha + \beta - \gamma)/2; \lambda) = 0$.

Proof. (i). For $\mu \in \Lambda(\alpha/2)$ we have $B(\lambda, \mu) = \alpha$ if and only if $Q(\lambda - \mu) = 0$ if and only if $\mu = \lambda$. (ii). The maps $\mu \mapsto \varepsilon\mu$ and $\mu' \mapsto \varepsilon^{-1}\mu'$ are inverse, hence bijections of the sets $N(\dots)$. (iii). The map $\mu \mapsto \sigma(\mu)$ gives the desired bijection. (iv). Let $\mu \in \Lambda(\beta/2)$ such that $B(\lambda, \mu) = \iota = (\alpha + \beta - \gamma)/2$. Then $\lambda - \mu$ has norm $\alpha/2 + \beta/2 - \iota = \gamma$. This is not possible because of the assumption $\Lambda(\gamma/2) = \emptyset$. □

After using the previous lemma, we have a small number of unknowns left. First we have the equation

$$\sum_{\iota \in \mathbb{Z}_F} n(\beta, \iota; \lambda) = |\Lambda(\beta/2)|.$$

Another way to see this is that the sum over all $n(\beta, \iota; \lambda)$ is the coefficient of $q^{\beta/2}$ of the theta series, i.e.

$$\sum_{\iota} n(\beta, \iota; \lambda) = a_{n,m}(\Theta_\Lambda), \text{ where } (n, m) = (\mathrm{tr}(\alpha_1\beta/2), \mathrm{tr}(\alpha_2\beta/2)).$$

(For notations see Section 3.4 and Chapter 4.)

We may also plug other coefficients into the theta series. These coefficients are norms of values of harmonic polynomials. In Section 5.5 we introduced zonal harmonic polynomials $P_d = P_d^\lambda$, which are basically Gegenbauer polynomials. We defined the homogenized Gegenbauer polynomials G_d^p with parameter $p := n/2 - 1$. For $\mu \in \Lambda(\beta/2)$ with $\iota = B(\lambda, \mu)$ we have

$$P_d(\mu) = G_d^p(\iota, \sqrt{\alpha\beta}).$$

Lemma 5.29 *The coefficient of $q^{\beta/2}$ in $\Theta_{\Lambda,\mathcal{N}P_d}$ is*

$$\sum_{\iota} \mathcal{N} G_d^p(\iota, \sqrt{\alpha\beta})\ n(\beta, \iota; \lambda).$$

Proof. Since the coefficient of $q^{\beta/2}$ is $\sum_{\mu\in\Lambda(\beta/2)} \mathcal{N} P_d(\mu)$ and P_d is zonal, this follows directly from the definition of the configuration numbers. □

In the sense of modular forms, $\Theta_{\Lambda,\mathcal{N}P_d}$ is a modular form of weight $k = \frac{n}{2} + d$, and it is a cusp form if $d > 0$, cf. Theorem 5.22. If $S_{\frac{n}{2}+d} = \{0\}$, then we get an additional equation for the unknowns $n(\iota, \beta; \lambda)$. More precisely, then we have

$$\sum_{\iota} \mathcal{N} G_d^p(\iota, \sqrt{\alpha\beta})\ n(\beta, \iota; \lambda) = 0.$$

Also we may consider other norms $\beta' \neq \beta$. If $\beta' = \beta \cdot \varepsilon$ for some $\varepsilon \in \mathbb{Z}_F^*$, then the configuration numbers for β' are the same as for β (see Lemma 5.28), and β' does not yield anything new. If not, we get more variables $n(\beta', \iota'; \lambda)$, but we get also more equations. This is especially interesting if $\dim S_{d+n/2}$ is small, then the previous lemma yields equations for the configuration numbers.

So, with these equations we can maybe restrict the sets of possible configuration numbers $n(\beta, \iota; \lambda)$ to very few.

In some cases, we can determine the lattices with given configuration numbers.

Furthermore, if the configuration numbers are given, we are able to check other properties, e.g. whether the layer $\Lambda(\alpha/2)$ is a t-design.

Proposition 5.30 (t-designs)
Let Λ be an extremal lattice, $\alpha \gg 0$, and $\lambda \in \Lambda(\alpha/2)$. Let $p := \frac{n-2}{2}$. If

$$\sum_{\iota} n(\alpha, \iota; \lambda)\ G_d^p(\iota, \alpha) = 0$$

for all $d = 1, \ldots, t$ and for all possibilities of configuration numbers

$$(\ n(\alpha, \iota; \lambda) \mid \iota \in F \ ),$$

then $\Lambda(\alpha/2)$ is a spherical t-design.

Proof. For all $\lambda' \in \Lambda(\alpha/2)$ and $1 \leq d \leq t$ we have

$$\sum_{\mu \in \Lambda(\alpha/2)} P_d^{\lambda}(\mu) = \sum_{\iota} n(\alpha, \iota; \lambda') G_d^{p}(\iota, \alpha) = 0.$$

Hence by Theorem 5.16, $\Lambda(\alpha/2)$ is a spherical t-design. $\square$

Example 5.31 *In the following chapters we construct some extremal lattices of Types (i), (ii), and (iii). Sometimes their layers are good spherical designs:*

- *The minimal vectors of the unique extremal lattices of dimensions 4 and 12 over $\mathbb{Q}[\sqrt{5}]$ both form a spherical 11-designs.*
- *The minimal vectors of extremal lattices of dimensions 4, 8 and 12 over $\mathbb{Q}[\sqrt{2}]$ each form a spherical 7-designs.*
- *The minimal vectors of one extremal Type (ii) lattice of dimension 4 over $\mathbb{Q}[\sqrt{3}]$ form a spherical 5-designs.*

Chapter 6

Extremal Lattices over $\mathbb{Q}[\sqrt{5}]$

In, this chapter, we look at the field $F = \mathbb{Q}[\sqrt{5}]$. The fundamental unit is $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$, and the different ideal is generated by $\delta := \frac{5+\sqrt{5}}{2} \gg 0$.

Further we look at lattices of Type (i), i.e. even unimodular lattices (because $\mathcal{N}(\varepsilon_0) = -1$). If (Λ, Q) is a lattice of Type (i), then the first trace lattice (Λ_1, Q_1) is the trace lattice with respect to $\alpha_1 = \delta^{-1} = \frac{5-\sqrt{5}}{10}$ and the second trace lattice (Λ_2, Q_2) is the trace lattice with respect to $\alpha_2 = 1$.

Let $A = (\alpha_1, \alpha_2)$. With respect to $\leq_A$, the totally positive elements of $\mathbb{Z}_F$ are ordered in the following way:

$$1 \leq_A \frac{3+\sqrt{5}}{2} \leq_A \frac{3-\sqrt{5}}{2} \leq_A 2 \leq_A \frac{5+\sqrt{5}}{2} \leq_A 3+\sqrt{5} \leq_A \frac{7+3\sqrt{5}}{2} \leq_A \dots$$

6.1 Extremal Lattices

For a detailed study of extremal lattices over $\mathbb{Q}[\sqrt{5}]$ see [Neb13].

All even unimodular lattices of dimensions 4, 8, and 12 are classified in the literature, cf. [Maa41] and [CosHsi87]. In fact, in dimension 4 there is a unique even unimodular lattice, the root lattice F_4, see 2.3. We will prove this with the methods of Section 5.5. In dimension 8, both lattices are trivially extremal. In dimension 12, there are 15 different isometry classes of lattices. Precisely one is extremal.

Extremal lattices are also known in dimensions 16, 24, 28 and 36, cf. [Neb13].

Table 6.1 lists the known extremal lattices, see [Neb13] for more details. The first column gives the rank n of the lattice (over $\mathbb{Q}[\sqrt{5}]$), and the second column gives the rank N of the trace lattices (over $\mathbb{Q}$). The third lists details of the trace lattices, we use the notation of the Lattice Data Base [NebSlo]. The last column gives the number or a lower bound of the number of extremal lattices.

Extremal Modular Forms The graduated algebra of Hilbert modular forms is

$$M = M(\mathrm{SL}_2(\mathbb{Z}[\sqrt{5}]) = \mathbb{C}[A_2, s_5, B_6, s_{15}].$$

Table 6.1: Extremal lattices over $\mathbb{Q}[\sqrt{5}]$

n	$N=2n$	trace lattices	# extr. lat.
4	8	(E_8, H_4)	1
8	16	$\Lambda_1 = E_8 \perp E_8$	2
12	24	$\Lambda_1 = \Lambda_{24}$	1
16	32		≥ 2
20	40	?	?
24	48	$\Lambda_1 = P_{48n}$	≥ 1
28	56		≥ 1
32	64	?	?
36	72	$\Lambda_1 = \Gamma_{72}$	≥ 1
40	80	?	?

The generators are the (symmetric) Eisenstein series A_2, the symmetric cusp forms B_6 and s_{15}, and the anti-symmetric cusp form s_5. Often one also writes $C_{10} = s_5^2$. As usual the weights are given by the index. There is the relation $s_{15}^2 \in \mathbb{C}[A_2, s_5, B_6]$. The generators have the following q-expansions (see Theorem 3.27).

$$\begin{array}{rcl}
A_2 & = & 1 + 120q_1q_2^2 + 120q_1q_2^2 + 120q_1^2q_2^3 + 600q_1^2q_2^4 + 720q_1^2q_2^5 + 600q_1^2q_2^6 \\
 & & +120q_1^2q_2^7 + 1440q_1^3q_2^7 + 1440q_1^3q_2^8 + 1200q_1^3q_2^9 + 720q_1^3q_2^{10} + O(q_1^4q_2^6) \\
s_5 & = & q_1q_2^2 - q_1q_2^3 - q_1^2q_2^3 - 10q_1^2q_2^4 + 10q_1^2q_2^6 + q_1^2q_2^7 \\
 & & +120q_1^3q_2^6 - 108q_1^3q_2^7 + 108q_1^3q_2^8 - 120q_1^3q_2^9 + O(q_1^4q_2^6) \\
B_6 & = & q_1q_2^2q_1q_2^3 + q_1^2q_2^3 - 910q_1^2q_2^5 + q_1^2q_2^7 - 910q_1^3q_2^5 + 25650q_1^3q_2^6 + 24092q_1^3q_2^7 \\
 & & +24092q_1^3q_2^8 + 25650q_1^3q_2^9 - 910q_1^3q_2^{10} + O(q_1^4q_2^7) \\
C_{10} & = & q_1^2q_2^4 - 2q_1^2q_2^5 + q_1^2q_2^6 - 2q_1^3q_2^5 - 18q_1^3q_2^6 + 20q_1^3q_2^8 \\
 & & -18q_1^3q_2^9 - 2q_1^3q_2^{10} + O(q_1^4q_2^6) \\
s_{15} & = & q_1^2q_2^5 - q_1^3q_2^5 - 275q_1^3q_2^7 - 275q_1^3q_2^8 - q_1^3q_2^{10} + O(q_1^4q_2^7)
\end{array}$$

The ring of even symmetric Hilbert modular forms is a polynomial ring,

$$M_{ev}^+ = \mathbb{C}[A_2, B_6, C_{10}].$$

We construct the Hilbert modular forms A_2, s_5, and B_6 with lattices. The theta series of the 4-dimensional root lattice F_4 is A_2. Let P_3 be a zonal function of degree 3 of a minimal vector of F_4, which we introduced in the Chapter 5. The spherical theta series $\Theta_{F_4, \mathcal{N}P}$ is s_5 up to multiplication with a unit. We explain the constructions in more details in later sections of this chapter.

Let Λ_{24} be the extremal lattice of dimension 24. So the coefficients $\Theta_{\Lambda_{24}}$ at $q_1q_2^2$

and $q_1 q_2^3$ are zero, and hence $\Theta_{\Lambda_{24}} \neq A_2^3$. Hence B_6 is a linear combination of its theta series and the theta series of $F_4 \perp F_4 \perp F_4$.

The extremal Hilbert modular forms are listed in [Neb13].

6.2 Dimension 4

A $\sqrt{5}$-structure of the $\mathbb{E}_8$ lattice. We look at 8-dimensional $\mathbb{Z}$-lattices with a $\sqrt{5}$-structure. Compare also [CosHsi87], [Hsi89], and [Neb13]. Let $\Lambda_1 := \mathbb{E}_8$ be the first trace lattice of a 4-dimensional totally positive even unimodular lattice (Λ, Q) over $F := \mathbb{Q}[\sqrt{5}]$. There is one $\sqrt{5}$-structure. The lattice Λ_1 has the endomorphism $1 + \zeta_5 + \zeta_5^{-1}$, where $\zeta_5 \in \mathrm{Aut}(\Lambda_1)$ is a 5th root of unity (i.e. has minimal polynomial $x^4 + x^3 + x^2 + x + 1$). Hence Λ_1 is a $\mathbb{Z}[\sqrt{5}]$-lattice. There is only one conjugacy class of 5th roots of unity, hence $\mathbb{E}_8$ has exactly one $\sqrt{5}$-structure coming from a ζ_5-structure. Thus $\mathbb{E}_8$ is a trace lattice of a lattice over the 5th cyclotomic field. Compare also Section 10.1.

The second trace lattice Λ_2 is isomorphic to H_4 (also called $Q_8(1)$, see [ConSlo88]). The trace lattice Λ_2 is extremal 5-modular.

In fact, this is the only even unimodular $\sqrt{5}$-structure of the $\mathbb{E}_8$ lattice, because there is only one even unimodular lattices of dimension 4 over $\mathbb{Q}[\sqrt{5}]$. It is the root lattice F_4. See also [Sch94], [Hsi89], and Section 1.1.

The theta series of $\Lambda = F_4$ is the (extremal) Hilbert modular form A_2, i.e.

$$\Theta_\Lambda = 1 + 120 q_1 q_2^2 + 120 q_1 q_2^3 + 120 q_1^2 q_2^3 + 600 q_1^2 q_2^4 + 720 q_1^2 q_2^5 + 600 q_1^2 q_2^6 + 120 q_1^2 q_2^7 + O(q_1^3 q_2^5).$$

The 240 minimal points of $\Lambda_1 = \mathbb{E}_8$ split into two sets which have length 2 and 3, respectively, in the other trace lattice Λ_2. All minimal vectors of Λ_2 belong to the first set. Over $\mathbb{Q}[\sqrt{5}]$ these are the sets of points with length 1 and $\frac{3+\sqrt{5}}{2}$, respectively. We denote them as $\Lambda(1)$ and $\Lambda(\frac{3+\sqrt{5}}{2})$. The real embeddings σ_1 and σ_2 of $\mathbb{Q}[\sqrt{5}]$ into $\mathbb{R}$ yield Euclidean spaces on $\mathbb{R}^4$, where the inner product is the $\mathbb{R}$-linear extension of the bilinear form associated to $\sigma_j \circ Q$. A quick computer calculation (in $\mathbb{Q}[\sqrt{5}]$) shows that $\Lambda(1)$ and $\Lambda(\frac{3+\sqrt{5}}{2})$ fulfill the condition of Proposition 5.15 for $t \leq 11$ and hence $\sigma_1(\Lambda(1)), \sigma_2(\Lambda(1))$, $\sigma_1(\Lambda(\frac{3+\sqrt{5}}{2}))$ and $\sigma_2(\Lambda(\frac{3+\sqrt{5}}{2}))$ are all 11-designs. Actually, each set is isometric to the unique 120-point spherical $11-$design in 4 dimensions, which is described in [BoyDan01]. It has minimal possible points. Though it is not tight (a tight design would have 118 elements; for the definition of tight designs see [Ban79]).

Uniqueness of the $\sqrt{5}$-Structure. We want to show that the lattice is unique with our methods of the last chapter. Suppose that Λ is an even unimodular lattice of dimension 4 over $\mathbb{Q}[\sqrt{5}]$. Since $\dim M_2(\mathrm{SL}_2(\mathbb{Z}_F)) = 1$ it is extremal and the trace lattice Λ_1 must be isomorphic to $\mathbb{E}_8$. The theta series is A_2. Let $\alpha := 2$ and choose $\lambda \in \Lambda(\alpha/2)$. We will see that the configuration numbers are determined by properties of the modular forms.

Proposition 6.1 *The non-trivial configuration numbers for $\beta = 2$ are given in the following table.*

Table 6.2: Configuration numbers

ι	0	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	2
$n(2, \iota; \lambda)$	30	40	24	24	2

Proof. First we want to find the possible ι's such that the number $n(\beta, \iota; \lambda)$ may be positive. Lattice points $\mu \in \Lambda(1)$ with $B(\lambda, \mu) = \iota = \iota_0 + \iota_1\sqrt{5}$ are also lattice points of the trace lattices Λ_1 and Λ_2. These lattices are positive definite, and the Cauchy-Schwartz inequality yields

$$\begin{aligned} \mathfrak{b}_1(\lambda, \mu) &= \operatorname{tr}(\frac{5-\sqrt{5}}{10}\iota) = \iota_0 - \iota_1 \le \sqrt{\mathfrak{b}_1(\lambda,\lambda)\mathfrak{b}_1(\mu,\mu)} = 2, \\ \mathfrak{b}_2(\lambda, \mu) &= \operatorname{tr}(\iota) = 2\iota_0 \le \sqrt{\mathfrak{b}_2(\lambda,\lambda)\mathfrak{b}_2(\mu,\mu)} = \sqrt{2}. \end{aligned}$$

This condition gives only finitely many possible ι's. Using also the fact that $Q(\lambda \pm \mu) = \frac{\alpha+\beta}{2} \pm \iota$ must be 0 or totally positive, we find that

$$\iota \in \{0, \pm 1, \pm\tfrac{1+\sqrt{5}}{2}, \pm\tfrac{1-\sqrt{5}}{2}, \pm 2\}.$$

Since $n(\beta, \iota; \lambda) = n(\beta, -\iota; \lambda)$, we set $S := \{0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, 2\}$.

The 5 unknowns in S must be fund. Hence we need (at least) 5 equations. Some of the unknowns are trivial or fulfill trivial relations, which we summarized in Lemma 5.28. In this case we have only the trivial one

$$n(2, 2; \lambda) = 2.$$

Hence 4 unknowns must be found. Of cause we have the equation

$$\sum_{\iota \in S} n(\beta, \iota; \lambda) = |\Lambda(\beta/2)| = 120.$$

We get more equations from spherical theta series with spherical polynomials which come from Gegenbauer polynomials, cf. Section 5.5. In this case, for degree 1 and 2, we have $P_1(\mu) = B(\lambda, \mu) = \iota$ and $P_2(\mu) = B(\lambda, \mu)^2 - 1 = \iota^2 - 1$. The spherical theta series $\Theta_{\Lambda, \mathcal{N}P_1}$ and $\Theta_{\Lambda, \mathcal{N}P_2}$ are cusp forms of weight 3 and 4, respectively. Since $S_3 = \{0\}$ and $S_4 = \{0\}$, we have the two equations

$$\sum_{\iota \in S} n(\beta, \iota; \lambda)\mathcal{N}(\iota) = 0 \text{ and } \sum_{\iota \in S} n(\beta, \iota; \lambda)\mathcal{N}(\iota^2 - 1) = 0.$$

Hence we get 3 equations, one for each $d = 0, 1, 2$. The matrix

$$\Big(P_d(\iota)\Big)_{\substack{d=0,1,2\\ \iota\in S\setminus\{2\}}}$$

has full rank. Thus the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} n(2,0;\lambda) \\ n(2,1;\lambda) \\ n(2,\frac{1+\sqrt{5}}{2};\lambda) \\ n(2,\frac{1-\sqrt{5}}{2};\lambda) \end{pmatrix} = \begin{pmatrix} 118 \\ -8 \\ -18 \end{pmatrix}$$

has the solutions $(30, 40, 48-a, a)^{tr}$ with $0 \le a \le 48$. We also know the configuration numbers for $\beta' = 3 + \sqrt{5} = \varepsilon_0^2\beta$, because

$$n(\beta, \varepsilon; \lambda) = n(\varepsilon_0^2\beta, \varepsilon_0\iota; \lambda) \text{ (cf. Lemma 5.28(ii))},$$

where $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$. The first trace lattice has to be $\mathbb{E}_8$ and the configuration numbers for $\mathbb{E}_8$ are known. So if we look at $\lambda \in \Lambda_1 \cong \mathbb{E}_8$, there are exactly 112 vectors μ in $\mathbb{E}_8$ with norm 1 and scalar product ± 1 to λ. From the theta series of Λ we know that

$$\mu \in \Lambda_1(1) \;\Leftrightarrow\; \mu \in \Lambda(1) \text{ or } \mu \in \Lambda(3+\sqrt{5}).$$

So combining these, we conclude that

$$\mu \in \Lambda_1(1) \text{ and } B_1(\lambda,\mu) = \pm 1 \iff \begin{array}{l} \mu \in \Lambda(1) \text{ and } B(\lambda,\mu) \in \{\pm 1, \pm\frac{1-\sqrt{5}}{2}\} \text{ or} \\ \mu \in \Lambda(3+\sqrt{5}) \text{ and } B(\lambda,\mu) \in \{\pm\frac{1+\sqrt{5}}{2}, \pm 1\}. \end{array}$$

Therefore

$$112 = n(2,1;\lambda) + n(2,\tfrac{1+\sqrt{5}}{2};\lambda) + 2n(2,\tfrac{1-\sqrt{5}}{2};\lambda) = 88 + a \implies a = 24.$$

$\square$

Corollary 6.2 (The Hilbert Modular Form s_5)
Let P_3 be the homogenized Gegenbauer polynomial G_3^1 to a vector $\lambda \in \Lambda(1)$, i.e.

$$P_3(\mu) = G_1^3(\iota, \sqrt{\alpha\beta}) = 8\iota^3 - 8\iota\beta,$$

where $\iota = B(\lambda,\mu)$ and $\beta = 2Q(\mu)$. Then the anti-symmetric Hilbert modular form s_5 is

$$s_5 = \frac{1}{7680}\Theta_{\Lambda,\mathcal{N}P_3}.$$

Proof. The sum over all elements $\mu \in \Lambda(1)$ is

$$\sum_{\iota} \mathcal{N}P_3(\iota, \sqrt{\alpha\beta})\; n(2, \iota; \lambda) = 7680.$$

Hence $\Theta_{\Lambda,\mathcal{N}P_3}$ is not equal to zero. This is of cause possible, because the cusp forms space is

$$S_5(\mathrm{SL}_2(\mathbb{Z}_F)) = \langle s_5 \rangle,$$

where s_5 is the anti-symmetric generator of $M(\mathrm{SL}_2(\mathbb{Z}_F))$. Especially, we can compute s_5 as the (normalized) spherical theta series $\Theta_{\Lambda,\mathcal{N}P_3}$. □

It remains the question how many even unimodular lattices of degree 4, up to isometry, exist. Of cause, we know the answer. But we can proof it using only the configuration numbers.

Theorem 6.3 *There is up to isometry only one even unimodular lattice of degree 4 over $\mathbb{Q}[\sqrt{5}]$. Its trace lattices are $\mathbb{E}_8$ and H_4. The fundamental unit $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ of $\mathbb{Q}[\sqrt{5}]$ acts on $\mathbb{E}_8$ and H_4 as $1 + \zeta_5 + \zeta_5^{-1}$, where $\zeta_5 \in \mathrm{Aut}(\mathbb{E}_8)$ is the unique (up to conjugation) 5th root of unity.*

Proof. Let $\Lambda \subset \mathbb{Q}[\sqrt{5}]^4$ be an even unimodular lattice. Let $\lambda \in \Lambda(1)$. Above we have seen that there are 12 lattice points in the layer $\Lambda(1)$ which have scalar product $\frac{1+\sqrt{5}}{2}$ with λ. Let $\mu_1, \ldots, \mu_{12}$ denote these points. Then

$$B(\tfrac{1+\sqrt{5}}{2}\lambda - \mu_i, \tfrac{1+\sqrt{5}}{2}\lambda - \mu_i) = (\tfrac{1+\sqrt{5}}{2})^2 2 - 2(\tfrac{1+\sqrt{5}}{2})^2 + 2 = 2$$

and hence

$$\frac{1+\sqrt{5}}{2}\lambda - \mu_i \in \Lambda(1).$$

Also

$$B(\frac{1+\sqrt{5}}{2}\lambda - \mu_i, \lambda) = \frac{1+\sqrt{5}}{2}2 - \frac{1+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}$$

and hence

$$\frac{1+\sqrt{5}}{2}\lambda - \mu_i = \mu_j \text{ for some } j.$$

We renumber the μ_i's such that $\mu_{6+i} = \frac{1+\sqrt{5}}{2}\lambda - \mu_i$ for $7 \leq i \leq 12$. Let $i \neq j$, then $B(\mu_i, \mu_j) \in \{0, \pm 1, \pm\frac{1+\sqrt{5}}{2}, \pm\frac{1-\sqrt{5}}{2}\}$ and

$$B(\mu_i, \frac{1+\sqrt{5}}{2}\lambda - \mu_j) = \frac{3+\sqrt{5}}{2} - B(\mu_i, \mu_j) \in \{0, \pm 1, \pm\frac{1+\sqrt{5}}{2}, \pm\frac{1-\sqrt{5}}{2}\}.$$

That is only possible for $B(\mu_i,\mu_j)=1$ and hence $B(\mu_i,\frac{1+\sqrt{5}}{2}\lambda-\mu_j)=\frac{1+\sqrt{5}}{2}$ or vice versa. Without loss of generality we may assume that

$$B(\mu_1,\mu_j)=\begin{cases}2 & \text{if } j=1,\\ 1 & \text{if } j=2,\dots,6\\ \frac{-1+\sqrt{5}}{2} & \text{if } j=7,\\ \frac{1+\sqrt{5}}{2} & \text{if } j=8,\dots,12.\end{cases}$$

We will show that $B=(\lambda,\mu_1,\mu_2,\mu_3)$ is a basis of F^4. Then it has Gram matrix

$$G_1:=\begin{pmatrix}2 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2}\\ \frac{1+\sqrt{5}}{2} & 2 & 1 & 1\\ \frac{1+\sqrt{5}}{2} & 1 & 2 & 1\\ \frac{1+\sqrt{5}}{2} & 1 & 1 & 2\end{pmatrix} \text{ or } G_2:=\begin{pmatrix}2 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2}\\ \frac{1+\sqrt{5}}{2} & 2 & 1 & 1\\ \frac{1+\sqrt{5}}{2} & 1 & 2 & \frac{1+\sqrt{5}}{2}\\ \frac{1+\sqrt{5}}{2} & 1 & \frac{1+\sqrt{5}}{2} & 2\end{pmatrix}.$$

Since $\det G_1, \det G_2\in\mathbb{Z}_F^*$, B is actually a lattice basis in both cases. Since $TG_2T^{\mathrm{tr}}=G_1$, where

$$T=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \frac{3+\sqrt{5}}{2} & -1 & \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\end{pmatrix}\in\mathrm{GL}_4(\mathbb{Q}[\sqrt{5}]),$$

the lattices are isometric. Hence all lattices are isometric, as desired. $\square$

Chapter 7

Extremal Lattices over $\mathbb{Q}[\sqrt{2}]$

The fundamental unit $\varepsilon_0 = 1 + \sqrt{2}$ of $F = \mathbb{Q}[\sqrt{2}]$ has norm -1. So we consider lattices of Type (i). We will construct and classify (extremal) Type (i) lattices over F.

Let (Λ, Q) be a Type (i) lattice. Then the first trace lattice (Λ_1, Q_1) is the trace lattice with respect to $\alpha_1 = \frac{2-\sqrt{2}}{4}$. It is even unimodular. Hence the dimension of (Λ, Q) is a multiple of 4. The second trace lattice (Λ_2, Q_2) is the trace lattice with respect to $\alpha_2 = \frac{1}{2}$. It is even 2-modular.

The roots of (Λ, Q) are vectors $\lambda \in \Lambda$ with $Q(\lambda) = \varepsilon_0^{2k}$, where $k \in \mathbb{Z}$. The reduced roots are the vectors of norm 1. The trace norms of reduced roots are 1 in both Λ_1 and Λ_2.

Let $A = (\alpha_1, \alpha_2)$. The totally positive elements ordered by $\leq_A$ are

$$1 \leq_A 2+\sqrt{2} \leq_A 3+2\sqrt{2} \leq_A 2 \leq_A 4+2\sqrt{2} \leq_A 5+3\sqrt{2} \leq_A 6+4\sqrt{2} \leq_A \ldots$$

7.1 Extremal Lattices

The following extremal lattices over $\mathbb{Q}[\sqrt{2}]$ are known.

In dimension 4, only one isometry class exists, c.f. [Tak85]. The lattice in this class is trivially extremal. It is the root lattice Δ_4', see Section 2.3.

All unimodular even lattices in dimension 8 were classified in [Tak85], [Hsi89], and [HsiHun89]. There are 6 isometry classes, and among them one is extremal.

I constructed additionally extremal lattices of dimension 12, 16, 20, and 24. See the sections of this chapter for more details.

The following table lists the known extremal lattices. The first column gives the rank n of the lattice (over $\mathbb{Q}[\sqrt{2}]$), the second gives the rank N of the trace lattices (over $\mathbb{Q}$), the third gives information about the trace lattices, and the forth gives the number or a lower bound of the number of extremal lattices.

Table 7.1: Extremal lattices over $\mathbb{Q}[\sqrt{2}]$

n	$N = 2n$	trace lattices	# extr. lat.
4	8	$(E_8, F_4 \perp F_4)$	1
8	16	$(E_8 \perp E_8, BW_{16})$	1
12	24	$\Lambda_1 = \Lambda_{24}$	5
16	32	$\Lambda_2 = Q_{32}$	≥ 1
20	40		≥ 1
24	48	$\Lambda_1 = P_{48n}$	≥ 1

Extremal Modular Forms The graduated algebra of Hilbert modular forms is

$$M = M(\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])) = \mathbb{C}[g_2, s_4, s_5, s_6, s_9].$$

The generators are the (symmetric) Eisenstein series g_2, the symmetric cusp forms s_2, s_6, and s_9 and the anti-symmetric cusp form s_5. There are relations $s_5^2 = s_4 s_6$ and $s_9^2 \in \mathbb{C}[g_2, s_4, s_6]$. As usual the weights are given by the index. The generators have the following q-expansions (see Theorem 3.28):

$$\begin{aligned}
g_2 &= 1 + 48q_1q_2 + 144q_1q_2^2 + 48q_1q_2^3 + 336q_1^2q_2^2 + 720q_1^2q_2^4 + 384q_1^2q_2^5 + 336q_1^2q_2^6 \\
&\quad +144q_1^3q_2^2 + 480q_1^3q_2^3 + 1152q_1^3q_2^4 + 864q_1^3q_2^5 + 1440q_1^3q_2^6 + 864q_1^3q_2^7 \\
&\quad +1152q_1^3q_2^8 + 480q_1^3q_2^9 + 144q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_4 &= q_1q_2 - 2q_1q_2^2 + q_1q_2^3 - 4q_1^2q_2^2 - 8q_1^2q_2^3 + 24q_1^2q_2^4 - 8q_1^2q_2^5 - 4q_1^2q_2^6 - 2q_1^3q_2^2 \\
&\quad +26q_1^3q_2^3 + 16q_1^3q_2^4 - 14q_1^3q_2^5 - 52q_1^3q_2^6 - 14q_1^3q_2^7 + 16q_1^3q_2^8 + 26q_1^3q_2^9 \\
&\quad -2q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_5 &= q_1q_2 - q_1q_2^3 + 16q_1^2q_2^2 - 56q_1^2q_2^3 + 56q_1^2q_2^5 - 16q_1^2q_2^6 \\
&\quad -42q_1^3q_2^3 + 378q_1^3q_2^5 - 378q_1^3q_2^7 + 42q_1^3q_2^9 + O(q_1^4q_2^3) \\
s_6 &= q_1q_2^2 - 2q_1^2q_2^2 - 16q_1^2q_2^3 + 12q_1^2q_2^4 - 16q_1^2q_2^5 - 2q_1^2q_2^6 + q_1^3q_2^2 + 32q_1^3q_2^3 + 40q_1^3q_2^4 \\
&\quad -32q_1^3q_2^5 + 170q_1^3q_2^6 - 32q_1^3q_2^7 + 40q_1^3q_2^8 + 32q_1^3q_2^9 + q_1^3q_2^{10} + O(q_1^4q_2^3) \\
s_9 &= q_1q_2^2 - 96q_1^2q_2^3 - 336q_1^2q_2^4 - 96q_1^2q_2^5 + O(q_1^3q_2^2)
\end{aligned}$$

The ring of even symmetric Hilbert modular forms is a polynomial ring,

$$M_{ev}^+ = \mathbb{C}[g_2, s_4, s_6].$$

The theta series of the 4-dimensional root lattice Δ_4' is g_2. Let Λ_8 be the extremal lattice in dimension 8. So the coefficient Θ_{Λ_8} at $q_1^1q_2^1$ is zero, and hence $\Theta_{\Lambda_8} \neq g_2^2$. Hence s_4 is a linear combination of Θ_{Λ_8} and the theta series of $\Delta_4' \perp \Delta_4'$. Analogously we construct s_6 with the theta series of an extremal lattice of dimension 12. Let P_3 be a zonal function of degree 3 of a minimal vector of Δ_4', which we introduced in the

Chapter 5. The spherical theta series $\Theta_{\Lambda,\mathcal{N}P_3}$ is s_5. We explain the constructions in more details in later sections of this chapter. We get the following theorem.

Theorem 7.1 (Generators)
The Hilbert modular forms g_2, s_4, s_5, and s_6 can be constructed with spherical theta series.

The q-expansions of the first extremal Hilbert modular forms are listed in Appendix C. Table 7.2 gives details of the extremal Hilbert modular forms of weight 2 to 20. The first column lists the rank n of the (possible) extremal lattice Λ, and the second column gives the weight k of the extremal Hilbert modular form. The valuation of the extremal Hilbert modular form of weight k is given in the third column. The last columns give the minima and kissing numbers of the trace lattices Λ_1 and Λ_2 of an extremal lattice Λ (if there is such a lattice). Its theta series would be the extremal modular form. More precisely, let f_k be the extremal Hilbert modular form of weight k. Then $f(q_1, 1) = 1 + 0q_1^1 + \cdots + 0q_1^{n_0-1} + a_{n_0}q_1^{n_0}$ for some $n_0 \in \mathbb{N}$. The hypothetical first trace lattice has minimum n_0 and kissing number a_{n_0}. Then the kissing number $\#\mathrm{Min}\Lambda_1$ is $a_{n_0} = \sum_{m\geq 0} a_{n,m}(f_k)$.

Table 7.2: Extremal Hilbert modular forms and lattices

n	k	valuation	$\min \Lambda_1$	$\#$Min Λ_1	$\min \Lambda_2$	$\#$Min Λ_2
4	2	$[1,1]$	1	240	1	48
8	4	$[1,2]$	1	480	2	4320
12	6	$[2,2]$	2	196560	2	3024
16	8	$[2,3]$	2	146880	3	261120
20	10	$[2,4]$	2	39600	3	84480
24	12	$[3,4]$	3	52416000	4	9828000
28	14	$[3,5]$	3	15590400	4	2232720
32	16	$[3,6]$	3	2611200	5	310210560
36	18	$[4,6]$	4	6218175600	5	57915648
40	20	$[4,7]$	4	1250172000	6	9092160000

Corollary 7.2 *Let Λ be an extremal lattice of dimension $n \leq 40$. From Table 7.2 follows directly that the first trace lattice Λ_1 is extremal unimodular and the second trace lattice Λ_2 is extremal 2-modular.*

Proof. We see that $\min \Lambda_1 = 1 + \lfloor \frac{2n}{24} \rfloor$ and $\min \Lambda_1 = 1 + \lfloor \frac{2n}{16} \rfloor$ is true for all $n \leq 20$. Lattices achieving this bounds are called extremal (cf. [Que95]). Hence Λ_1 and Λ_2 are extremal unimodular or 2-modular, respectively. $\square$

7.2 Dimension 4

Takada showed in [Tak85] that there is exactly one isometry class of even unimodular lattices of dimension 4 over $\mathbb{Q}[\sqrt{2}]$. See also [HsiHun89], [Sch94], or [Kir16]. We will give a different proof with our methods.

Assume that we have an extremal lattice (Λ, Q) of degree 4 over $F = \mathbb{Q}[\sqrt{2}]$. Then its theta series is the Eisenstein series g_2 of weight 2,

$$\Theta_\Lambda = g_2 = 1 + 48q_1q_2 + 144q_1q_2^2 + 48q_1q_2^3 + O(q_1^2q_2^2).$$

The first trace lattice Λ_1 is the $\mathbb{E}_8$ lattice, because Λ_1 is even unimodular of dimension 8.

Construction We construct a $\mathbb{Z}_F$-lattice with a root of unity automorphism of the $\mathbb{Z}$-lattice $\mathbb{E}_8$, compare also Section 10.1. The automorphism group of $\mathbb{E}_8$ has one conjugacy class of $8th$ roots of unity, i.e. elements with minimal polynomial $x^4 + 1$. If $\zeta_8 \in \mathrm{Aut}(\mathbb{E}_8)$ is such a root of unity, then $\zeta_8 + \zeta_8^{-1}$ is an endomorphism of $\mathbb{E}_8$ which has the minimal polynomial $x^2 - 2$. So the endomorphism $\zeta_8 + \zeta_8^{-1}$ defines a $\mathbb{Z}[\sqrt{2}]$-structure Λ on $\mathbb{E}_8$. This means that Λ is an even unimodular lattice of dimension 4 over $\mathbb{Q}[\sqrt{2}]$, the first trace lattice is $\mathbb{E}_8$, and multiplication with $\sqrt{2}$ is given by $\zeta_8 + \zeta_8^{-1}$ on $\mathbb{E}_8$. Actually, Λ is the root lattice Δ_4', see Example 2.11.

Uniqueness We want to find all even unimodular lattices of rank 4 over $\mathbb{Q}[\sqrt{2}]$. Assume that (Λ, Q) is such a lattice. We follow the ideas of Section 5.5. We fix a lattice point of Λ, say $\lambda \in \Lambda(1)$, and seek to find the numbers

$$n(\beta, \iota; \lambda) = |\{\mu \in \Lambda(\beta/2) \mid B(\lambda, \mu) = \pm\iota\}|$$

for fixed β and all possible ι's.

Proposition 7.3 (Configuration Numbers)
The non-zero configuration numbers for $\beta = 2$ are given in the following table.

Table 7.3: Configuration numbers

ι	0	1	$\sqrt{2}$	2
$n(2, \iota; \lambda)$	18	16	12	2

Proof. If $\mu \in \Lambda(1)$ then the scalar product $\iota = B(\lambda, \mu)$ is one of the following,

$$\iota \in \{0, \pm 1, \pm\sqrt{2}, \pm 2\}.$$

Since $\iota = \pm 2$ if and only if $\mu = \pm\lambda$, we have to compute the configuration numbers $n(2, \iota; \lambda)$ for $\iota \in S := \{0, 1, \sqrt{2}\}$. We know the theta series of Λ, so

$$\sum_{\iota \in S \cup \{2\}} n(2, \iota; \lambda) = 48 = |\Lambda(1)|.$$

A harmonic polynomial of degree one is $P_1(\mu) = B(\lambda, \mu) = \iota$. Since $S_3(\mathrm{SL}_2(\mathbb{Z}_F)) = \{0\}$, we have $\Theta_{\Lambda, \mathcal{N}P_1} = 0$. Hence we have the equation

$$\sum_{\iota} n(2, \iota; \lambda)\mathcal{N}(\iota) = 0.$$

The minimal vectors of the second trace lattice are $\Lambda_2(1) = \Lambda(1)$, which one can see from Θ_Λ. The second trace lattice is 2-modular of minimum 1 and dimension $N = 8$, hence it is extremal. According to Example 5.17 the minimal vectors of an extremal 2-modular lattice of dimension $N \equiv 8 \pmod{16}$ form a spherical 3-design. The homogenized Gegenbauer polynomial of degree 2 and parameter $\frac{8}{2} - 1 = 3$ is $G_2^3(t, s) = 24t^2 - 3s^2$. We have $t = \mathrm{tr}(\iota/2)$ and $s = \sqrt{4}$. We normalize and get the equation

$$\sum_{\iota} n(2, \iota; \lambda)(2\,\mathrm{tr}(\tfrac{\iota}{2})^2 - 1) = 0.$$

The linear equation system is

$$\left(\begin{array}{rrrr|r} 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 48 \\ 0 & 1 & -2 & 4 & 0 \\ -1 & 1 & -1 & 7 & 0 \end{array}\right)$$

The unique solution is $(18, 16, 12, 2)^{tr}$. □

Corollary 7.4 (The Hilbert Modular Form s_5)
Let $\lambda \in \Lambda(1)$ be any vector. Let P_3^λ the harmonic polynomial of degree 3 zonal to λ, defined in Definition 5.10. So for $\mu \in \Lambda$, we have

$$P_3(\mu) = G_3^1(\iota, \sqrt{2\beta}) = 8\iota^3 - 8\iota\beta,$$

where $\iota = B(\lambda, \mu)$, $\beta = 2Q(\mu)$, and $G_3^1(t, s) = 8t^3 - 4ts^2$ is the homogenized Gegenbauer polynomial of parameter 1 and degree 3.

The anti-symmetric Hilbert modular form s_5 is

$$s_5 = \frac{1}{3072}\Theta_{\Lambda, \mathcal{N}P_3}.$$

Proof. The sum over all elements $\mu \in \Lambda(1)$, i.e. $\beta = 2$, is

$$\sum_{\iota} \mathcal{N}(8\iota^3 - 16\iota)\; n(2, \iota; \lambda) = 3072.$$

Hence $\Theta_{\Lambda, \mathcal{N}P_3}$ is not equal to zero. Since $S_5(\mathrm{SL}_2(\mathbb{Z}_F)) = \langle s_5 \rangle$, this is possible and s_5 is equal to the (normalization of the) spherical theta series $\Theta_{\Lambda, \mathcal{N}P_3}$. □

Corollary 7.5 (Spherical Designs)
The set $\Lambda(1)$ *is a spherical* 7*-design in* $\mathbb{R}^4$*.*

Proof. We identify $\Lambda(1)$ with $\Lambda(1)^{(1)} = \sigma_1(\Lambda(1)) \subseteq \mathbb{R}^4$ or $\Lambda^{(2)} \subseteq \mathbb{R}^4$. The inner product is the $\mathbb{R}$-linear extension of $\sigma_1 \circ B$ or $\sigma_2 \circ B$, where B is the bilinear form of Λ.

Using Gegenbauer polynomials we may check if $\Lambda(1)$ is a spherical t-designs, see Proposition 5.30. We check if

$$\sum_{\iota} n(\beta, \iota; \lambda) G_d^1(\iota, 2) = 0,$$

where G_d^1 is the homogenized Gegenbauer polynomial of degree d and parameter $4/2 - 1 = 1$. This equation is true for homogenized Gegenbauer polynomials up to degree 7 (the odd degrees are trivial) and is not true for degree 8. Hence $\Lambda(1)$ is a spherical 7-design.

□

Theorem 7.6 (Uniqueness of the $\sqrt{2}$-structure of $\mathbb{E}_8$) *There is up to isometry only one even unimodular lattice of degree* 4 *over* $\mathbb{Q}[\sqrt{2}]$*. Its trace lattices are* $\mathbb{E}_8$ *and* $F_4 \perp F_4$*. The fundamental unit* $\varepsilon_0 = 1 + \sqrt{2}$ *of* $\mathbb{Q}[\sqrt{2}]$ *acts on* $\mathbb{E}_8$ *and* F_4 *as* $\zeta_8 + \zeta_8^{-1}$*, where* $\zeta_8 \in \mathrm{Aut}(\mathbb{E}_8)$ *is the unique automorphism (up to conjugation) with minimal polynomial* $x^4 + 1$*.*

Proof. Let $\Lambda \subset \mathbb{Q}[\sqrt{2}]^4$ be an even unimodular lattice. Let $\lambda \in \Lambda(1)$. Above we have seen that there are 8 lattice vectors in the layer $\Lambda(1)$ which have scalar product 1 with λ. If μ is such a vector, one checks that also $\lambda - \mu$ is. If μ' is another of these vectors, the scalar products must fulfill

$$B(\mu, \mu'), B(\mu, \lambda - \mu') \in \{0, \pm 1, \pm\sqrt{2}, \pm 2\}.$$

Hence there are only the cases $B(\mu, \mu') = 0$ if and only if $B(\mu, \lambda - \mu') = 1$, and vice versa, and the trivial cases $B(\mu, \mu') = 2$ if and only if $\mu = \mu'$, and $B(\mu, \mu') = -1$ if and only if $\mu = \lambda - \mu'$. We order the 8 vectors $\mu_1, \ldots, \mu_8$ such that $\mu_{4+i} = \lambda - \mu_i$ for $i = 1, 2, 3, 4$ and $B(\mu_1, \mu_i) = 0$ for $i = 2, 3, 4$.

The Gram matrix of $(\lambda, \mu_1, \mu_2, \mu_3)$ is

$$G_1 := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \text{ or } G_2 := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Both matrices are non-singular and could thus be Gram matrices of sublattices of Λ. Denote by M a lattice with Gram matrix G_1. The determinant is 4, so M may have an even unimodular overlattice Λ. A computer calculation shows that there is exactly one even unimodular overlattice, up to isometry. The determinant of G_5 is 5. Since $5\mathbb{Z}_F$ is a prime ideal, a lattice with Gram matrix G_5 cannot have unimodular overlattices.

In conclusion, all even unimodular lattices of dimension 4 are isomorphic to the overlattice of G_1, which is the root lattice Δ_4'. □

7.3 Dimension 8

There are 6 isometry classes in dimension 8; they were constructed in [Tak85], [Hsi89], and [HsiHun89]. One of them is extremal. Hsia and Hung used Kneser's neighboring method and Siegel's mass formula in [HsiHun89] to classify the lattices. We will proof the existence and uniqueness of the extremal lattice with our methods.

Construction The automorphism group of $\mathbb{E}_8 \perp \mathbb{E}_8$ has two conjugacy classes of $8th$ roots of unity (i.e. elements with minimal polynomial x^4+1). The elements of one of them act diagonal on the two copies of $\mathbb{E}_8$. Denote a representative of the other one with ζ_8. The endomorphism $\zeta_8 + \zeta_8^{-1}$ has minimal polynomial x^2-2. Hence it defines an even unimodular $\sqrt{2}$-structure on $\mathbb{E}_8 \perp \mathbb{E}_8$. That means that there is an even unimodular lattice Λ over $\mathbb{Q}[\sqrt{2}]$ of dimension 8 such that $\mathbb{E}_8 \perp \mathbb{E}_8$ is the first trace lattice. Multiplication by $\sqrt{2}$ on Λ is applying of $\zeta_8 + \zeta_8^{-1}$ on $\mathbb{E}_8 \perp \mathbb{E}_8$. We find that Λ has no roots, i.e. lattice points of norm 2. Since the space of Hilbert modular forms of weight 4 is 2-dimensional, Λ is extremal.

The Gram matrix of Λ is given in Appendix C.

Uniqueness Assume that we have an extremal lattice (Λ, Q) of degree 8. Then its theta series is the extremal Hilbert modular form of weight 4,

$$\begin{aligned}\Theta_\Lambda = 1 &+ 480q_1q_2^2 + 3360q_1^2q_2^2 + 15360q_1^2q_2^3 + 24480q_1^2q_2^4 + 15360q_1^2q_2^5 + 3360q_1^2q_2^6 \\ &+ 480q_1^3q_2^2 + O(q_1^3q_2^3).\end{aligned}$$

The first trace lattice Λ_1 is extremal unimodular. Hence it is one of the two even extremal lattices of dimension 18, i.e. $\Lambda_1 = E_8 \perp E_8$ or $\Lambda_1 = D_{16}^+$. All minimal vectors of the first trace lattice are minimal over the number field. They form a 3-design with respect to the first trace form.

The second trace lattice is 2-modular with minimum 2. It is the Barnes Wall lattice, because there is only one isometry class of extremal 2-modular lattices, cf. [SchVen94].

Again, we fix a minimal lattice point $\lambda \in \Lambda$. Then λ has square length $\alpha = 2(2+\sqrt{2})$. For $\beta = \alpha$ we will determine the numbers

$$n(\beta, \iota; \lambda) = |\{\mu \in \Lambda(\beta/2) \mid B(\lambda, \mu) = \pm\iota\}|.$$

Proposition 7.7 (Configuration Numbers of Extremal Lattices)
The non-zero configuration numbers for $\beta = 2(2+\sqrt{2})$ are given in the following table.

Table 7.4: Configuration numbers

ι	0	$1+\sqrt{2}$	$2+\sqrt{2}$	$2+2\sqrt{2}$	$4+2\sqrt{2}$
$n(\beta,\iota;\lambda)$	210	128	112	28	2

Proof. If $\mu \in \Lambda(2+\sqrt{2})$, then the scalar product $\iota = B(\lambda,\mu)$ is one of the following,

$$\iota \in \{0, \pm 1, \pm(1+\sqrt{2}), \pm(2+\sqrt{2}), \pm(2+2\sqrt{2}), \pm(3+2\sqrt{2}), \pm(4+2\sqrt{2})\}.$$

Of cause, $\iota = \pm\alpha$ is only possible if $\mu = \pm\lambda$. Suppose that $\iota = 1$. Then $\lambda - \mu$ would have norm $2+\sqrt{2}-1+2+\sqrt{2} = 3+2\sqrt{2}$. That is not possible because the coefficient of Θ_Λ at $q_1 q_2^3$ is zero ($\mathrm{tr}(\frac{2-\sqrt{2}}{4}\cdot(3+2\sqrt{2})) = 1$ and $\mathrm{tr}(\frac{1}{2}\cdot(3+2\sqrt{2})) = 3$). By the same argument, $\iota = \pm(3+2\sqrt{2})$ is not possible.

Therefore we have to compute the configuration numbers $n(\beta,\iota;\lambda)$ for

$$\iota \in S := \{0, 1+\sqrt{2}, 2+\sqrt{2}, 2+2\sqrt{2}\}$$

We know the theta series of Λ, so we have the equation

$$\sum_{\iota \in S \cup \{\alpha\}} n(\beta,\iota;\lambda) = 480 = |\Lambda(\beta)|. \tag{1}$$

A harmonic polynomial of degree 1 is $P_1(\mu) = B(\lambda,\mu) = \iota$. The space of Hilbert modular forms of degree 5 is $S_5 = \langle s_5 \rangle$. Since $s_5 = q_1 q_2 - q_1 q_2^3 + O(q_1^2 q_2^2)$ and the coefficient of Θ_{Λ,P_1} at $q_1 q_2$ is zero, the spherical theta series Θ_{Λ,P_1} cannot be a multiple of s_5. Hence we have $\Theta_{\Lambda,\mathcal{N}P_1} = 0$ and the equation

$$\sum_{\iota \in S \cup \{\alpha\}} n(\beta,\iota;\lambda)\mathcal{N}(\iota) = 0. \tag{2}$$

Let P_3^λ be harmonic polynomial of degree 3 zonal to λ, see Definition 5.10. It is the Gegenbauer polynomial G_3^3 homogenized with $4+2\sqrt{2}$, i.e.

$$P_3(\mu) = G_3^3(\iota, 4+2\sqrt{2}) = 80\iota^3 - 24\iota \cdot (4+2\sqrt{2}).$$

Since $S_7 = \langle g_2 s_5 \rangle$ and $g_2 s_5 = q_1 q_2 - q_1 q_2^3 + O(q_1^2 q_2^2)$, the spherical theta series $\Theta_{\Lambda,P}$ has to be zero. So

$$\sum_{\iota \in S \cup \{\alpha\}} n(\beta, \iota; \lambda) \mathcal{N}(G_3^3(\iota, 4+2\sqrt{2})) = 0. \tag{3}$$

That does not work for spherical polynomials of degree 2, because $S_6 = \langle g_2 s_4, s_6 \rangle$ and s_6 starts with $q_1 q_2^2$. Hence $\Theta_{\lambda,P_2} = c s_6$, where $c \in \mathbb{C}$ is a constant, which is (likely) nonzero.

From the theta series we see that $\Lambda(\beta)$ is the set of minimal vectors of the first trace lattice. Since the first trace lattice is unimodular, $\Lambda(\beta)$ forms a spherical 3-design over $\mathbb{Z}$ with respect to the first trace form. If $G_2^7(t,s)$ is the homogenized Gegenbauer polynomial of degree 2 and parameter 7, then

$$\sum_{\iota \in S \cup \{\alpha\}} n(\beta, \iota; \lambda) G_2^7(\mathrm{tr}(\tfrac{2-\sqrt{2}}{4}\iota), 2) = 0. \tag{4}$$

The linear equation system containing the equations (1) to (4) determines the 4 unknowns uniquely. □

Corollary 7.8 (Spherical Designs) *The set $\Lambda(2+\sqrt{2})$ is a spherical 7-design.*

Proof. So we check if

$$\sum_{\iota} n(\beta, \iota; \lambda) G_d^3(\iota, 2) = 0,$$

where G_d^3 is the homogenized Gegenbauer polynomial of degree d and parameter 3. This equation is true for homogenized Gegenbauer polynomials up to degree 7 and is not true for degree 8. Hence $\Lambda(2+\sqrt{2}) \subset \mathbb{R}^8$ is a spherical 7-design.

□

Theorem 7.9 (Uniqueness of the Extremal 8-dim. Lattice over $\mathbb{Q}[\sqrt{2}]$) *There is up to isometry only one extremal even unimodular lattice of degree 8 over $\mathbb{Q}[\sqrt{2}]$. Its trace lattices are $\mathbb{E}_8 \perp \mathbb{E}_8$ and the Barnes-Wall lattice BW_{16}.*

Proof. Let Λ be an extremal lattice of dimension 8. Let $\lambda \in \Lambda(2+\sqrt{2})$. Above we have seen that there are 14 lattice vectors in the layer $\Lambda(2+\sqrt{2})$ which have scalar product $2+2\sqrt{2}$ with λ. If μ is such a vector, one checks that also $\sqrt{2}\lambda - \mu$

is such a vector. For μ' another of these vectors, the scalar products $B(\mu,\mu')$ and $B(\mu,\sqrt{2}\lambda-\mu')$ must be in

$$\{0,\pm(1+\sqrt{2}),\pm(2+\sqrt{2}),\pm(2+2\sqrt{2}),\pm(3+2\sqrt{2}),\pm(4+2\sqrt{2})\}.$$

That is possible only for $B(\mu,\mu')=2+\sqrt{2}$, $B(\mu,\mu')=2$ (if and only if $\mu=\mu'$), and $B(\mu,\mu')=-1$ (if and only if $\mu=\lambda-\mu'$).

Therefore we can order the 14 vectors as $\mu_1,\dots,\mu_{14}$ such that $\mu_{7+i}=\lambda-\mu_i$ and $B(\mu_i,\mu_j)=2+\sqrt{2}$, where $1\le i,j\le 7$ and $i\ne j$.

Therefore the Gram matrix of $(\lambda,\mu_1,\dots,\mu_7)$ has $4+2\sqrt{2}$ on the diagonal, $2+2\sqrt{2}$ on the first row and column (except the first entry), and $2+\sqrt{2}$ everywhere else,

$$G:=\begin{pmatrix} 4+2\sqrt{2} & 2+2\sqrt{2} & 2+2\sqrt{2} & \dots & 2+2\sqrt{2}\\ 2+2\sqrt{2} & 4+2\sqrt{2} & 2+\sqrt{2} & \dots & 2+\sqrt{2}\\ 2+2\sqrt{2} & 2+\sqrt{2} & 4+2\sqrt{2} & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 2+\sqrt{2}\\ 2+2\sqrt{2} & 2+\sqrt{2} & \dots & 2+\sqrt{2} & 4+2\sqrt{2}\end{pmatrix}.$$

It has determinant $18464+13056\sqrt{2}=\sqrt{2}^{10}\cdot(1+\sqrt{2})^8$. Let M be the lattice with Gram matrix G. A computer calculation shows that M has two even unimodular overlattices, up to isometry. One of them is extremal, i.e. has no lattice vectors of norm 1. Therefore every extremal even unimodular lattice is isometric to that lattice. □

7.4 Dimension 12

The mass of the genus of even unimodular lattices of dimension 12 is around 8.5. Kneser's neighboring method at the prime $\sqrt{2}\mathbb{Z}_F$ provides all 577 isometry classes. Among them 99 classes contain lattices without roots, i.e. the norm of any element is not in $\mathbb{Z}_F^*$. And 5 of them are extremal, i.e. the norm of any vector is neither in $\mathbb{Z}_F^*$ nor in $(2+\sqrt{2})\mathbb{Z}_F^*$. We give their Gram matrices in Appendix C.

For every extremal lattice, the first trace lattice is the Leech lattice. That condition is necessary, but it is not sufficient for $\sqrt{2}$. The second trace lattices are extremal 2-modular. They are pairwise non-isometric.

The minimal vectors of every extremal lattice form a spherical 7-design (when they are embedded into $\mathbb{R}^{12}$).

We cannot determine the configuration numbers with our methods.

7.5 Higher Dimensions

The classification of extremal lattices in dimensions higher than 12 is not complete. We list further extremal lattices in dimensions 16, 20, and 24. The existence or non-existence of extremal lattices in any other dimension is not known.

Dimension 16 The extremal Hilbert modular form of weight 8 is

$$\begin{aligned} f_8 &= 1 + 34560q_1^2q_2^3 + 77760q_1^2q_2^4 + 34560q_1^2q_2^5 + 192000q_1^3q_2^3 + 4147200q_1^3q_2^4 \\ &\quad +15966720q_1^3q_2^5 + 24145920q_1^3q_2^6 + 15966720q_1^3q_2^7 + 4147200q_1^3q_2^8 \\ &\quad +192000q_1^3q_2^9 + O(q_1^4q_2^3) \end{aligned}$$

Let L_2 be the 32-dimension 2-modular extremal lattice Q_{32}, which was found by Quebbemann in [Que87a]. We use the root of unity automorphism method, see Section 10.1, to construct an extremal lattice of dimension 16. There are several conjugacy classes of $8th$ roots of unity, i.e. elements with minimal polynomial x^4+1. Let G_2 be the Gram matrix of L_2. Exactly one of the $8th$ roots of unity ζ_8 yield an even unimodular lattice L_1 given by the Gram matrix

$$G_1 = G_2\frac{2+\zeta_8+\zeta_8^{-1}}{2}.$$

(Here $\frac{2+\zeta_8+\zeta_8^{-1}}{2}$ corresponds to $\frac{2-\sqrt{2}}{2}$.)

Since L_1 and L_2 are even, the $\sqrt{2}$-structure given by $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$ is an even unimodular lattice over $\mathbb{Q}[\sqrt{2}]$. Call it Λ. The first and second trace lattices of Λ are $\Lambda_1 = L_1$ and $\Lambda_2 = L_2$.

The lattice L_1 has minimum 2, hence it is extremal. Therefore Λ is an extremal lattice of dimension 16. So $\Theta_\Lambda = f_8$. The theta series is also given by the merged theta series of L_1 and L_2, i.e.

$$\Theta_\Lambda = \sum_{n,m\geq 0} a_{n,m}q_1^nq_2^m,$$

where the coefficients are $a_{n,m} = |\{\lambda \in \mathbb{Q}^{32} \mid \lambda^{\mathrm{tr}}G_1\lambda = 2n \text{ and } \lambda^{\mathrm{tr}}G_2\lambda = 2m\}|$. The Gram matrices G_2 and $\frac{2+\zeta_8+\zeta_8^{-1}}{2}G_2$ are given in Appendix C.

Dimension 20 The extremal Hilbert modular form of weight 10 is

$$\begin{aligned} f_{10} &= 1 + 39600q_1^2q_2^4 + 84480q_1^3q_2^3 + 3928320q_1^3q_2^4 + 21542400q_1^3q_2^5 \\ &\quad +36748800q_1^3q_2^6 + 21542400q_1^3q_2^7 + 3928320q_1^3q_2^8 + 84480q_1^3q_2^9 + O(q_1^4q_2^4). \end{aligned}$$

Let L_2 be the extremal 2-modular lattice of dimension 40 constructed by Bachoc in [Bac97]. It is a 10-dimensional unimodular lattices over the Hurwitz order. Let G_2 be its Gram matrix.

There is a $\sqrt{2}$-structure $\nu \in \mathrm{End}(L_2)$ which defines an even unimodular lattice L_1 given by the Gram matrix $G_1 = G_2 \frac{2-\nu}{2}$ such that the merged theta series is f_{10}.

The endomorphism ν is also a similarity between $L_2^{\#}$ and L_2 of norm 2. It is up to conjugation over $\mathrm{Aut}(L_2)$ the only similarity which defines an extremal $\sqrt{2}$-structure. See Section 10.2 for more details and a proof. The Gram matrices G_1 and G_2 are given in Appendix C.

Dimension 24 The extremal Hilbert modular form of weight 12 is

$$\begin{aligned} f_{12} = \quad & 1 + 1572480 q_1^3 q_2^4 + 12579840 q_1^3 q_2^5 + 24111360 q_1^3 q_2^6 + 12579840 q_1^3 q_2^7 \\ & + 1572480 q_1^3 q_2^8 + O(q_1^4 q_2^4) \end{aligned}$$

Let $L_1 = P_{48n}$ be the 48-dimension unimodular extremal lattice which was found by Nebe in [Neb98b]. Let G_1 be its Gram matrix. There are 6 conjugacy classes of $8th$ roots of unity in the automorphism group. Let ζ_8 be a $8th$ root of unity such that its conjugacy class has length 60 (the smallest one). It yields an 2-modular lattice L_2 given by the Gram matrix

$$G_2 = (2 + \zeta_8 + \zeta_8^{-1}) G_1.$$

The lattice has minimum 8, hence it is extremal 2-modular. And therefore $\zeta_8 + \zeta_8^{-1}$ is a $\sqrt{2}$-structure which is an extremal even unimodular lattice of dimension 24 over $\mathbb{Q}[\sqrt{2}]$. Its theta series is f_{12}. The Gram matrices G_1 and $(2+\zeta_8+\zeta_8^{-1})G_1$ are given in Appendix C.

The other roots of unity yield 2-modular lattices with minimum 6, so the $\sqrt{2}$-structures are not extremal.

Chapter 8

Extremal Type (ii) Lattices over $\mathbb{Q}[\sqrt{3}]$

8.1 Extremal Lattices

In this chapter we discuss Type (ii) lattices over $F = \mathbb{Q}[\sqrt{3}]$, because the fundamental unit $\varepsilon_0 = 2+\sqrt{3}$ has norm 1. Let (Λ, Q) be a Type (ii) lattice, i.e. (Λ, Q) is Galois-invariant, even, and unimodular. The first trace lattice (Λ_1, Q_1) is formed with respect to $\alpha_1 = \frac{1}{2}$. It is 3-modular. The second trace lattice (Λ_2, Q_2) is the trace lattices with respect to $\alpha_2 = \frac{3-\sqrt{3}}{6}$. It is 2-modular. The roots of (Λ, Q) are the vectors with norm ε_0^k, where $k \in \mathbb{Z}$. One can reduce that to the norms 1 and ε_0. Over the trace lattices, we have for $\lambda \in \Lambda$:

$$\begin{aligned} Q(\lambda) = 1 \quad &\Leftrightarrow \quad Q_1(\lambda) = 1,\ Q_2(\lambda) = 1, \\ Q(\lambda) = 2+\sqrt{3} \quad &\Leftrightarrow \quad Q_1(\lambda) = 2,\ Q_2(\lambda) = 1. \end{aligned}$$

Let $A = (\alpha_1, \alpha_2)$. Ordered with respect to $\leq_A$, the first totally positive integer elements are

$$1 \ \leq_A \ 2+\sqrt{3} \ \leq_A 2 \ \leq_A \ 2-\sqrt{3} \ \leq_A 3 \ \leq_A \ \ldots$$

Extremal Lattices The following extremal lattices over $\mathbb{Q}[\sqrt{3}]$ are known.

In dimension 2, there is just the root lattice G_2 exists. In dimension 4, there are the root lattices $G_2 \perp G_2$ and F_4. See also Section 2.3. All unimodular even lattices in dimension 6 and 8 were classified in [Hun91]. There is one extremal lattice in dimension 6, and there are 3 extremal lattices in dimension 8.

I constructed additionally all 21 extremal lattices of dimension 10.

The following table lists the extremal lattices. The first column gives the rank n of the lattice (over $\mathbb{Q}[\sqrt{3}]$), and the second column gives the rank N of the trace lattices (over $\mathbb{Q}$).

Table 8.1: Extremal lattices over $\mathbb{Q}[\sqrt{3}]$

n	$N = 2n$	trace lattices	# extr. lat.
2	4	$(A_2 \perp A_2, F_4)$	1
4	8		2
6	12	$\Lambda_1 = K_{12}$	1
8	16	$\Lambda_2 = BW_{16}$	3
10	20		21
12	24		?
14	28		?

Extremal Modular Forms The theta series of an Type (ii) lattice of dimension n is a (Galois) symmetric Hilbert modular form $\mathbb{H} \times \overline{\mathbb{H}} \to \mathbb{C}$ of weight $k = \frac{n}{2}$. If the lattices is fundamentally invariant, then the theta series is fundamentally symmetric. The graduated algebra of Galois and fundamentally symmetric Hilbert modular forms on $\mathbb{H} \times \overline{\mathbb{H}}$ is a polynomial ring in 3 generators, i.e.

$$\overline{M}^{+,\varepsilon_0}\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]\right) = \mathbb{C}[g_1, g_4, g_6].$$

The generators are the Eisenstein series g_1, g_4, and g_6 of weight 1, 4, and 6. This was proved by [Gun65], see also Section 3.2. We will show the following.

Theorem 8.1 *One can construct symmetric Hilbert modular forms g_1, s_3, and g_4 of weight 1, 3, and 4, respectively, with spherical theta series.*

The algebra of symmetric Hilbert modular forms of level $\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])$ on $\mathbb{H} \times \overline{\mathbb{H}}$ is a polynomial ring in g_1, s_3, and g_4, i.e.

$$\overline{M}^{+}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]) = \mathbb{C}[g_1, s_3, g_4].$$

The generators have the following q-expansions:

$$\begin{aligned} g_1 &= 1 + 12q_1q_2 + 12q_1^2q_2 + 12q_1^2q_2^2 + 12q_1^2q_2^3 + 12q_1^3q_2^3 + O(q_1^4q_2^2) \\ s_3 &= q_1q_2 - q_1^2q_2 - 4q_1^2q_2^2 - q_1^2q_2^3 + 9q_1^3q_2^3 + O(q_1^4q_2^2) \\ g_4 &= 23 + 240q_1q_2 + 240q_1^2q_2 + 17520q_1^2q_2^2 + 240q_1^2q_2^3 + 60480q_1^3q_2^2 \\ &\quad +181680q_1^3q_2^3 + 60480q_1^4q_2^3 + O(q_1^4q_2^2) \\ g_6 &= 1681 + 504q_1q_2 + 504q_1^2q_2 + 532728q_1^2q_2^3 + 504q_1^2q_2^3 + 4058208q_1^3q_2^2 \\ &\quad +29883672q_1^3q_2^3 + 4058208q_1^3q_2^4 + O(q_1^4q_2^2) \end{aligned}$$

A Type (ii) lattice (Λ, Q) is called extremal if $\Theta_{(\Lambda,Q)}$ is extremal in $\overline{M}_k^+(\mathrm{SL}_2(\mathbb{Z}_F))$, see Definition 4.6. That means that $\Theta_{(\Lambda,Q)}$ is the extremal form among all (Galois) symmetric Hilbert modular forms of weight $k = \frac{\dim \Lambda}{2}$.

The following Table 8.2 gives details of the extremal symmetric Hilbert modular forms of weight 1 to 20. The first column gives the dimension n of the (possible) extremal lattice Λ, and the second column gives the weight k of the extremal Hilbert modular form. The third column gives the valuation of the extremal Hilbert modular form of weight k. The last columns give the minima and kissing numbers of the trace lattices Λ_1 and Λ_2 of an extremal lattice Λ (if such lattices exist). Its theta series would be the extremal symmetric modular form.

Table 8.2: Extremal symmetric Hilbert modular forms and lattices

n	k	valuation	$\min \Lambda_1$	#Min Λ_1	$\min \Lambda_2$	#Min Λ_2
2	1	$[1, 1]$	1	12	1	24
4	2	$[1, 1]$	1	24	2	48
6	3	$[2, 1]$	2	756	1	72
8	4	$[2, 2]$	2	720	2	4320
10	5	$[2, 2]$	2	540	2	3960
12	6	$[3, 2]$	3	26208	2	3024
14	7	$[3, 2]$	3	17472	2	1512

Corollary 8.2 *Let (Λ, Q) be an extremal Type (ii) lattice of dimension $n \leq 14$. Then the trace lattice (Λ_1, Q_1) is extremal 3-modular, and the trace lattice (Λ_2, Q_2) is extremal 2-modular.*

Proof. Form the table we observe that $\min \Lambda_1 = 1 + \lfloor \frac{2n}{12} \rfloor$ and $\min \Lambda_2 = 1 + \lfloor \frac{2n}{16} \rfloor$ for $n \leq 14$, n even. Modular lattices achieving this bound are extremal, see [Que95]. Hence Λ_1 and Λ_2 are extremal 3- or 2-modular, respectively.

□

Theorem 8.3 *We list the extremal symmetric Hilbert modular forms of weight $k \leq 12$ in Appendix D. The extremal modular forms of weight 8 to 12 each have some negative Fourier coefficients. Hence they cannot be theta series, and extremal Galois invariant lattices of dimension 16 to 24 thus do not exist.*

8.2 Dimension 2

The Gram matrix

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$$

defines an even unimodular lattice G_2 of dimension 2. It is a root lattice. Its theta series is the Eisenstein series g_1,

$$\Theta_{G_2}(z) = g_1(z) = 1 + 12q_1q_2 + 12q_1^2q_2 + 12q_1^2q_2^2 + 12q_1q_2^3 + O(q_1^3q_2^2).$$

The first trace lattice is the 3-modular lattice $A_2 \perp A_2$, and the second trace lattice is the 2-modular lattice F_4.

The lattice G_2 represent the only isometry class of unimodular lattices of dimension 2, see for example [Hun91]. We will show this result with our methods.

Assume that we have an extremal lattice (Λ, Q) of rank 2 over $F = \mathbb{Q}[\sqrt{3}]$. Then its theta series is the Eisenstein series g_1 of weight 1.

Like in the cases before, for a fixed lattice point $\lambda \in \Lambda(1)$ we want to determine the configuration numbers

$$n(\beta, \iota; \lambda) = |\{\mu \in \Lambda(\beta/2) \mid B(\lambda, \mu) = \pm\iota\}|$$

for fixed β (e.g. $\beta = 2$) and all possible ι's.

Proposition 8.4 (Configuration numbers)
The non-trivial configuration numbers for $\beta = 2$ are given in the following table.

Table 8.3: Configuration numbers

ι	0	1	$\sqrt{3}$	2
$n(2, \iota; \lambda)$	2	4	4	2

Proof. If $\mu \in \Lambda(1)$ then the scalar product $\iota = B(\lambda, \mu)$ is one of the following,

$$\iota \in \{0, \pm 1, \pm\sqrt{3}, \pm 2\}.$$

Clearly $n(2, 2; \lambda) = 2$. We have

$$\sum_{\iota} n(2, \iota; \lambda) = 12 = |\Lambda(1)|,$$

where the sum is over $\iota \in \{0, 1, \sqrt{3}, 2\}$.

The Hilbert modular forms space of weight 2 is just $\langle g_1^2 \rangle$. Hence there are not any cusp forms and $\Theta_{\Lambda, \mathcal{N}P_1} = 0$. Hence

$$\sum_{\iota} n(2, \iota; \lambda)\mathcal{N}(\iota) = 0.$$

The first trace lattice is 3-modular of dimension 4, therefore each layer forms a spherical 3-design. Especially $\Lambda_1(2) = \Lambda(2)$, as one sees from the theta series, is a 3-design. Hence

$$\sum_{\iota} n(2,\iota;\lambda)\left(\left(\frac{\mathrm{tr}(\iota)}{2}\right)^2 - 1\right) = 0.$$

These equations determine the numbers uniquely. □

Theorem 8.5 (Uniqueness) *There is up to isometry only one even unimodular lattice of degree* 2 *over* $\mathbb{Q}[\sqrt{3}]$.

Proof. Let $\Lambda \subset \mathbb{Q}[\sqrt{3}]^2$ be an even unimodular lattice. Let $\lambda \in \Lambda(1)$. Above we have seen that there are 4 lattice vectors in the layer $\Lambda(1)$ which have scalar product $\sqrt{3}$ with λ. Let μ be such a vector. Then $\langle\lambda,\mu\rangle$ is a full sublattice of Λ. But the sublattice has Gram matrix

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$$

and hence is the root lattice G_2. But G_2 is even unimodular, and so $\Lambda = \langle\lambda,\mu\rangle = G_2$. □

8.3 Dimension 4

The space of symmetric modular forms of weight 2 is $M_2^+ = \langle g_1^2\rangle$. Hence $G_2 \perp G_2$ is an (extremal) Type (ii) lattice. Also, there is the root lattice F_4, see Section 2.3. We show hat these two represent all isometry classes, compare also [Hun91].

Assume that we have an extremal lattice (Λ, Q) of rank 4 over $F = \mathbb{Q}[\sqrt{3}]$. Then its theta series is

$$\Theta_\Lambda = g_1^2 = 1 + 24q_1q_2 + 24q_1^2q_2 + 168q_1^2q_2^2 + 24q_1q_2^3 + O(q_1^3q_2^2).$$

For a fixed lattice point $\lambda \in \Lambda(1)$ we will determine the configuration numbers $n(1,\iota;\lambda)$.

Proposition 8.6 (Configuration numbers)
There are two possibilities for the non-trivial numbers for $\beta = 2$. *They are given in the following table.*

Proof. We consider lattice points $\mu \in \Lambda(1)$. Like for dimension 2, we have the following scalar products

$$\iota = B(\lambda,\mu) \in S := \{0, \pm 1, \pm\sqrt{3}, \pm 2\}.$$

Table 8.4: Configuration numbers

ι	0	1	$\sqrt{3}$	2
$n(2,\iota;\lambda)$	6	16	0	2
$n(2,\iota;\lambda)$	14	4	4	2

And we have the equations $n(2,2;\lambda) = 2$ and

$$\sum_{\iota \in S} n(2,\iota;\lambda) = 24.$$

The first trace lattice is 3-modular of dimension 8. So according to Example 5.17,

$$\Lambda_1(b) \cup \sqrt{3} \cdot \Lambda_1^{\#}(b/3)$$

is a spherical 3-design for all $b \in \mathbb{N}$. We have $\Lambda_1^{\#} = 2(2\sqrt{3})^{-1}\Lambda_1$, and hence $\Lambda_1^{\#}(b/3) = \frac{1}{\sqrt{3}}\Lambda_1(b)$. In our case we have $b = 1$. Let $a := 1$ be the norm of λ in Λ_1. The Gegenbauer polynomial of parameter $8/2 - 1 = 3$ and degree 2 is $24t^2 - 3$. Homogenized with $\sqrt{2a \cdot 2b} = 2$, we get the equation

$$\sum_{\iota} n(2,\iota;\lambda) \left(2 \operatorname{tr}\left(\frac{\iota}{2}\right)^2 - 1 + 6 \operatorname{tr}\left(\frac{\iota}{2\sqrt{3}}\right)^2 - 1 \right) = 0.$$

These equations leave three possibilities of configuration numbers:

ι	0	1	$\sqrt{3}$	2
$n(2,\iota;\lambda)$	6	16	0	2
$n(2,\iota;\lambda)$	10	10	2	2
$n(2,\iota;\lambda)$	14	4	4	2

The second configuration is not possible, because if $\mu \in \Lambda(1)$ with $B(\lambda, \mu) = 1$, then also $\lambda - \mu \in \Lambda(1)$ and $B(\lambda, \lambda - \mu) = 1$. The same is of cause true for $-\mu$ and $-(\lambda - \mu)$. So $n(2,1;\lambda)$ is a multiple of 4. □

Theorem 8.7 (Uniqueness)
There are up to isometry precisely two Type (ii) lattice of degree 4 over $\mathbb{Q}[\sqrt{3}]$, the root lattices $G_2 \perp G_2$ and F_4.

Proof. Let $\Lambda \subset \mathbb{Q}[\sqrt{3}]^2$ be an even unimodular lattice. We have seen that for $\lambda \in \Lambda(1)$ there are two different possibilities for the numbers $n(2,\iota;\lambda)$, where $\iota = 0, 1, \sqrt{3}$.

First assume that there is a vector $\lambda \in \Lambda$ which has the configuration numbers of the second column in Table 8.4.

Let $\mu \in \Lambda(1)$ such that $B(\lambda,\mu) = \sqrt{3}$. Then $\langle \lambda, \mu \rangle$ is a sublattice of Λ and isometric to G_2. Therefore $\Lambda \cong G_2 \perp G_2$.

So assume that all vectors of $\Lambda(1)$ have the configuration numbers of the first column of Table 8.4. Let $\lambda \in \Lambda(1)$. Then there are exactly 8 vectors $\mu \in \Lambda(1)$ with $B(\lambda,\mu) = 1$. We can order them as

$$\mu_1, \dots, \mu_4,\ \lambda - \mu_1, \dots, \lambda - \mu_4.$$

Since $n(2,\sqrt{3};\lambda) = 0$, the scalar products $B(\mu_i,\mu_j)$ and $B(\mu_i, \lambda - \mu_j)$ might be 0, 1, or -1 (assuming $i \neq j$ and $i - j \not\equiv 0 \pmod 4$). More precisely, we have $B(\mu_i,\mu_j) = 0$ and $B(\mu_i, \lambda - \mu_j) = 1$ or vice versa. Without loss of generality assume $B(\mu_1,\mu_j) = 0$ for $j = 2, 3, 4$. So the Gram matrix of $(\lambda, \mu_1, \mu_2, \mu_3)$ is

$$G_1 := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \text{ or } G_2 := \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

Both matrices are non-singular and could thus be Gram matrices of full sublattices of Λ. Denote by M a lattice with Gram matrix G_1. The determinant is 4, so M may have an even unimodular overlattice Λ. A computer calculation shows that all even unimodular overlattices are isometric to F_4. The determinant of G_2 is 5 and $5\mathbb{Z}[\sqrt{3}]$ is a prime ideal, hence a lattice with Gram matrix G_2 cannot have unimodular overlattices. $\square$

So we have two unimodular lattices, $G_2 \perp G_2$ and F_4. All lattice points of norm 1 in every lattice have the same same configuration numbers. By calculations we find that $(G_2 \perp G_2)(1)$ is a spherical 3-design, and $F_4(1)$ is a spherical 5-design.

We find that $\Theta_{F_4,\mathcal{N}P_1}$ is non-trivial. Hence $\Theta_{F_4,\mathcal{N}P_1}$ is not fundamentally symmetric, because the space of fundamental and Galois symmetric forms of weight 3 is just $\langle g_1^3 \rangle$. We get the following corollary.

Corollary 8.8 (The Hilbert modular form s_3)
The Hilbert modular form s_3 is given by

$$s_3 := \frac{1}{24}\Theta_{F_4,\mathcal{N}P_1}.$$

It is not fundamentally symmetric but Galois symmetric. Its q-expansion is

$$s_3 = q_1 q_2 - q_1^2 q_2 - 4 q_1^2 q_2^2 - q_1^2 q_2^3 + 9 q_1^3 q_2^3 + O(q_1^4 q_2^2).$$

8.4 Dimension 6

The space of symmetric modular forms of weight 3 is $M_3^+ = \langle g_1^3, s_3\rangle$. So the extremal modular form is

$$f_6 = 1 + 72q_1^2q_2 + 612q_1^2q_2^2 + 72q_1^2q_2^3 + 864q_1^3q_2^2 + 2304q_1^3q_2^3 + 864q_1^3q_2^4 + O(q_1^4q_2^2).$$

It is not fundamentally symmetric, so an extremal Type (ii) lattice would not be fundamentally invariant.

There are 6 isometry classes of even unimodular lattices, cf. [Hun91]. Two are fundamentally symmetric, precisely $G_2 \perp G_2 \perp G_2$ and $G_2 \perp F_4$. One lattice is extremal, i.e. has no vectors of norm 1. It still has roots, in difference to the $\sqrt{5}$ and $\sqrt{2}$ cases, because it has vectors of norm $2+\sqrt{3} \in \mathbb{Z}_F^*$.

We will prove that there is exactly one extremal lattice. Let Λ be extremal of rank 6. Since $\Theta_\Lambda = f_6$, the first trace lattice Λ_1 has minimum 2. Since it is 3-modular, it is extremal. The only extremal even 3-modular lattice in dimension 12 is the Coxeter-Todd lattice K_{12}, hence $\Lambda_1 = K_{12}$.

With the computational method of Section 10.2 one can show that K_{12} has exactly one Type (ii) $\sqrt{3}$-structure. See there for details. Hence we get the following theorem.

Theorem 8.9 *There is precisely one extremal Type (ii) lattice over $\mathbb{Q}[\sqrt{3}]$ of dimension 6. Its first trace lattice is the Coxeter-Todd lattice. Its Gram matrix is given in Appendix D.*

8.5 Dimension 8

The even unimodular lattices of dimension 12 were classified by Hung in [Hun91]. There are 31 isometry classes of even unimodular lattices. Among them 3 classes contain extremal lattices. These lattices are characterized by the lack of roots, i.e. elements of norm 1 or $2+\sqrt{3}$. We give their Gram matrices in Appendix D.

We cannot determine the configuration numbers with our methods.

Proof (of Theorem 8.1). Let

$$\overline{M}^+ := \overline{M}^+\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])\right),\ \overline{M}^+_{ev} := \sum_{k\geq 0} \overline{M}^+_{2k},$$
$$\overline{M}^{+,\varepsilon_0} := \overline{M}^{+,\varepsilon_0}\left(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])\right),\ \text{and } \overline{M}^{+,\varepsilon_0}_{ev} := \sum_{k\geq 0} \overline{M}^{+,\varepsilon_0}_{2k}.$$

Gundlach showed in [Gun65] that

$$\overline{M}^{+,\varepsilon_0} = \mathbb{C}[g_1, g_4, g_6],$$

where g_1 is an Eisenstein series and s_4 and s_6 are cusp forms. The generator g_1 is the theta series of the 2-dimensional lattice G_2, and g_4 can be constructed from the theta series of an extremal lattice of dimension 8.

Forms in $\overline{M}^+$ are invariant under the group $G := \mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])$, forms in $\overline{M}^{+,\varepsilon_0}$ are invariant under the group $H := \left\langle \mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]), \left(\begin{smallmatrix} \varepsilon_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle$. The scalar matrices $Z(\mathbb{Z}_F) = \{aI_2 \mid a \in \mathbb{Z}_F^*\}$ act trivial on modular forms. Since

$$[HZ(\mathbb{Z}_F) : GZ(\mathbb{Z}_F)] = 2,$$

we have $[\overline{M}^{+,\varepsilon_0} : \overline{M}^+] = 2$.

We constructed a Galois symmetric Hilbert modular form s_3 which is not fundamentally symmetric in Corollary 8.8:

$$s_3 = \tfrac{1}{24}\Theta_{F_4,\mathcal{N}P_1}(z).$$

Therefore $\overline{M}^+$ is generated by g_1, g_4, g_6 and a form s of weight 1, 2, or 3.

The Hilbert series of $\overline{M}_{ev}$ is given in [vdG88, Proposition VIII.1.1]:

$$\begin{aligned} \sum_{k\geq 0} \dim \overline{M}_{2k} t^{2k} &= \frac{(1-t^8)(1-t^{20})}{(1-t^2)(1-t^4)^2(1-t^6)(1-t^{10})} \\ &= 1 + t^2 + 3t^4 + 4t^6 + 6t^8 + 9t^{10} + 12t^{12} + O(t^{14}). \end{aligned}$$

Since $\dim \overline{M}_2^+ = 1$, the form s cannot have weight 1 or 2.

g_6 can be expressed in the other generators,

$$g_6 = -\tfrac{3055}{2} g_1^4 + \tfrac{279}{2} g_1^2 g_4 + 155520 s_3^2.$$

Other algebraic dependencies are not possible. So $\overline{M}^+$ is a polynomial ring in g_1, s_3, and g_4. □

8.6 Dimension 10

The mass of the genus of even unimodular lattices of dimension 10 is

$$\frac{11 \cdot 13 \cdot 23 \cdot 41^2 \cdot 1801}{2^{17} \cdot 3^6 \cdot 5 \cdot 7} \approx 3.0.$$

Kneser's neighboring method at the prime $\sqrt{3}\mathbb{Z}_F$ provides all 430 isometry classes. Among them 21 are extremal lattices. We give their Gram matrices in Appendix D.

We cannot determine the configuration numbers with our methods.

There are no known extremal Type (ii) lattices in dimensions 12 or 14. There are no extremal Type (ii) lattices in dimensions 16, 18, and 20, see Theorem 8.3.

Chapter 9

Extremal Type (iii) Lattices over $\mathbb{Q}[\sqrt{3}]$

Let (Λ, Q) be an extremal Type (iii) lattice over $F = \mathbb{Q}[\sqrt{3}]$, i.e. (Λ, Q) is Galois invariant, trace even, and trace unimodular. The first trace lattice (Λ_1, Q_1) is the trace lattices with respect to $\alpha_1 = 1$. It is even unimodular. The second trace lattice (Λ_2, Q_2) is the trace lattices with respect to $\alpha_2 = 3 + \sqrt{3}$. It is even 6-modular, see Proposition 2.3.

The norms of elements of trace even unimodular lattices are totally positive in $\mathbb{Z}_F^{\#}$. Let $A = (\alpha_1, \alpha_2)$. Ordered by, $\leq_A$, the minimal totally positive elements of $\mathbb{Z}_F^{\#}$ are

$$\frac{3-\sqrt{3}}{6} \leq_A \frac{1}{2} \leq_A \frac{3+\sqrt{3}}{6} \leq_A \frac{2-\sqrt{3}}{2} \leq_A \frac{3-\sqrt{3}}{3} \leq_A \ldots$$

We also assume that (Λ, Q) is fundamentally invariant, see Definition 1.4. We call a Type (iii) lattice *extremal fundamentally invariant*, if it is fundamentally invariant and its theta series is extremal in $M_k^{+,\varepsilon_0}(\mathrm{SL}(\mathbb{Z}[\sqrt{3}]))$.

Quebbemann defined strongly modular lattices in [Que97]. A $\mathbb{Z}-$lattice L is called strongly 6-modular if

$$L \cong \sqrt{2}(L^{\#} \cap \tfrac{1}{2}L) \text{ and } L \cong \sqrt{3}(L^{\#} \cap \tfrac{1}{3}L).$$

Proposition 9.1 *If (Λ, Q) is a fundamentally invariant Type (iii) lattice, then Λ_2 is strongly 6-modular.*

Proof. To avoid confusion let $a := \sqrt{2}, b := \sqrt{3} \in \mathbb{R}$ be scalars. An elements $\nu \in F[\sqrt{3}]$ acts as an endomorphism of $\mathbb{R} \otimes \Lambda_2$, we write ν for the endomorphism.

Since Λ is Galois and fundamentally invariant, we have $\sigma, \tau \in \mathrm{End}(\Lambda)$ with $Q(\sigma(\lambda)) = \overline{Q(\lambda)}$ and $Q(\tau(\lambda)) = \varepsilon_0 Q(\lambda)$ for all $\lambda \in \Lambda$. Also $\sigma, \tau \in \mathrm{End}(\Lambda_2)$. And $f := \sigma(3+\sqrt{3})$ is a similarity of norm 6 with $f(\Lambda_2^{\#}) = \Lambda_2$ (cf. Proposition 2.3).

We define $\varphi := b(1-\sqrt{3})\tau f^{-1}$ and claim that φ is an isometry from Λ_2 to $b(\Lambda_2^{\#} \cap \frac{1}{3}\Lambda_2)$.

Let $\lambda \in \Lambda_2$. Then $f^{-1}(\lambda) \in \Lambda_2^{\#}$ and hence $\varphi(\lambda) \in b\Lambda^{\#}$. And

$$3(1-\sqrt{3})\tau f^{-1}(\lambda) = 3(1-\sqrt{3})\tfrac{3-\sqrt{3}}{6}\tau\sigma^{-1}(\lambda) = (2-\sqrt{3})\sqrt{3}\tau\sigma^{-1}(\lambda) \in \Lambda.$$

So $\varphi(\lambda) \in b(\Lambda_2^{\#} \cap \frac{1}{3}\Lambda_2)$ for all $\lambda \in \Lambda$. The lattices have the same determinant (cf. [Que97, Proposition 1]). Is remains to show that φ is an isometry. For all $\lambda \in \Lambda$ we have

$$\begin{aligned}
Q_2(\varphi(\lambda)) &= 3\,\mathrm{tr}\left((3+\sqrt{3})Q\left((1-\sqrt{3})\tfrac{3-\sqrt{3}}{6}\tau\sigma^{-1}(\lambda)\right)\right) \\
&= \tfrac{3}{6^2}\,\mathrm{tr}\left((3+\sqrt{3})(1-\sqrt{3})^2(3-\sqrt{3})^2 Q(\tau\sigma^{-1}(\lambda))\right) \\
&\overset{(1-\sqrt{3})^2=2\overline{\varepsilon_0}}{=} \mathrm{tr}\left(\overline{\varepsilon_0}(3-\sqrt{3})\varepsilon_0\overline{Q(\lambda)}\right) \\
&= \mathrm{tr}\left((3+\sqrt{3})Q(\lambda)\right) = Q_2(\lambda).
\end{aligned}$$

So $\varphi : \Lambda_2 \to b(\Lambda_2^{\#} \cap \frac{1}{3}\Lambda_2)$ is an isometry.

Let $\psi := a\sqrt{3}f^{-1}$. Let $\lambda \in \Lambda_2$. Then $f^{-1}(\lambda) \in \Lambda_2^{\#}$ and hence $\psi(\lambda) \in a\Lambda^{\#}$. And

$$2\sqrt{3}f^{-1}(\lambda) = 2\sqrt{3}\tfrac{3-\sqrt{3}}{6}\sigma^{-1}(\lambda) = (-1+\sqrt{3})\sigma^{-1}(\lambda) \in \Lambda.$$

So $\psi(\lambda) \in a(\Lambda_2^{\#} \cap \frac{1}{2}\Lambda_2)$ for all $\lambda \in \Lambda$. Again, we have to show that ψ is an isometry. For all $\lambda \in \Lambda$ we have

$$\begin{aligned}
Q_2(\psi(\lambda)) &= 2\mathrm{tr}\left((3+\sqrt{3})Q(\sqrt{3}\tfrac{3-\sqrt{3}}{6}\sigma^{-1}(\lambda))\right) \\
&= \tfrac{1}{6}\mathrm{tr}\left((3+\sqrt{3})(3-\sqrt{3})^2 Q(\sigma^{-1}(\lambda))\right) \\
&= \mathrm{tr}\left((3-\sqrt{3})\overline{Q(\lambda)}\right) \\
&= \mathrm{tr}\left((3+\sqrt{3})Q(\lambda)\right) = Q_2(\lambda).
\end{aligned}$$

So $\psi : \Lambda_2 \to a(\Lambda_2^{\#} \cap \frac{1}{2}\Lambda_2)$ is an isometry.

Therefore Λ_2 is strongly 6-modular. $\square$

9.1 Extremal Fundamentally Invariant Lattices

In dimension 4, there is precisely one Type (iii) lattice. We prove the uniqueness. In dimension 8, there is at least one extremal fundamentally invariant lattice. The Leech lattice has one conjugacy class of 12*th* roots of unity automorphism. This defines a structure which is an extremal fundamentally invariant Type (iii) lattice over $\mathbb{Q}[\sqrt{3}]$ of dimension 12, compare Section 10.1. The Gram matrices of these three extremal lattices are given in Appendix D. Extremal fundamentally invariant lattices in higher dimensions are not known.

The theta series of a Type (iii) lattice is a Galois symmetric Hilbert modular form $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$. If the lattice is fundamentally invariant, then the theta series is fundamentally symmetric.

The graduated algebra of Galois and fundamentally symmetric Hilbert modular forms on $\mathbb{H} \times \mathbb{H}$ is

$$M^{+,\varepsilon_0}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}])) = \mathbb{C}[g_2, g_3, g_4].$$

The generators are the Eisenstein series g_2, g_3, and g_4, see [Gun65] or Section 3.2.

The generators have the following q-expansions:

$$\begin{aligned}
g_2 &= 1 + 72q_1q_2^2 + 96q_1q_2^3 + 72q_1q_2^4 + 96q_1^2q_2^3 + 360q_1^2q_2^4 + 288q_1^2q_2^5 \\
&\quad +672q_1^2q_2^6 + 288q_1^2q_2^7 + 360q_1^2q_2^8 + 96q_1^2q_2^9 + O(q_1^3q_2^4), \\
g_3 &= 1 - 108q_1q_2^2 - 288q_1q_2^3 - 108q_1q_2^4 - 288q_1^2q_2^3 - 1836q_1^2q_2^4 - 4320q_1^2q_2^5 \\
&\quad -3744q_1^2q_2^6 - 4320q_1^2q_2^7 - 1836q_1^2q_2^8 - 288q_1^2q_2^9 + O(q_1^3q_2^4), \\
g_4 &= 23 + 2160q_1q_2^2 + 6720q_1q_2^3 + 2160q_1q_2^4 + 6720q_1^2q_2^3 + 140400q_1^2q_2^4 \\
&\quad +319680q_1^2q_2^5 + 490560q_1^2q_2^6 + 319680q_1^2q_2^7 + 140400q_1^2q_2^8 \\
&\quad +6720q_1^2q_2^9 + O(q_1^3q_2^4).
\end{aligned}$$

The following table gives details of the extremal Galois and fundamentally symmetric Hilbert modular forms of weight 2 to 20. The first column gives the rank n of the (possible) extremal fundamentally invariant Type (iii) lattice Λ, and the second column gives the weight k of the extremal Galois and fundamentally symmetric Hilbert modular form. The third column gives the valuation of the extremal Hilbert modular form of weight k. The last columns give the minima and kissing numbers of the trace lattices Λ_1 and Λ_2. Its theta series would be the extremal modular form.

Table 9.1: Extremal Hilbert modular forms and lattices

n	k	valuation	$\min \Lambda_1$	#Min Λ_1	$\min \Lambda_2$	#Min Λ_2
4	2	$[1, 2]$	1	240	2	72
8	4	$[1, 3]$	1	480	2	960
12	6	$[2, 4]$	2	10584	4	10584
16	8	$[2, 5]$	2	146880	5	103680
20	10	$[2, 6]$	2	39600	6	997920

Corollary 9.2 *Let Λ be an extremal fundamentally invariant Type (iii) lattice of dimension $n \leq 20$. Then the trace lattice (Λ_1, Q_1) is extremal unimodular, and the trace lattice (Λ_2, Q_2) is extremal strongly 6-modular.*

Proof. We see that $\min \Lambda_1 = 1 + \lfloor \frac{2n}{24} \rfloor$ and $\min \Lambda_2 = 1 + \lfloor \frac{2n}{8} \rfloor$ for all $n \leq 20$. So Λ_1 is extremal unimodular. And Λ_2 is extremal strongly 6-modular, compare [Que97]. □

Theorem 9.3 *We list the extremal Galois and fundamentally symmetric Hilbert modular forms of even weight $k \leq 20$ in Appendix D. The extremal modular forms of weight 12 to 20 each have some non-integral Fourier coefficients. Hence they cannot be theta series, and extremal fundamentally invariant Type (iii) lattices of dimension 24 to 40 do not exist.*

9.2 Dimension 4

The space of modular forms of weight 2 is $M_2 = \langle g_2 \rangle$.

An automorphism $\zeta_{12} \in \mathrm{Aut}(\mathbb{E}_8)$ (i.e. with minimal polynomial $x^4 - x^2 + 1$, unique up to conjugation) yields a $\sqrt{3}$-structure with endomorphism $\sqrt{3} = \zeta_{12} + \zeta_{12}^{-1}$. It is trace even unimodular, Galois and fundamental invariant. Its theta series has to be g_2. So it is trivially extremal.

We will proof that it is unique.

For that, assume that (Λ, Q) is a fundamentally invariant Type (iii) lattice of rank 4. Then its theta series is

$$\begin{aligned} g_2 &= 1 + 72q_1q_2^2 + 96q_1q_2^3 + 72q_1q_2^4 + 96q_1^2q_2^3 + 360q_1^2q_2^4 + 288q_1^2q_2^5 \\ &\quad + 672q_1^2q_2^6 + 288q_1^2q_2^7 + 360q_1^2q_2^8 + 96q_1^2q_2^9 + O(q_1^3q_2^4). \end{aligned}$$

Let $\alpha := 2\min\Lambda = 1 - \frac{1}{3}\sqrt{3}$, i.e. $(\mathrm{tr}(\alpha_1\alpha/2), \mathrm{tr}(\alpha_2\alpha/2)) = (1, 2)$ and $|\Lambda(\alpha/2)| = 72$. Let $\lambda \in \Lambda(\alpha/2)$ be fixed. We determine the configuration numbers $n(\alpha, \iota; \lambda)$.

Proposition 9.4 (Configuration numbers)
The non-trivial numbers for $\beta = \alpha$ are given in the following table, where $a, b \in 2\mathbb{Z}$ with $0 \leq a \leq 18$ and $0 \leq b \leq 32$.

Table 9.2: Configuration numbers

ι	0	$\frac{1}{2} - \frac{1}{6}\sqrt{3}$	$\frac{1}{2} - \frac{1}{3}\sqrt{3}$	$\frac{1}{2} - \frac{1}{2}\sqrt{3}$	$\frac{1}{6}\sqrt{3}$	$1 - \frac{1}{3}\sqrt{3}$
$n(\alpha, \iota; \lambda)$	a	20	b	$18 - a$	$32 - b$	2

Proof. If $\mu \in \Lambda(\alpha/2)$, then the possible scalar products $\iota = B(\lambda, \mu)$ are (without loss of generality $\mathrm{tr}(\iota) \geq 0$)

$$\iota \in S := \{0, \tfrac{1}{2} - \tfrac{1}{6}\sqrt{3}, \tfrac{1}{2} - \tfrac{1}{3}\sqrt{3}, \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}, \tfrac{1}{6}\sqrt{3}, 1 - \tfrac{1}{3}\sqrt{3}\}.$$

Clearly, $n(\alpha,\alpha;\lambda)=2$ and

$$\sum_{\iota\in S} n(2,\iota;\lambda)=24.$$

The second trace lattice (Λ_2,Q_2) is 6-modular and has minimum 2 and kissing number 72 because of $\Theta_\Lambda(1,q_2)=1+72q_2^2+192q_2^3+504q_2^4+O(q_2^5)$.

Quebbemann described the strongly 6-modular lattices in dimension 8 in [Que97]. There are precisely 2 extremal strongly modular lattices, but only $A_2\otimes D_4$ has kissing number 72. So $\Lambda_2=A_2\otimes D_4$. The scalar products in S are in Λ_2 the following.

ι	0	$\frac12-\frac16\sqrt3$	$\frac12-\frac13\sqrt3$	$\frac12-\frac12\sqrt3$	$\frac16\sqrt3$	$1-\frac13\sqrt3$
$\mathrm{tr}((3+\sqrt3)\iota)$	0	2	1	0	1	4

So $n(\alpha,\frac12-\frac16\sqrt3;\lambda)=20$, because there are 20 vectors in $\Lambda_2(2)$ with scalar product 2 with $\lambda\in\Lambda_2$.

Also $n(\alpha,0;\lambda)+n(\alpha,\frac12-\frac12\sqrt3;\lambda)=18$ and $n(\alpha,\frac12-\frac13\sqrt3;\lambda)+n(\alpha,\frac16\sqrt3;\lambda)=32$, so the configuration numbers have the claimed form. □

Theorem 9.5 (Uniqueness)
There is up to isometry only one fundamentally invariant Type (iii) lattice of rank 4 over $\mathbb{Q}[\sqrt3]$.

Proof. Let Λ be such a lattice. We have seen that for $\lambda\in\Lambda(\alpha/2)$ there are 10 lattice points $\mu_1,\dots,\mu_{10}\in\Lambda(\alpha/2)$ with $\iota:=B(\lambda,\mu_j)=\frac12-\frac16\sqrt3$.

Actually, we can order the vectors such that $\mu_{5+j}=\lambda-\mu_j$ for $j=1,\dots,5$.

For $j=2,\dots,5$, there are only a few possibilities for the scalar products $B(\mu_1,\mu_j)$ and $B(\mu_1,\lambda-\mu_j)$. Since $B(\mu_1,\lambda-\mu_j)=\iota-B(\mu,\mu_j)$, we find that

$$B(\mu_1,\mu_j)\in T:=\{0,\tfrac12-\tfrac13\sqrt3,\tfrac16\sqrt3\}.$$

We get Gram matrices for $(\lambda,\mu_1,\mu_2,\mu_3)$:

$$G_{a,b,c}:=\begin{pmatrix}\alpha&\iota&\iota&\iota\\ \iota&\alpha&a&b\\ \iota&a&\alpha&c\\ \iota&b&c&\alpha\end{pmatrix},$$

where $a,b,c\in T$. If $a,b,c\neq0$, then $\det G_{a,b,c}$ has negative norm. This is not possible, so without loss of generality $a=0$. If $b=c=\frac12-\frac13\sqrt3$ or $b=c=\frac16\sqrt3$, then $\det G_{a,b,c}=0$. But the lattice generated by (λ,μ_1,μ_2) does not contain 10 vectors with scalar product ι with λ. This is a contradiction.

By computer calculations, the other possibilities for $b,c\in T$ yield exactly one trace even overlattice up to isometry.

So there is exactly one isometry class of trace even unimodular lattices. □

Chapter 10

Algorithmic Approaches

In this chapter we describe two computational methods to find (trace) unimodular lattices to a given lattices over the rationals.

10.1 Roots of Unity Automorphisms

We already described the first method for $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{2}]$, and $\mathbb{Q}[\sqrt{3}]$. In general, let $F = \mathbb{Q}[\sqrt{D}]$ be a real quadratic number field, where $D > 1$ is square-free. The discriminant of F is d_F. We use the following lemma.

Lemma 10.1 *Let ζ_{d_F} be a primitive d_Fth root of unity over $\mathbb{Q}$.*

Then F is a subfield of the d_Fth cyclotomic field, and more precisely there is a $p \in \mathbb{Z}[x]$ such that $F = \mathbb{Q}[p(\zeta_{d_F} + \zeta_{d_F}^{-1})]$.

Proof. Let K_ℓ be the ℓth cyclotomic field, where $\ell \in \mathbb{N}$. Let $D = 2^\varepsilon \cdot p_1 \dots p_s$ be the prime factorization, where $\varepsilon \in \{0,1\}$ and p_i is a odd prime. So $\mathbb{Q}[\sqrt{(-1)^{(p_i-1)/2}p_i}] \subseteq K_{p_i} \subseteq K_{d_F}$. Hence

$$\mathbb{Q}[\sqrt{(-1)^a D 2^{-\varepsilon}}] \subseteq K_{d_F},. \text{ where } a := |\{i \mid p_i \cong 3 \pmod 4\}|.$$

If $D \equiv 1 \pmod 4$, then $\varepsilon = 0$, a is even, and hence $\mathbb{Q}[\sqrt{D}] \subseteq K_{d_F}$. If $D \equiv 3 \pmod 4$, then $\varepsilon = 0$ and a is odd, but K_{d_F} additionally contains the $4th$ root of unity. So $\mathbb{Q}[\sqrt{-1}\sqrt{-D}] = \mathbb{Q}[\sqrt{D}] \subseteq K_{d_F}$. If $D \equiv 2 \pmod 4$, then $d_F = 4D$ is divisible by 8. Hence K_{4d} contains $8th$ roots of unity and hence $\mathbb{Q}[\sqrt{-1}]$, $\mathbb{Q}[\sqrt{-D}] \subseteq K_{4d}$. Therefore $\mathbb{Q}[\sqrt{D}] \subseteq K_{4D}$.

Since $\mathbb{Q}[\sqrt{D}]$ is totally real,

$$\mathbb{Q}[\sqrt{D}] \subseteq K_{d_F} \cap \mathbb{R} = \mathbb{Q}[\zeta_{d_F} + \zeta_{d_F}^{-1}].$$

So there is a $p \in \mathbb{Q}[x]$ such that $p(\zeta_{d_F} + \zeta_{d_F}^{-1}) = \omega$, where

$$\omega = \begin{cases} \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4, \\ \sqrt{D} & \text{if } D \equiv 2,3 \pmod 4. \end{cases}$$

Actually $p \in \mathbb{Z}[x]$, because the ring of integers of K_{d_F} is $\mathbb{Z}[\zeta_{d_F}]$ and ω is clearly integral. $\square$

Proposition 10.2 *Let $(L, \mathfrak{q})$ be a $\mathbb{Z}$-lattice of dimension $N = 2n$. If there is a d_Fth root of unity automorphism $\zeta_{d_F} \in \mathrm{Aut}(L, \mathfrak{q})$ whose minimal polynomial is the d_Fth cyclotomic polynomial, then L has an F-structure.*

Proof. Define $\nu := p(\zeta_{d_F} + \zeta_{d_F}^{-1})$, where $p = \sum a_i x^i \in \mathbb{Z}[x]$ is defined like in the previous lemma. Then $\nu \in \mathrm{End}(L, \mathfrak{q})$ and its minimal polynomial is the same minimal polynomial as the generator of $\mathbb{Z}_F$. ν is self-adjoint because

$$\begin{aligned}(p(\zeta_{d_F})\lambda, \mu) &= \sum a_i \left((\zeta^i\lambda, \mu) + (\zeta^{-i}\lambda, \mu)\right) \\ &= \sum a_i \left((\lambda, \zeta^{-i}\mu) + (\lambda, \zeta^i\mu)\right) = (\lambda, p(\zeta)\mu).\end{aligned}$$

So ν defines an F-structure on L. The quadratic form is given by Theorem 2.6. $\square$

10.2 Structures from D-Modular Lattices

In this section let $(L, \mathfrak{q})$ be a D-modular lattice, where $F = \mathbb{Q}[\sqrt{D}]$ is a real quadratic number field. Let $N = 2n$ be the dimension of L. We fix a lattice basis E of L. We identify all endomorphisms with matrices given by E. Let $\mathcal{G}$ be the Gram matrix of E.

We want to give an algorithm to find all unimodular F-structures of L. Concretely:

Type (i) Find all even unimodular lattices (Λ, Q) such that $(L, \mathfrak{q}) = (\Lambda_2, Q_2)$ is the second trace lattice.

Type (ii) Find all even unimodular lattices (Λ, Q) such that $(L, \mathfrak{q}) = (\Lambda_1, Q_1)$ is the first trace lattice.

Let $\alpha := e^{-1} = \sqrt{\frac{D}{d_F}} \in \{1, 2\}$, i.e. $\alpha = \alpha_2$ for Type (i) and $\alpha = \alpha_1$ for Type (ii).

Let $\nu \in \mathrm{End}(L, \mathfrak{q})$ be self-adjoint with minimal polynomial $x^2 - D$. If $D \equiv 1 \pmod 4$, we assume additionally that $\frac{1+\nu}{2} \in \mathrm{End}(L, \mathfrak{q})$. Let A be the set of all such endomorphisms.

So ν (or $\frac{1+\nu}{2}$) gives an F-structure (Λ, Q) of L such that L is the trace lattice with respect to α, cf. Theorem 2.6.

Let $L' := \sqrt{D} L^{\#} \in \mathbb{R} \otimes L$ the rescaled dual lattice. Since L is D-modular, L' and L are isometric.

Lemma 10.3 *Let $\nu \in A$.*

(i) ν is a similarity of norm D with $\nu(L^{\#}) = L$.

(ii) If $\sigma : L' \to L$ is an isometry with $\sigma^{\mathrm{tr}}\mathcal{G}\sigma = D\mathcal{G}^{-1}$, then

$$\tfrac{1}{D}\nu\sigma\mathcal{G} \in \mathrm{Aut}(L, \mathfrak{q}).$$

(iii) Let $\nu' \in A$. The F-structures defined by ν and ν' are isometric if and only if there is a $g \in \mathrm{Aut}(L, \mathfrak{q})$ with $g^{\mathrm{tr}}\nu g = \nu'$.

Proof. (i). The dual lattice is

$$L^{\#} = e\mathbb{Z}_F^{\#}L = \sqrt{D}^{-1}\mathbb{Z}_F L.$$

Since multiplication with $\sqrt{D}$ is evaluation of ν in L, ν maps $L^{\#}$ onto L.

(ii). Let $g := \frac{1}{D}\nu\sigma\mathcal{G}$. Since $\nu^{\mathrm{tr}}\mathcal{G}\nu = D\mathcal{G}$ and $\sigma^{\mathrm{tr}}\mathcal{G}\sigma = D\mathcal{G}^{-1}$, we have $g^{\mathrm{tr}}\mathcal{G}g = \mathcal{G}$. So $g \in \mathrm{Aut}(L, \mathfrak{q})$.

(iii). See Proposition 1.12. □

The following theorem states a way to compute all F-structures.

Theorem 10.4 *Let $\sigma : L' \to L$ be an isometry with $\sigma^{\mathrm{tr}}\mathcal{G}\sigma = D\mathcal{G}^{-1}$. Let*

$$B := \{Dg\mathcal{G}^{-1}\sigma^{-1} \mid g \in \mathrm{Aut}(L, \mathfrak{q})\}.$$

Then $A \subseteq B$, and the isometry classes of unimodular F-structures of $(L, \mathfrak{q})$ are in bijection with the conjugacy classes of A over $\mathrm{Aut}(L, \mathfrak{q})$.

Proof. Let $\nu \in A$. By (ii) of the previous lemma, $g := \frac{1}{D}\nu\sigma\mathcal{G} \in \mathrm{Aut}(L, \mathfrak{q})$. So $\nu = Dg\mathcal{G}^{-1}\sigma^{-1} \in B$. The second statement follows from (iii) of the previous lemma. □

Example 10.5 (Extremal Lattices of Dimension 20 over $\mathbb{Q}[\sqrt{2}]$)
Bachoc constructed an extremal 2-modular lattice of dimension 40 with a 10-dimensional unimodular lattice over the Hurwitz order in [Bac97]. Let L_2 be the lattice, and let G_2 be its Gram matrix (see Appendix C).

There is precisely one extremal even unimodular lattice of dimension 20 over $\mathbb{Q}[\sqrt{2}]$ such that the trace lattice with respect to $\frac{1}{2}$ is L_2.

We prove this with the previous theorem. We find five conjugacy classes of similarities between $L_2^{\#}$ and L_2. Let $\nu_1, \ldots, \nu_5$ be representatives. Then $\nu_i^2 = 2$ and each ν_i defines a $\sqrt{2}$-structure $\Lambda^{(i)}$ over L_2, i.e. a unimodular lattice of dimension 20 over $\mathbb{Q}[\sqrt{2}]$.

Each structure $\Lambda^{(i)}$ is determined by the trace lattices $\Lambda_1^{(i)}$ and $\Lambda_2^{(i)} = L_2$. The first trace lattice is given by the Gram matrix $\frac{2-\nu_i}{2}G_2$. Two of the five structures are even lattices. Both their first trace lattices are extremal even unimodular, but only one of them is an extremal Type (i) $\mathbb{Z}[\sqrt{2}]$-lattice. The Gram matrices of its trace lattices are given in Appendix C.

Example 10.6 (Extremal Lattices of Dimension 6 over $\mathbb{Q}[\sqrt{3}]$)
In the same way we can show that the Coxeter-Todd lattice K_{12} is the trace lattice of precisely one Type (ii) lattice over $\mathbb{Q}[\sqrt{3}]$.

Let G_1 be the Gram matrix of K_{12}. By applying the last theorem we find that K_{12} has three conjugacy classes of $\sqrt{3}$-endomorphisms. If ν is such an endomorphism, then $G_1\frac{3-\nu}{3}$ is the Gram matrix of a 2-modular lattice. The lattice to one conjugacy class is even, call it L_2. By Theorem 2.6, there is an even unimodular lattice Λ of dimension 6 over $\mathbb{Q}[\sqrt{3}]$ such that $\Lambda_1 = K_{12}$ and $\Lambda_2 = L_2$.

We compute that Λ is Galois-invariant and hence an extremal Type (ii) lattice. Its Gram matrix is given in Appendix D.

Since the other two $\sqrt{3}$-endomorphisms of K_{12} yield non-even lattices, Λ is unique.

Appendix A

Algorithms

```
//1. Hilbert Series of even Hilbert modular forms
//2. Eisenstein series of weight 2 in q-Expansion
//3. q-Expansion of Hecke eigenform

////////////////////////////////////
//1. Hilbert series, see Gerhard van der Geer, Hilbert Modular Surfaces, Sringer
//1988, p. 188
function HilbertSeries( D, precision ) //D>1 square-free. F=Q[sqrt(D)] real
//quadratic number field. Returns Hilbert series 1 + \sum_{k=1}^\infty dim M_2k(
//SL_2(Z_K) )
 F:=QuadraticField(D);
 _<t>:=PolynomialRing(Rationals());
 N:=ideal<Integers(F)|1>;
 D:=Discriminant(Integers(F));
 h:=ClassNumber(F);
 if D eq 5 then
  HSeries := (1+t^10)/(1-t)/(1-t^3)/(1-t^5);
  return PowerSeriesRing(Rationals(),precision)!HSeries, HSeries;
 end if;
 if D eq 12 then
  HSeries:= (1-t^6)*(1-t^8)/(1-t)/(1-t^2)/(1-t^3)/(1-t^4);
  return PowerSeriesRing(Rationals(),precision)!HSeries, HSeries;
 end if;
 if D eq 8 then
  a3plus := 1;
 else
  if D mod 3 eq 0 then
   a3plus:=1/2*h*(-3*D);
  else
   if (D mod 9 eq 3 ) then
    a3plus:=4*h*(-D/3);
   else
    a3plus:=3*h*(-3*D);
   end if;
  end if;
 end if;
 d1:=h+Dimension(HilbertCuspForms(F,N,[2,2]));
 d2:=h+Dimension(HilbertCuspForms(F,N,[4,4]));
 d3:=h+Dimension(HilbertCuspForms(F,N,[6,6]));
 chiYGamma := d1-ClassNumber(F)+1; //because dim M_2 = b_1 + 2, b_1 =
chi(YGamma)+h-3
```

```
 zetaFm1 := 1/4*(d3-2*d2+chiYGamma+h-2/3*a3plus);
 chi := d3-3*d2+3*chiYGamma+2*h-a3plus;
 b:=[ 1 , chiYGamma+h-3 , 4*zetaFm1-chi-1/3*a3plus-h+3, 4*zetaFm1+2/3*a3plus-2,
4*zetaFm1-chi-1/3*a3plus-h+3, chiYGamma+h-3 ,1 ];
 HSeries:= &+[ b[i+1]*t^i : i in [0..6] ] / (1-t)^2 / (1-t^3);
 return PowerSeriesRing(Rationals(),precision)!HSeries, HSeries;
end function;


/////////////////////////////////////
//2. The Eisenstein series of weight 2, see Rolf Mueller, Hilbertsche
//Modulformen und Modulfunktionen zu Q[sqrt(8)], Mathematische Annalen 266
//(1984)
function HilbertEisensteinSeries( D , powerbound )//D>1 square-free. Computes
Hilbert Eisenstein Series up to q1^powerbound.
 F:=QuadraticField(D);
 zeta:=SiegelZeta(F);
 d:=Discriminant(Integers(F));
 PR<q1,q2>:=PolynomialRing(F,2);
 if (Floor(d-Sqrt(d)))/2 in Integers() then  //(n *c - m)/2 is power of q1, (n c
- m)/2 should be positive. c is a constant
  c:=d - Floor(d-Sqrt(d)) +2;
 else
  c:=d - Floor(d-Sqrt(d))+1;
 end if;
 kappa:=Integers()!4*zeta^-1;

 g2:=PR!1;
 for n in [1..powerbound] do//computing coefficients
  if n*D/2 in Integers() then
   g2+:= kappa * HilbertSigma(F,[0,n]) * q1^n * q2^(Integers()!(c*n/2));
  end if;
  for m in [m : m in [1..Floor(n*Sqrt(d))] | (m-n*d)/2 in Integers()] do
   g2+:= kappa * HilbertSigma(F,[m,n]) * q1^n * q2^(Integers()!(c*n/2-m/2) ) +
kappa * HilbertSigma(F,[m,n]) * q1^n * q2^(Integers()!(c*n/2+m/2));
  end for;
 end for;
 return g2;
end function;

SiegelZeta:=function(F)  //F tot. real number field. Returns Zeta function over
F evaluated at -1, accorduing to Siegel, see van de Geer, p.20.
 d:=Discriminant(Integers(F));
 bound:=Floor( Sqrt(d-4));
 zetaF:=0;
 for n in [-bound..bound] do
  x:=((d-n^2)/4);
  if x in Integers() then
```

```
   Factors:=Factorisation(Integers()!x);
   sigma1:=1;
   for I in Factors do
    InnerSum:=1 + &+[I[1]^i : i in [1..I[2]]];
    sigma1*:=InnerSum;
   end for;
   zetaF+:=sigma1;
  end if;
  end for;
 return zetaF/60;
end function;

HilbertSigma:=function(F,mn); // mn=[m,n]. Return sigma_(k-1)(ZF,nu), see
Mueller, p.87.
 ZF:=Integers(F); N:=ideal< ZF | 1>;
 if Discriminant(Integers(F))/4 in Integers() then
  Nu:=ideal<ZF | mn[1]/2 + mn[2]*F.1>;
 else
  Nu:=ideal<ZF | mn[1]/2 + mn[2]/2*F.1 >;
 end if;
 Factors:=Factorisation(Nu);
 sigma:=1;
 for I in Factors do
  InnerSum:=1+ &+[Abs(Norm(I[1]^i)) : i in [1..I[2]]];
  sigma*:=InnerSum;
 end for;
 return sigma;
end function;


////////////////////////////////////
//3. q-Expansion of Hecke eigenform, see MAGMA'S Hilbert modular forms package
//by Dembele and Voight
function MyqExpansion(D,k, eigenform, powerbound)  //returns q1q2 expansion of
//eigenform up to q1^powerbound
 F:=QuadraticField(D);
 d:=Discriminant(Integers(F));
 if (Floor(d-Sqrt(d))) mod 2 eq 0  then
  c:=d - Floor(d-Sqrt(d))+2;
 else
  c:=d - Floor(d-Sqrt(d))+1;
 end if; //(n *c - m)/2 is power of q1, (n c - m)/2 should be positive.
 if d mod 4 eq 0 then
  eps:=1;
 else
  eps:=2;
 end if; //Discriminant determs formula for elements of inverse different
 PR1<q1,q2>:=PolynomialRing(F,2);
```

```
 s:=PR1!0;
 for n in [1..powerbound] do
  if n*d mod 2 eq 0  then
   ideal:=ideal<Integers(F) | n/eps*F.1>;
   a:=MyHeckeEigenvalue(F,k,eigenform,ideal);
   if a notin Integers() and not Parent(a) subset BaseRing(PR1) then
    PR1<q1,q2>:=PolynomialRing(Parent(a), 2);
   end if;
   s:=PR1!s + PR1!a* PR1.1^n * PR1.2^(Integers()!(c*n/2));
  end if;
  for m in [m : m in [1..Floor(n*Sqrt(d))] | (m-n*d)/2 in Integers()] do
   ideal:=ideal<Integers(F) | m/2+n/eps*F.1>;
   a:=MyHeckeEigenvalue(F,k,eigenform,ideal);
   if a notin Integers() and not Parent(a) subset BaseRing(PR1) then
    PR1<q1,q2>:=PolynomialRing(Parent(a), 2);
   end if;
   s:=PR1!s + PR1!a* PR1.1^n * PR1.2^(Integers()!(c*n/2+m/2) );

   ideal:=ideal<Integers(F) | -m/2+n/eps*F.1>;
   a:=MyHeckeEigenvalue(F,k,eigenform,ideal);  //equal to a above iff s is
symmetric
   if a notin Integers() and not Parent(a) subset BaseRing(PR1) then
    PR1<q1,q2>:=PolynomialRing(Parent(a), 2);
   end if;
   s:=PR1!s + PR1!a* PR1.1^n * PR1.2^(Integers()!(c*n/2-m/2));
  end for;
 end for;
 return Monic(s);
end function;

function MyHeckeEigenvalue(F,k, eigenform , ideal ) // eigenform f, ideal<>1 in
//Z_F. Returns Hecke eigenvalue of f, see 11.2..
//Acording to [Dembele, Shimura78 (prop. 2.1)] right??
 if ideal eq ideal<Integers(F)|1> then
  return 1;  // Because MAGMA gives the norminated HEF - but HeckeEigenvalue
doesnot work for ideals notequal prim ideals
 end if;
 factors:=Factorization(ideal);
 if #factors eq 1 then
  P:=factors[1];
  if P[2] eq 1 then
   return HeckeEigenvalue( eigenform , P[1] );
  else
   return HeckeEigenvalue(eigenform,P[1]) *
MyHeckeEigenvalue(F,k,eigenform,P[1]^(P[2]-1)) - Norm(P[1])^(k-1) *
MyHeckeEigenvalue(F,k,eigenform, P[1]^(P[2]-2));
  end if;
 end if;
```

```
 return &*[ MyHeckeEigenvalue(F,k, eigenform, a[1]^(a[2]) ) : a in factors ];
end function;

Monic:=function(f) //returns monic form of f
 if f eq 0 then return f; end if;
 ab:=q1q2Valuation(f);
 return f/q1q2Coefficient(f,ab);
end function;

q1q2Valuation:=function(f);  //f in P. Return Order of f
 if f eq Parent(f)!0 then
  return [-1,-1];
 end if;
 m:=Monomials(f)[#Monomials(f)];
 return
[Degree(Evaluate(m,[Parent(f).1,1])),Degree(Evaluate(m,[1,Parent(f).2]))];
end function;

q1q2Coefficient:=function( f, ab ) // f in C[[q1,q2]], return coefficient of
q1^a q2^b
 if ab eq [-1,-1] then
  return 0;
 else
  return Coefficient( Coefficient(f , Parent(f).1, ab[1]) , Parent(f).2, ab[2]);
 end if;
end function;
```

Appendix B

Data for $\mathbb{Q}[\sqrt{5}]$

Extremal Hilbert modular forms

k	Extremal Hilbert modulal form
2	$1 + 120q_1^1q_2^2 + 120q_1^1q_2^3 + 120q_1^2q_2^3 + 600q_1^2q_2^4 + 720q_1^2q_2^5 + 600q_1^2q_2^6 + 120q_1^2q_2^7 + 720q_1^3q_2^5 + 1200q_1^3q_2^6 + 1440q_1^3q_2^7 + 1440q_1^3q_2^8 + 1200q_1^3q_2^9 + 720q_1^3q_2^{10} + 600q_1^4q_2^6 + 1440q_1^4q_2^7 + 2520q_1^4q_2^8 + 2400q_1^4q_2^9 + 3600q_1^4q_2^{10} + 2400q_1^4q_2^{11} + 2520q_1^4q_2^{12} + 1440q_1^4q_2^{13} + 600q_1^4q_2^{14} + O(q_1^5)$
4	$1 + 240q_1^1q_2^2 + 240q_1^1q_2^3 + 240q_1^2q_2^3 + 15600q_1^2q_2^4 + 30240q_1^2q_2^5 + 15600q_1^2q_2^6 + 240q_1^2q_2^7 + 30240q_1^3q_2^5 + 175200q_1^3q_2^6 + 319680q_1^3q_2^7 + 319680q_1^3q_2^8 + 175200q_1^3q_2^9 + 30240q_1^3q_2^{10} + 15600q_1^4q_2^6 + 319680q_1^4q_2^7 + 998640q_1^4q_2^8 + 1646400q_1^4q_2^9 + 1965600q_1^4q_2^{10} + 1646400q_1^4q_2^{11} + 998640q_1^4q_2^{12} + 319680q_1^4q_2^{13} + 15600q_1^4q_2^{14} + O(q_1^5)$
6	$1 + 37800q_1^2q_2^4 + 120960q_1^2q_2^5 + 37800q_1^2q_2^6 + 120960q_1^3q_2^5 + 2217600q_1^3q_2^6 + 6048000q_1^3q_2^7 + 6048000q_1^3q_2^8 + 2217600q_1^3q_2^9 + 120960q_1^3q_2^{10} + 37800q_1^4q_2^6 + 6048000q_1^4q_2^7 + 39501000q_1^4q_2^8 + 93139200q_1^4q_2^9 + 120582000q_1^4q_2^{10} + 93139200q_1^4q_2^{11} + 39501000q_1^4q_2^{12} + 6048000q_1^4q_2^{13} + 37800q_1^4q_2^{14} + O(q_1^5)$
8	$1 + 21600q_1^2q_2^4 + 103680q_1^2q_2^5 + 21600q_1^2q_2^6 + 103680q_1^3q_2^5 + 6355200q_1^3q_2^6 + 25920000q_1^3q_2^7 + 25920000q_1^3q_2^8 + 6355200q_1^3q_2^9 + 103680q_1^3q_2^{10} + 21600q_1^4q_2^6 + 25920000q_1^4q_2^7 + 356940000q_1^4q_2^8 + 1188518400q_1^4q_2^9 + 1702036800q_1^4q_2^{10} + 1188518400q_1^4q_2^{11} + 356940000q_1^4q_2^{12} + 25920000q_1^4q_2^{13} + 21600q_1^4q_2^{14} + O(q_1^5)$
10	$1 + 39600q_1^2q_2^5 + 39600q_1^3q_2^5 + 6270000q_1^3q_2^6 + 37620000q_1^3q_2^7 + 37620000q_1^3q_2^8 + 6270000q_1^3q_2^9 + 39600q_1^3q_2^{10} + 37620000q_1^4q_2^7 + 1097415000q_1^4q_2^8 + 5161860000q_1^4q_2^9 + 8186112000q_1^4q_2^{10} + 5161860000q_1^4q_2^{11} + 1097415000q_1^4q_2^{12} + 37620000q_1^4q_2^{13} + O(q_1^5)$
12	$1 + 2620800q_1^3q_2^6 + 23587200q_1^3q_2^7 + 23587200q_1^3q_2^8 + 2620800q_1^3q_2^9 + 23587200q_1^4q_2^7 + 1447664400q_1^4q_2^8 + 9599990400q_1^4q_2^9 + 16864848000q_1^4q_2^{10} + 9599990400q_1^4q_2^{11} + 1447664400q_1^4q_2^{12} + 23587200q_1^4q_2^{13} + O(q_1^5)$
14	$1 + 537600q_1^3q_2^6 + 7257600q_1^3q_2^7 + 7257600q_1^3q_2^8 + 537600q_1^3q_2^9 + 7257600q_1^4q_2^7 + 950632200q_1^4q_2^8 + 8876044800q_1^4q_2^9 + 17289417600q_1^4q_2^{10} + 8876044800q_1^4q_2^{11} + 950632200q_1^4q_2^{12} + 7257600q_1^4q_2^{13} + O(q_1^5)$
16	$1 + 1305600q_1^3q_2^7 + 1305600q_1^3q_2^8 + 1305600q_1^4q_2^7 + 344107200q_1^4q_2^8 + 4536960000q_1^4q_2^9 + 9760012800q_1^4q_2^{10} + 4536960000q_1^4q_2^{11} + 344107200q_1^4q_2^{12} + 1305600q_1^4q_2^{13} + O(q_1^5)$
18	$1 + 75411000q_1^4q_2^8 + 1378944000q_1^4q_2^9 + 3309465600q_1^4q_2^{10} + 1378944000q_1^4q_2^{11} + 75411000q_1^4q_2^{12} + O(q_1^5)$
20	$1 + 304920000q_1^4q_2^9 + 640332000q_1^4q_2^{10} + 304920000q_1^4q_2^{11} + O(q_1^5)$

Appendix C

Data for $\mathbb{Q}[\sqrt{2}]$

Extremal Hilbert modular forms

k	Extremal Hilbert modulal form
2	$1+48q_1^1q_2^1+144q_1^1q_2^2+48q_1^1q_2^3+336q_1^2q_2^2+384q_1^2q_2^3+720q_1^2q_2^4+384q_1^2q_2^5$ $+336q_1^2q_2^6+144q_1^3q_2^2+480q_1^3q_2^3+1152q_1^3q_2^4+864q_1^3q_2^5+1440q_1^3q_2^6$ $+864q_1^3q_2^7+1152q_1^3q_2^8+480q_1^3q_2^9+144q_1^3q_2^{10}+O(q_1^4q_2^3)$
4	$1+480q_1^1q_2^2+3360q_1^2q_2^2+15360q_1^2q_2^3+24480q_1^2q_2^4+15360q_1^2q_2^5+3360q_1^2q_2^6$ $+480q_1^3q_2^2+30720q_1^3q_2^3+134400q_1^3q_2^4+215040q_1^3q_2^5+288960q_1^3q_2^6$ $+215040q_1^3q_2^7+134400q_1^3q_2^8+30720q_1^3q_2^9+480q_1^3q_2^{10}+O(q_1^4q_2^3)$
6	$1+3024q_1^2q_2^2+48384q_1^2q_2^3+93744q_1^2q_2^4+48384q_1^2q_2^5+3024q_1^2q_2^6+161280q_1^3q_2^3$ $+1548288q_1^3q_2^4+3967488q_1^3q_2^5+5419008q_1^3q_2^6+3967488q_1^3q_2^7+1548288q_1^3q_2^8$ $+161280q_1^3q_2^9+O(q_1^4q_2^3)$
8	$1+34560q_1^2q_2^3+77760q_1^2q_2^4+34560q_1^2q_2^5+192000q_1^3q_2^3+4147200q_1^3q_2^4$ $+15966720q_1^3q_2^5+24145920q_1^3q_2^6+15966720q_1^3q_2^7+4147200q_1^3q_2^8$ $+192000q_1^3q_2^9+O(q_1^4q_2^3)$
10	$1+39600q_1^2q_2^4+84480q_1^3q_2^3+3928320q_1^3q_2^4+21542400q_1^3q_2^5$ $+36748800q_1^3q_2^6+21542400q_1^3q_2^7+3928320q_1^3q_2^8+84480q_1^3q_2^9+O(q_1^4q_2^4)$
12	$1+1572480q_1^3q_2^4+12579840q_1^3q_2^5+24111360q_1^3q_2^6+12579840q_1^3q_2^7$ $+1572480q_1^3q_2^8+O(q_1^4q_2^4)$
14	$1+4838400q_1^3q_2^5+5913600q_1^3q_2^6+4838400q_1^3q_2^7+O(q_1^4q_2^4)$
16	$1+2611200q_1^3q_2^6+O(q_1^4q_2^5)$
18	$1+327499200q_1^4q_2^6+O(q_1^4q_2^7)$
20	$1+487872000q_1^4q_2^7+O(q_1^4q_2^8)$

Dimension 4
Name: Δ_4', F_4.
Gram matrix:

$$\begin{pmatrix} 2 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & 2 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 2 \end{pmatrix}$$

Dimension 8
Name: Λ_8
Gram matrix:

$$\begin{pmatrix}
4+2\sqrt{2} & 2+2\sqrt{2} & -10-7\sqrt{2} & 2+1\sqrt{2} & 1+1\sqrt{2} & -2+1\sqrt{2} & -7+4\sqrt{2} & 37-26\sqrt{2} \\
2+2\sqrt{2} & 4+2\sqrt{2} & -14-10\sqrt{2} & 1+1\sqrt{2} & 2+1\sqrt{2} & -2\sqrt{2} & 4-4\sqrt{2} & -27+18\sqrt{2} \\
-10-7\sqrt{2} & -14-10\sqrt{2} & 116+82\sqrt{2} & -10-7\sqrt{2} & -14-10\sqrt{2} & 6+4\sqrt{2} & 3+1\sqrt{2} & -1+6\sqrt{2} \\
2+1\sqrt{2} & 1+1\sqrt{2} & -10-7\sqrt{2} & 4 & 2 & -2+2\sqrt{2} & -17+12\sqrt{2} & 84-60\sqrt{2} \\
1+1\sqrt{2} & 2+1\sqrt{2} & -14-10\sqrt{2} & 2 & 4 & -1-1\sqrt{2} & -6+4\sqrt{2} & 22-16\sqrt{2} \\
-2+1\sqrt{2} & -2\sqrt{2} & 6+4\sqrt{2} & -2+2\sqrt{2} & -1-1\sqrt{2} & 12-4\sqrt{2} & 39-30\sqrt{2} & -210+150\sqrt{2} \\
-7+4\sqrt{2} & 4-4\sqrt{2} & 3+1\sqrt{2} & -17+12\sqrt{2} & -6+4\sqrt{2} & 39-30\sqrt{2} & 252-172\sqrt{2} & -1279+903\sqrt{2} \\
37-26\sqrt{2} & -27+18\sqrt{2} & -1+6\sqrt{2} & 84-60\sqrt{2} & 22-16\sqrt{2} & -210+150\sqrt{2} & -1279+903\sqrt{2} & 6720-4750\sqrt{2}
\end{pmatrix}$$

Dimension 12
There are five extremal lattices. We give there Gram matricies. The first is constructed by using the ζ_5-structure of the Leech lattice (root of unity automorphism).

$$\begin{pmatrix}
4 & 2 & 1+1\sqrt{2} & 2+2\sqrt{2} & 2 & -1+1\sqrt{2} & -1-1\sqrt{2} & -17+12\sqrt{2} & -13+10\sqrt{2} & 24-16\sqrt{2} & -244+173\sqrt{2} & -1769+1252\sqrt{2} \\
2 & 4 & 2+1\sqrt{2} & 1+2\sqrt{2} & -1-2\sqrt{2} & -1+1\sqrt{2} & -3\sqrt{2} & -18+12\sqrt{2} & 4-2\sqrt{2} & -17+13\sqrt{2} & 604-426\sqrt{2} & 1936-1368\sqrt{2} \\
1+1\sqrt{2} & 2+1\sqrt{2} & 8+4\sqrt{2} & 4+3\sqrt{2} & -4-4\sqrt{2} & 2-5\sqrt{2} & -4 & 7-1\sqrt{2} & 0 & -24+22\sqrt{2} & 741-524\sqrt{2} & 3942-2786\sqrt{2} \\
2+2\sqrt{2} & 1+2\sqrt{2} & 4+3\sqrt{2} & 8+2\sqrt{2} & -2\sqrt{2} & -6+3\sqrt{2} & -2+1\sqrt{2} & 14-10\sqrt{2} & -18+13\sqrt{2} & -47+34\sqrt{2} & -323+228\sqrt{2} & 815-574\sqrt{2} \\
2 & -1-2\sqrt{2} & -4-4\sqrt{2} & -2\sqrt{2} & 48+22\sqrt{2} & 18-2\sqrt{2} & 16+12\sqrt{2} & 1-2\sqrt{2} & -32+35\sqrt{2} & 6+1\sqrt{2} & 393-284\sqrt{2} & 3622-2560\sqrt{2} \\
-1+1\sqrt{2} & -1+1\sqrt{2} & 2-5\sqrt{2} & -6+3\sqrt{2} & 18-2\sqrt{2} & 88-56\sqrt{2} & -16+13\sqrt{2} & 41-33\sqrt{2} & -18+15\sqrt{2} & 264-189\sqrt{2} & 2660-1883\sqrt{2} & 10686-7554\sqrt{2} \\
-1-1\sqrt{2} & -3\sqrt{2} & -4 & -2+1\sqrt{2} & 16+12\sqrt{2} & -16+13\sqrt{2} & 36-4\sqrt{2} & -62+41\sqrt{2} & 31-18\sqrt{2} & 64-39\sqrt{2} & -900+636\sqrt{2} & -8711+6159\sqrt{2} \\
-17+12\sqrt{2} & -18+12\sqrt{2} & 7-1\sqrt{2} & 14-10\sqrt{2} & 1-2\sqrt{2} & 41-33\sqrt{2} & -62+41\sqrt{2} & 358-246\sqrt{2} & -42+30\sqrt{2} & -380+270\sqrt{2} & 1631-1155\sqrt{2} & 31590-22339\sqrt{2} \\
-13+10\sqrt{2} & 4-2\sqrt{2} & 0 & -18+13\sqrt{2} & -32+35\sqrt{2} & -18+15\sqrt{2} & 31-18\sqrt{2} & -42+30\sqrt{2} & 376-262\sqrt{2} & -61+46\sqrt{2} & 4682-3311\sqrt{2} & 4007-2833\sqrt{2} \\
24-16\sqrt{2} & -17+13\sqrt{2} & -24+22\sqrt{2} & -47+34\sqrt{2} & 6+1\sqrt{2} & 264-189\sqrt{2} & 64-39\sqrt{2} & -380+270\sqrt{2} & -61+46\sqrt{2} & 4034-2818\sqrt{2} & -28547+20180\sqrt{2} & -205335+145196\sqrt{2} \\
-244+173\sqrt{2} & 604-426\sqrt{2} & 741-524\sqrt{2} & -323+228\sqrt{2} & 393-284\sqrt{2} & 2660-1883\sqrt{2} & -900+636\sqrt{2} & 1631-1155\sqrt{2} & 4682-3311\sqrt{2} & -28547+20180\sqrt{2} & 716940-506884\sqrt{2} & 3530381-2496385\sqrt{2} \\
-1769+1252\sqrt{2} & 1936-1368\sqrt{2} & 3942-2786\sqrt{2} & 815-574\sqrt{2} & 3622-2560\sqrt{2} & 10686-7554\sqrt{2} & -8711+6159\sqrt{2} & 31590-22339\sqrt{2} & 4007-2833\sqrt{2} & -205335+145196\sqrt{2} & 3530381-2496385\sqrt{2} & 20342032-14383974\sqrt{2}
\end{pmatrix}$$

$$
\begin{pmatrix}
4 & 2 & 0 & -3+2\sqrt{2} & 32-22\sqrt{2} & 8-6\sqrt{2} & 180-128\sqrt{2} & -12+10\sqrt{2} & 0 & -24+16\sqrt{2} & -4851-3432\sqrt{2} & -118840-84032\sqrt{2} \\
2 & 4 & 1 & -2+2\sqrt{2} & 32-22\sqrt{2} & 7-6\sqrt{2} & 103-74\sqrt{2} & 2 & -1-2\sqrt{2} & 69-50\sqrt{2} & -4787-3385\sqrt{2} & -117970-83417\sqrt{2} \\
0 & 1 & 4 & -2+1\sqrt{2} & 32-22\sqrt{2} & 11-9\sqrt{2} & -2+1\sqrt{2} & 12-10\sqrt{2} & -2-4\sqrt{2} & 185-131\sqrt{2} & -4934-3490\sqrt{2} & -119016-84156\sqrt{2} \\
-3+2\sqrt{2} & -2+2\sqrt{2} & -2+1\sqrt{2} & 8-4\sqrt{2} & -76+54\sqrt{2} & -20+14\sqrt{2} & -214+151\sqrt{2} & -3+3\sqrt{2} & -3-3\sqrt{2} & -206+146\sqrt{2} & -2007-1415\sqrt{2} & -48933-34600\sqrt{2} \\
32-22\sqrt{2} & 32-22\sqrt{2} & 32-22\sqrt{2} & -76+54\sqrt{2} & 1446-1022\sqrt{2} & 469-332\sqrt{2} & 4105-2903\sqrt{2} & 0 & -22+13\sqrt{2} & 3679-2602\sqrt{2} & -2751-2156\sqrt{2} & -70601-49792\sqrt{2} \\
8-6\sqrt{2} & 7-6\sqrt{2} & 11-9\sqrt{2} & -20+14\sqrt{2} & 469-332\sqrt{2} & 170-116\sqrt{2} & 1339-946\sqrt{2} & -10+9\sqrt{2} & 2\sqrt{2} & 1160-822\sqrt{2} & 4776+3350\sqrt{2} & 116019+82097\sqrt{2} \\
180-128\sqrt{2} & 103-74\sqrt{2} & -2+1\sqrt{2} & -214+151\sqrt{2} & 4105-2903\sqrt{2} & 1339-946\sqrt{2} & 24252-17148\sqrt{2} & -2016+1424\sqrt{2} & 36-23\sqrt{2} & -4373+3093\sqrt{2} & 3101+1455\sqrt{2} & 62737+44917\sqrt{2} \\
-12+10\sqrt{2} & 2 & 12-10\sqrt{2} & -3+3\sqrt{2} & 0 & -10+9\sqrt{2} & -2016+1424\sqrt{2} & 348-230\sqrt{2} & -29+6\sqrt{2} & 2391-1696\sqrt{2} & 77+167\sqrt{2} & 4936+3458\sqrt{2} \\
0 & -1-2\sqrt{2} & -2-4\sqrt{2} & -3-3\sqrt{2} & -22+13\sqrt{2} & 2\sqrt{2} & 36-23\sqrt{2} & -29+6\sqrt{2} & 56+16\sqrt{2} & -164+118\sqrt{2} & 20651+14593\sqrt{2} & 495662+350479\sqrt{2} \\
-24+16\sqrt{2} & 69-50\sqrt{2} & 185-131\sqrt{2} & -206+146\sqrt{2} & 3679-2602\sqrt{2} & 1160-822\sqrt{2} & -4373+3093\sqrt{2} & 2391-1696\sqrt{2} & -164+118\sqrt{2} & 26878-19000\sqrt{2} & 129-183\sqrt{2} & -1505-964\sqrt{2} \\
-4851-3432\sqrt{2} & -4787-3385\sqrt{2} & -4934-3490\sqrt{2} & -2007-1415\sqrt{2} & -2751-2156\sqrt{2} & 4776+3350\sqrt{2} & 3101+1455\sqrt{2} & 77+167\sqrt{2} & 20651+14593\sqrt{2} & 129-183\sqrt{2} & 33058936+23373378\sqrt{2} & 803558094+568201151\sqrt{2} \\
-118840-84032\sqrt{2} & -117970-83417\sqrt{2} & -119016-84156\sqrt{2} & -48933-34600\sqrt{2} & -70601-49792\sqrt{2} & 116019+82097\sqrt{2} & 62737+44917\sqrt{2} & 4936+3458\sqrt{2} & 495662+350479\sqrt{2} & -1505-964\sqrt{2} & 803558094+568201151\sqrt{2} & 19536428860+13814341276\sqrt{2}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & 0 & -3+2\sqrt{2} & 32-22\sqrt{2} & 8-6\sqrt{2} & -12+10\sqrt{2} & 2\sqrt{2} & -1-2\sqrt{2} & 458-324\sqrt{2} & -9-9\sqrt{2} & -44+31\sqrt{2} & 551238-389784\sqrt{2} \\
0 & 4 & -2+1\sqrt{2} & 32-22\sqrt{2} & 13-9\sqrt{2} & 12-10\sqrt{2} & -3\sqrt{2} & 1\sqrt{2} & 488-346\sqrt{2} & 36+37\sqrt{2} & 34-25\sqrt{2} & 600341-424505\sqrt{2} \\
-3+2\sqrt{2} & -2+1\sqrt{2} & 8-4\sqrt{2} & -76+54\sqrt{2} & -21+16\sqrt{2} & -3+3\sqrt{2} & -6\sqrt{2} & 1\sqrt{2} & -1144+809\sqrt{2} & -4-5\sqrt{2} & 107-76\sqrt{2} & -1393990+985700\sqrt{2} \\
32-22\sqrt{2} & 32-22\sqrt{2} & -76+54\sqrt{2} & 1446-1022\sqrt{2} & 501-354\sqrt{2} & 0 & -66+45\sqrt{2} & 52-36\sqrt{2} & 21664-15319\sqrt{2} & -92+69\sqrt{2} & -95+66\sqrt{2} & 26344878-18628642\sqrt{2} \\
8-6\sqrt{2} & 13-9\sqrt{2} & -21+16\sqrt{2} & 501-354\sqrt{2} & 190-128\sqrt{2} & 7-1\sqrt{2} & -19+5\sqrt{2} & 20-17\sqrt{2} & 7516-5314\sqrt{2} & -197-63\sqrt{2} & -255+183\sqrt{2} & 9130158-6455995\sqrt{2} \\
-12+10\sqrt{2} & 12-10\sqrt{2} & -3+3\sqrt{2} & 0 & 7-1\sqrt{2} & 348-230\sqrt{2} & -33+9\sqrt{2} & -35+1\sqrt{2} & 336-240\sqrt{2} & -185-24\sqrt{2} & 422-297\sqrt{2} & 587763-415610\sqrt{2} \\
2\sqrt{2} & -3\sqrt{2} & -6\sqrt{2} & -66+45\sqrt{2} & -19+5\sqrt{2} & -33+9\sqrt{2} & 64+4\sqrt{2} & -4+9\sqrt{2} & -976+696\sqrt{2} & 43+45\sqrt{2} & 243-183\sqrt{2} & -1205328+852305\sqrt{2} \\
-1-2\sqrt{2} & 1\sqrt{2} & 1\sqrt{2} & 52-36\sqrt{2} & 20-17\sqrt{2} & -35+1\sqrt{2} & -4+9\sqrt{2} & 64+14\sqrt{2} & 750-529\sqrt{2} & -3+23\sqrt{2} & -1892+1362\sqrt{2} & 865293-611863\sqrt{2} \\
458-324\sqrt{2} & 488-346\sqrt{2} & -1144+809\sqrt{2} & 21664-15319\sqrt{2} & 7516-5314\sqrt{2} & 336-240\sqrt{2} & -976+696\sqrt{2} & 750-529\sqrt{2} & 325012-229816\sqrt{2} & -1587+1086\sqrt{2} & -115+50\sqrt{2} & 395421149-279604975\sqrt{2} \\
-9-9\sqrt{2} & 36+37\sqrt{2} & -4-5\sqrt{2} & -92+69\sqrt{2} & -197-63\sqrt{2} & -185-24\sqrt{2} & 43+45\sqrt{2} & -3+23\sqrt{2} & -1587+1086\sqrt{2} & 8820+4132\sqrt{2} & -12828+8903\sqrt{2} & -2149987+1520317\sqrt{2} \\
-44+31\sqrt{2} & 34-25\sqrt{2} & 107-76\sqrt{2} & -95+66\sqrt{2} & -255+183\sqrt{2} & 422-297\sqrt{2} & 243-183\sqrt{2} & -1892+1362\sqrt{2} & -115+50\sqrt{2} & -12828+8903\sqrt{2} & 391434-272896\sqrt{2} & 7147120-5053528\sqrt{2} \\
551238-389784\sqrt{2} & 600341-424505\sqrt{2} & -1393990+985700\sqrt{2} & 26344878-18628642\sqrt{2} & 9130158-6455995\sqrt{2} & 587763-415610\sqrt{2} & -1205328+852305\sqrt{2} & 865293-611863\sqrt{2} & 395421149-279604975\sqrt{2} & -2149987+1520317\sqrt{2} & 7147120-5053528\sqrt{2} & 481310861398-340338173794\sqrt{2}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & 1+1\sqrt{2} & 2+2\sqrt{2} & 0 & 1-1\sqrt{2} & 0 & 1-1\sqrt{2} & -16+11\sqrt{2} & -17+12\sqrt{2} & -10+7\sqrt{2} & -196+139\sqrt{2} & -1367+966\sqrt{2} \\
1+1\sqrt{2} & 8+4\sqrt{2} & 4+2\sqrt{2} & 1\sqrt{2} & -3+1\sqrt{2} & -6-6\sqrt{2} & -2+2\sqrt{2} & 4-3\sqrt{2} & 1 & -53+42\sqrt{2} & -72+52\sqrt{2} & -280+199\sqrt{2} \\
2+2\sqrt{2} & 4+2\sqrt{2} & 8+4\sqrt{2} & -3-1\sqrt{2} & 2\sqrt{2} & -6-6\sqrt{2} & -2+2\sqrt{2} & 5-3\sqrt{2} & 2 & -72+49\sqrt{2} & -78+56\sqrt{2} & -346+244\sqrt{2} \\
0 & 1\sqrt{2} & -3-1\sqrt{2} & 6 & 3-4\sqrt{2} & 2+3\sqrt{2} & -1+3\sqrt{2} & 3-1\sqrt{2} & -18+12\sqrt{2} & 44-33\sqrt{2} & -41+29\sqrt{2} & -258+183\sqrt{2} \\
1-1\sqrt{2} & -3+1\sqrt{2} & 2\sqrt{2} & 3-4\sqrt{2} & 50-30\sqrt{2} & -18+6\sqrt{2} & -15+9\sqrt{2} & -75+55\sqrt{2} & 16-12\sqrt{2} & -132+95\sqrt{2} & -331+232\sqrt{2} & -3729+2635\sqrt{2} \\
0 & -6-6\sqrt{2} & -6-6\sqrt{2} & 2+3\sqrt{2} & -18+6\sqrt{2} & 44+20\sqrt{2} & -8-8\sqrt{2} & 11-19\sqrt{2} & -22+20\sqrt{2} & -49+34\sqrt{2} & -208+136\sqrt{2} & -598+420\sqrt{2} \\
1-1\sqrt{2} & -2+2\sqrt{2} & -2+2\sqrt{2} & -1+3\sqrt{2} & -15+9\sqrt{2} & -8-8\sqrt{2} & 40-8\sqrt{2} & -7+15\sqrt{2} & 14-12\sqrt{2} & 262-195\sqrt{2} & 428-299\sqrt{2} & 2309-1631\sqrt{2} \\
-16+11\sqrt{2} & 4-3\sqrt{2} & 5-3\sqrt{2} & 3-1\sqrt{2} & -75+55\sqrt{2} & 11-19\sqrt{2} & -7+15\sqrt{2} & 304-208\sqrt{2} & 14-12\sqrt{2} & 319-232\sqrt{2} & 1749-1235\sqrt{2} & 15591-11024\sqrt{2} \\
-17+12\sqrt{2} & 1 & 2 & -18+12\sqrt{2} & 16-12\sqrt{2} & -22+20\sqrt{2} & 14-12\sqrt{2} & 14-12\sqrt{2} & 402-282\sqrt{2} & 132-95\sqrt{2} & 3104-2197\sqrt{2} & 17307-12238\sqrt{2} \\
-10+7\sqrt{2} & -53+42\sqrt{2} & -72+49\sqrt{2} & 44-33\sqrt{2} & -132+95\sqrt{2} & -49+34\sqrt{2} & 262-195\sqrt{2} & 319-232\sqrt{2} & 132-95\sqrt{2} & 7340-5106\sqrt{2} & 13064-9269\sqrt{2} & 79305-56091\sqrt{2} \\
-196+139\sqrt{2} & -72+52\sqrt{2} & -78+56\sqrt{2} & -41+29\sqrt{2} & -331+232\sqrt{2} & -208+136\sqrt{2} & 428-299\sqrt{2} & 1749-1235\sqrt{2} & 3104-2197\sqrt{2} & 13064-9269\sqrt{2} & 48668-34382\sqrt{2} & 302179-213664\sqrt{2} \\
-1367+966\sqrt{2} & -280+199\sqrt{2} & -346+244\sqrt{2} & -258+183\sqrt{2} & -3729+2635\sqrt{2} & -598+420\sqrt{2} & 2309-1631\sqrt{2} & 15591-11024\sqrt{2} & 17307-12238\sqrt{2} & 79305-56091\sqrt{2} & 302179-213664\sqrt{2} & 1978680-1399134\sqrt{2}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & 2 & 1+2\sqrt{2} & 3+4\sqrt{2} & 21-15\sqrt{2} & -30+20\sqrt{2} & 12-10\sqrt{2} & -128-91\sqrt{2} & 51273-36256\sqrt{2} & 15+10\sqrt{2} & -10908+7714\sqrt{2} & 1023941-724037\sqrt{2} \\
2 & 4 & 2+2\sqrt{2} & 3+4\sqrt{2} & 18-12\sqrt{2} & -30+20\sqrt{2} & 12-10\sqrt{2} & -172-121\sqrt{2} & 47870-33848\sqrt{2} & 67+48\sqrt{2} & -10164+7186\sqrt{2} & 956295-676203\sqrt{2} \\
1+2\sqrt{2} & 2+2\sqrt{2} & 48+16\sqrt{2} & 1+2\sqrt{2} & -51+44\sqrt{2} & -4\sqrt{2} & 169-148\sqrt{2} & 259+185\sqrt{2} & -138824+98175\sqrt{2} & 311+232\sqrt{2} & 29596-20926\sqrt{2} & -2769367+1958238\sqrt{2} \\
3+4\sqrt{2} & 3+4\sqrt{2} & 1+2\sqrt{2} & 26+8\sqrt{2} & -1\sqrt{2} & -8\sqrt{2} & -73+68\sqrt{2} & -592-419\sqrt{2} & 22393-15838\sqrt{2} & 15+11\sqrt{2} & -4732+3352\sqrt{2} & 446527-315749\sqrt{2} \\
21-15\sqrt{2} & 18-12\sqrt{2} & -51+44\sqrt{2} & -1\sqrt{2} & 494-342\sqrt{2} & -375+265\sqrt{2} & -508+402\sqrt{2} & 170+56\sqrt{2} & 1194847-844885\sqrt{2} & -46-249\sqrt{2} & -254415+179903\sqrt{2} & 23849386-16864061\sqrt{2} \\
-30+20\sqrt{2} & -30+20\sqrt{2} & -4\sqrt{2} & -8\sqrt{2} & -375+265\sqrt{2} & 656-460\sqrt{2} & -234+156\sqrt{2} & 124+80\sqrt{2} & -1053079+744641\sqrt{2} & 35-9\sqrt{2} & 223608-158118\sqrt{2} & -21035751+14874523\sqrt{2} \\
12-10\sqrt{2} & 12-10\sqrt{2} & 169-148\sqrt{2} & -73+68\sqrt{2} & -508+402\sqrt{2} & -234+156\sqrt{2} & 4748-2840\sqrt{2} & 506+89\sqrt{2} & -1844086+1303895\sqrt{2} & -389-1134\sqrt{2} & 391960-277309\sqrt{2} & -36778358+26006193\sqrt{2} \\
-128-91\sqrt{2} & -172-121\sqrt{2} & 259+185\sqrt{2} & -592-419\sqrt{2} & 170+56\sqrt{2} & 124+80\sqrt{2} & 506+89\sqrt{2} & 26520+18668\sqrt{2} & 10478-7408\sqrt{2} & -368-456\sqrt{2} & -2845+1514\sqrt{2} & 207557-146675\sqrt{2} \\
51273-36256\sqrt{2} & 47870-33848\sqrt{2} & -138824+98175\sqrt{2} & 22393-15838\sqrt{2} & 1194847-844885\sqrt{2} & -1053079+744641\sqrt{2} & -1844086+1303895\sqrt{2} & 10478-7408\sqrt{2} & 3195613724-2259640104\sqrt{2} & 69378-50037\sqrt{2} & -679717634+480633070\sqrt{2} & 63790841336-45106936496\sqrt{2} \\
15+10\sqrt{2} & 67+48\sqrt{2} & 311+232\sqrt{2} & 15+11\sqrt{2} & -46-249\sqrt{2} & 35-9\sqrt{2} & -389-1134\sqrt{2} & -368-456\sqrt{2} & 69378-50037\sqrt{2} & 153888+104644\sqrt{2} & -27930+2497\sqrt{2} & 1379727-981435\sqrt{2} \\
-10908+7714\sqrt{2} & -10164+7186\sqrt{2} & 29596-20926\sqrt{2} & -4732+3352\sqrt{2} & -254415+179903\sqrt{2} & 223608-158118\sqrt{2} & 391960-277309\sqrt{2} & -2845+1514\sqrt{2} & -679717634+480633070\sqrt{2} & -27930+2497\sqrt{2} & 144581338-102232872\sqrt{2} & -13568498626+9594377914\sqrt{2} \\
1023941-724037\sqrt{2} & 956295-676203\sqrt{2} & -2769367+1958238\sqrt{2} & 446527-315749\sqrt{2} & 23849386-16864061\sqrt{2} & -21035751+14874523\sqrt{2} & -36778358+26006193\sqrt{2} & 207557-146675\sqrt{2} & 63790841336-45106936496\sqrt{2} & 1379727-981435\sqrt{2} & -13568498626+9594377914\sqrt{2} & 1273393541158-900425207620\sqrt{2}
\end{pmatrix}
$$

Dimension 16 The extremal lattice is given by the following two Gram matricies G_1^{16} and G_2^{16} of degree 32 over $\mathbb{Q}$. The first yields an extremal even unimodular lattice, the second yields an extremal even 2-modular lattice. The $\sqrt{2}$ structure is given by the formula $(G_1^{16})^{-1}G_2^{16} = 2 - \sqrt{2}$.

Dimension 20 The followng matrix G_1^{20} and G_2^{20} are the Gram matrix of an extremal unimodular and an extremal 2-modular lattice in dimension 40. The 2-modular lattice is constructed in [Bac97]. The unimodular lattice is constructed via a similarity of the 2-modular lattice. The similarity is the $\sqrt{2}$-structure and is given by $(G_1^{20})^{-1}G_2^{20} = 2 - \sqrt{2}$.

Dimension 24 The extremal lattice is given by the following two Gram matricies G_1^{24} and G_2^{24} of degree 48 over $\mathbb{Q}$. The first yields an extremal even unimodular lattice, the second yields an extremal even 2-modular lattice. The $\sqrt{2}$ structure is given by the formula $(G_1^{24})^{-1}G_2^{24} = 2 - \sqrt{2}$.

$$
G_1^{16} = \begin{pmatrix}
6 & 0 & -3 & -3 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & -2 & -3 & -3 & 2 & 1 & 2 & 5 & -3 & 1 & -2 & -2 & -2 & -3 & -4 & 0 & -1 & -2 & 2 & 3 & -1 & 1 \\
0 & 6 & -3 & -3 & 0 & 0 & 0 & 0 & -2 & 2 & 3 & -2 & -1 & -1 & 2 & -3 & 0 & 0 & -4 & -4 & -2 & -2 & 2 & -1 & -1 & -1 & 0 & -1 & 3 & -1 & 2 & 0 \\
-3 & -3 & 6 & 3 & 0 & 0 & 0 & 0 & 1 & -1 & -3 & 1 & 2 & 2 & -1 & 0 & 0 & -3 & 2 & 2 & 1 & 1 & -1 & 2 & 1 & 1 & -1 & 2 & -2 & -2 & -1 & 1 \\
-3 & -3 & 3 & 6 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 1 & 3 & 3 & -1 & 1 & -1 & -4 & 3 & 1 & 1 & 1 & 1 & 3 & 3 & 1 & 1 & 2 & -3 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & -3 & -3 & -2 & 2 & -2 & 3 & 2 & 2 & 2 & -3 & 2 & 2 & 1 & 2 & 4 & 4 & -1 & -1 & -2 & 2 & 3 & 3 & -2 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 6 & 3 & 3 & 2 & -2 & -2 & 1 & -2 & -2 & 2 & -1 & 2 & 2 & -3 & -2 & 1 & -2 & 3 & -5 & 0 & 2 & -2 & -2 & 1 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & -3 & 3 & 6 & 3 & 3 & -3 & 1 & -1 & -3 & -3 & -1 & 1 & -1 & -1 & -1 & -3 & -2 & -2 & 4 & -3 & 0 & 0 & -3 & -3 & 0 & 2 & 2 & -1 \\
0 & 0 & 0 & 0 & -3 & 3 & 3 & 6 & 3 & -3 & -1 & -2 & -3 & -3 & 1 & 2 & 1 & 1 & -2 & -3 & -1 & -1 & 2 & -3 & 1 & 1 & -2 & -3 & 2 & 0 & -1 & -3 \\
2 & -2 & 1 & -1 & -2 & 2 & 3 & 3 & 6 & -3 & 0 & -2 & -3 & -3 & 0 & 2 & 1 & 2 & -1 & 0 & -1 & -1 & 0 & -3 & -1 & 1 & -3 & -3 & 1 & 0 & -1 & -1 \\
-2 & 2 & -1 & 1 & 2 & -2 & -3 & -3 & -3 & 6 & 1 & 1 & 3 & 3 & 0 & -2 & -1 & -2 & 1 & 0 & 1 & 1 & 0 & 3 & 1 & -1 & 2 & 2 & 0 & -2 & 1 & 1 \\
1 & 3 & -3 & -2 & -2 & -2 & 1 & -1 & 0 & 1 & 6 & -2 & -1 & -1 & -1 & 0 & -2 & -1 & -1 & -2 & -3 & -2 & 1 & 1 & 1 & -2 & -1 & -1 & 2 & 0 & 1 & -1 \\
-2 & -2 & 1 & 1 & 3 & 1 & -1 & -2 & -2 & 1 & -2 & 6 & 2 & 2 & -1 & -1 & 0 & 0 & 3 & 2 & 4 & 3 & 0 & 0 & 2 & 1 & 1 & 3 & -2 & 0 & 0 & 0 \\
-3 & -1 & 2 & 3 & 2 & -2 & -3 & -3 & -3 & 3 & -1 & 2 & 6 & 3 & 0 & -2 & -1 & -3 & 3 & 2 & 2 & 2 & -1 & 4 & 1 & 1 & 3 & 3 & -3 & -1 & 0 & 1 \\
-3 & -1 & 2 & 3 & 2 & -2 & -3 & -3 & -3 & 3 & -1 & 2 & 3 & 6 & -1 & -1 & -1 & -3 & 3 & 2 & 2 & 2 & -1 & 4 & 2 & -1 & 1 & 3 & -2 & -2 & 1 & 1 \\
2 & 2 & -1 & -1 & 2 & 2 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 6 & -2 & 3 & 3 & -4 & -1 & 0 & -1 & 0 & -3 & -3 & 2 & 0 & -1 & 2 & 0 & 0 & 1 \\
1 & -3 & 0 & 1 & -3 & -1 & 1 & 2 & 2 & -2 & 0 & -1 & -2 & -1 & -2 & 6 & -1 & 0 & 2 & 1 & -1 & 0 & -1 & 1 & 2 & -1 & -1 & -2 & 0 & 0 & -3 & -2 \\
2 & 0 & 0 & -1 & 2 & 2 & -1 & 1 & 1 & -1 & -2 & 0 & -1 & -1 & 3 & -1 & 4 & 3 & -2 & 1 & 1 & 0 & -1 & -3 & -2 & 2 & 0 & 0 & 1 & 0 & 0 & 1 \\
5 & 0 & -3 & -4 & 2 & 2 & -1 & 1 & 2 & -2 & -1 & 0 & -3 & -3 & 3 & 0 & 3 & 8 & -3 & 1 & 1 & 0 & -2 & -5 & -4 & 1 & 0 & -2 & 2 & 2 & -2 & 0 \\
-3 & -4 & 2 & 3 & 1 & -3 & -1 & -2 & -1 & 1 & -1 & 3 & 3 & 3 & -4 & 2 & -2 & -3 & 8 & 3 & 3 & 5 & -1 & 4 & 3 & 0 & 3 & 3 & -5 & -1 & -1 & 1 \\
1 & -4 & 2 & 1 & 2 & -2 & -3 & -3 & 0 & 0 & -2 & 2 & 2 & 2 & -1 & 1 & 1 & 1 & 3 & 6 & 2 & 2 & -4 & 2 & -1 & 0 & 1 & 2 & -2 & 0 & -2 & 2 \\
-2 & -2 & 1 & 1 & 4 & 1 & -2 & -1 & -1 & 1 & -3 & 4 & 2 & 2 & 0 & -1 & 1 & 1 & 3 & 2 & 6 & 4 & 0 & -1 & 1 & 2 & 2 & 2 & -2 & -1 & 0 & 0 \\
-2 & -2 & 1 & 1 & 4 & -2 & -2 & -1 & -1 & 1 & -2 & 3 & 2 & 2 & -1 & 0 & 0 & 0 & 5 & 2 & 4 & 8 & -1 & 1 & 0 & 1 & 4 & 3 & -4 & -1 & -1 & 2 \\
-2 & 2 & -1 & 1 & -1 & 3 & 4 & 2 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & -2 & -1 & -4 & 0 & -1 & 6 & -2 & 1 & 0 & -1 & -2 & 0 & 1 & 4 & -1 \\
-3 & -1 & 2 & 3 & -1 & -5 & -3 & -3 & -3 & 3 & 1 & 0 & 4 & 4 & -3 & 1 & -3 & -5 & 4 & 2 & -1 & 1 & -2 & 8 & 3 & -2 & 2 & 3 & -2 & -2 & -1 & 1 \\
-4 & -1 & 1 & 3 & -2 & 0 & 0 & 1 & -1 & 1 & 1 & 2 & 1 & 2 & -3 & 2 & -2 & -4 & 3 & -1 & 1 & 0 & 1 & 3 & 8 & 0 & 0 & 2 & 0 & -3 & 0 & -4 \\
0 & -1 & 1 & 1 & 2 & 2 & 0 & 1 & 1 & -1 & -2 & 1 & 1 & -1 & 2 & -1 & 2 & 1 & 0 & 0 & 2 & 1 & 0 & -2 & 0 & 4 & 1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & 3 & -2 & -3 & -2 & -3 & 2 & -1 & 1 & 3 & 1 & 0 & -1 & 0 & 0 & 3 & 1 & 2 & 4 & -1 & 2 & 0 & 1 & 6 & 3 & -3 & 0 & 0 & 2 \\
-2 & -1 & 2 & 2 & 3 & -2 & -3 & -3 & -3 & 2 & -1 & 3 & 3 & 3 & -1 & -2 & 0 & -2 & 3 & 2 & 2 & 3 & -2 & 3 & 2 & 1 & 3 & 6 & -2 & -1 & 0 & 2 \\
2 & 3 & -2 & -3 & -2 & 1 & 0 & 2 & 1 & 0 & 2 & -2 & -3 & -2 & 2 & 0 & 1 & 2 & -5 & -2 & -2 & -4 & 0 & -2 & 0 & -1 & -3 & -2 & 6 & -1 & -1 & -2 \\
3 & -1 & -2 & 0 & 0 & 1 & 2 & 0 & 0 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 2 & -1 & 0 & -1 & -1 & 1 & -2 & -3 & 0 & 0 & -1 & -1 & 6 & 1 & 1 \\
-1 & 2 & -1 & 1 & 1 & 2 & 2 & -1 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & -3 & 0 & -2 & -1 & -2 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 6 & 1 \\
1 & 0 & 1 & 0 & 3 & -2 & -1 & -3 & -1 & 1 & -1 & 0 & 1 & 1 & 1 & -2 & 1 & 0 & 1 & 2 & 0 & 2 & -1 & 1 & -4 & 0 & 2 & 2 & -2 & 1 & 1 & 6
\end{pmatrix}
$$

$$
G_2^{16} = \begin{pmatrix}
6 & 0 & -3 & -3 & 0 & 0 & 0 & 0 & 2 & -2 & 1 & -2 & -3 & -3 & 2 & 1 & 3 & 3 & -2 & 1 & -2 & -2 & -2 & -3 & -3 & 1 & -1 & -1 & 2 & 2 & 0 & 1 \\
0 & 6 & -3 & -3 & 0 & 0 & 0 & 0 & -2 & 2 & 3 & -2 & -1 & -1 & 2 & -3 & 1 & 1 & -2 & -3 & -2 & -2 & 2 & -1 & -1 & -1 & 1 & -1 & 2 & -2 & 2 & 1 \\
-3 & -3 & 6 & 3 & 0 & 0 & 0 & 0 & 1 & -1 & -3 & 1 & 2 & 2 & -1 & 0 & -2 & -2 & 1 & 0 & 1 & 1 & -1 & 2 & 1 & 1 & -1 & 2 & -2 & -1 & -2 & 0 \\
-3 & -3 & 3 & 6 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & 1 & 3 & 3 & -1 & 1 & -3 & -3 & 1 & 1 & 1 & 1 & 1 & 3 & 2 & 0 & 0 & 1 & -2 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 6 & 0 & -3 & -3 & -2 & 2 & -2 & 3 & 2 & 2 & 2 & -3 & 2 & 2 & 1 & 2 & 3 & 3 & -2 & -1 & -2 & 2 & 3 & 3 & -2 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 6 & 3 & 3 & 2 & -2 & -2 & 1 & -2 & -2 & 2 & -1 & 2 & 2 & -3 & -2 & -1 & -1 & 2 & -3 & 0 & 2 & -1 & -1 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & -3 & 3 & 6 & 3 & 3 & -3 & 1 & -1 & -3 & -3 & -1 & 1 & -1 & -1 & -1 & -3 & -2 & -2 & 3 & -2 & 1 & 1 & -3 & -2 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & -3 & 3 & 3 & 6 & 3 & -3 & -1 & -2 & -3 & -3 & 1 & 2 & 1 & 1 & -2 & -3 & -1 & -1 & 3 & -1 & 1 & 1 & -2 & -3 & 2 & 0 & 0 & -2 \\
2 & -2 & 1 & -1 & -2 & 2 & 3 & 3 & 6 & -3 & 0 & -2 & -3 & -3 & 0 & 2 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & -2 & 0 & 2 & -3 & -2 & 1 & 0 & 0 & -1 \\
-2 & 2 & -1 & 1 & 2 & -2 & -3 & -3 & -3 & 6 & 1 & 1 & 3 & 3 & 0 & -2 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & -2 & 2 & 1 & 0 & -2 & 1 & 1 \\
1 & 3 & -3 & -2 & -2 & -2 & 1 & -1 & 0 & 1 & 6 & -2 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -2 & -2 & 1 & 0 & 1 & -1 & -1 & -1 & 2 & -1 & 1 & -1 \\
-2 & -2 & 1 & 1 & 3 & 1 & -1 & -2 & -2 & 1 & -2 & 6 & 2 & 2 & -1 & -1 & 0 & 0 & 2 & 2 & 3 & 3 & -1 & 0 & 1 & 1 & 1 & 3 & -2 & 1 & -1 & 0 \\
-3 & -1 & 2 & 3 & 2 & -2 & -3 & -3 & -3 & 3 & -1 & 2 & 6 & 3 & 0 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & -1 & 3 & 0 & 0 & 3 & 2 & -3 & 0 & -1 & 0 \\
-3 & -1 & 2 & 3 & 2 & -2 & -3 & -3 & -3 & 3 & -1 & 2 & 3 & 6 & -1 & -1 & -2 & -2 & 2 & 2 & 2 & 2 & -1 & 3 & 1 & -2 & 1 & 2 & -1 & -1 & 0 & 1 \\
2 & 2 & -1 & -1 & 2 & 2 & -1 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 6 & -2 & 3 & 3 & -3 & -1 & -1 & -1 & 0 & -2 & -3 & 2 & 1 & -1 & 1 & 0 & 0 & 2 \\
1 & -3 & 0 & 1 & -3 & -1 & 1 & 2 & 2 & -2 & 0 & -1 & -2 & -1 & -2 & 6 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & -1 & -2 & -2 & 1 & 0 & -2 & -2 \\
3 & 1 & -2 & -3 & 2 & 2 & -1 & 1 & 1 & -1 & -1 & 0 & -2 & -2 & 3 & -1 & 6 & 3 & -1 & 1 & 0 & 0 & -1 & -3 & -2 & 2 & 1 & 0 & 1 & 0 & 1 & 2 \\
3 & 1 & -2 & -3 & 2 & 2 & -1 & 1 & 1 & -1 & -1 & 0 & -2 & -2 & 3 & -1 & 3 & 6 & -2 & 0 & 0 & 0 & -1 & -3 & -3 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-2 & -2 & 1 & 1 & 1 & -3 & -1 & -2 & -1 & 1 & 0 & 2 & 2 & 2 & -3 & 1 & -1 & -2 & 6 & 2 & 3 & 3 & -1 & 2 & 2 & 0 & 2 & 2 & -3 & -1 & -1 & 1 \\
1 & -3 & 0 & 1 & 2 & -2 & -3 & -3 & -1 & 1 & -1 & 2 & 2 & 2 & -1 & 1 & 1 & 0 & 2 & 6 & 2 & 2 & -3 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
-2 & -2 & 1 & 1 & 3 & -1 & -2 & -1 & -1 & 1 & -2 & 3 & 2 & 2 & -1 & 0 & 0 & 0 & 3 & 2 & 6 & 3 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & -1 & 0 \\
-2 & -2 & 1 & 1 & 3 & -1 & -2 & -1 & -1 & 1 & -2 & 3 & 2 & 2 & -1 & 0 & 0 & 0 & 3 & 2 & 3 & 6 & -1 & 1 & 0 & 0 & 3 & 2 & -3 & 0 & -1 & 1 \\
-2 & 2 & -1 & 1 & -2 & 2 & 3 & 3 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -3 & 0 & -1 & 6 & 0 & 1 & -1 & -1 & -3 & 1 & 0 & 3 & -1 \\
-3 & -1 & 2 & 3 & -1 & -3 & -2 & -1 & -2 & 2 & 0 & 0 & 3 & 3 & -2 & 1 & -3 & -3 & 2 & 1 & 0 & 1 & 0 & 6 & 2 & -2 & 1 & 1 & -1 & -1 & -1 & 0 \\
-3 & -1 & 1 & 2 & -2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & -3 & 2 & -2 & -3 & 2 & -1 & 1 & 0 & 1 & 2 & 6 & 0 & -1 & 1 & 0 & -2 & 0 & -3 \\
1 & -1 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & -2 & -1 & 1 & 0 & -2 & 2 & -1 & 2 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 0 & 6 & 0 & 2 & -1 & 1 & -1 & 0 \\
-1 & 1 & -1 & 0 & 3 & -1 & -3 & -2 & -3 & 2 & -1 & 1 & 3 & 1 & 1 & -2 & 1 & 1 & 2 & 1 & 1 & 3 & -1 & 1 & -1 & 0 & 6 & 2 & -3 & 0 & 0 & 2 \\
-1 & -1 & 2 & 1 & 3 & -1 & -2 & -3 & -2 & 1 & -1 & 3 & 2 & 2 & -1 & -2 & 0 & -1 & 2 & 1 & 1 & 2 & -3 & 1 & 1 & 2 & 2 & 6 & -2 & 0 & -1 & 1 \\
2 & 2 & -2 & -2 & -2 & 0 & 0 & 2 & 1 & 0 & 2 & -2 & -3 & -1 & 1 & 1 & 1 & 1 & -3 & -1 & -1 & -3 & 1 & -1 & 0 & -1 & -3 & -2 & 6 & -1 & 0 & -1 \\
2 & -2 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 & -1 & 6 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -2 & 1 & -1 & -1 & -1 & -1 & -1 & 3 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 6 & 1 \\
1 & 1 & 0 & -1 & 3 & -1 & -1 & -2 & -1 & 1 & -1 & 0 & 0 & 1 & 2 & -2 & 2 & 1 & 1 & 1 & 0 & 1 & -1 & 0 & -3 & 0 & 2 & 1 & -1 & 0 & 1 & 6
\end{pmatrix}
$$

$$G_1^{20} = \begin{pmatrix}
6 & 0 & 1 & 1 & 1 & 2 & 1 & 3 & 0 & 1 & 0 & -1 & 0 & 0 & -2 & 2 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 2 & 1 & -1 & 1 & 2 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\
0 & 6 & -1 & 1 & 1 & 1 & 0 & 0 & -2 & 2 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -2 & 0 & 3 & 0 & 2 & -1 & -2 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 1 \\
1 & -1 & 6 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 & -2 & 0 & 2 & 1 & 1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & -2 & 1 & -1 & -2 \\
1 & 1 & 0 & 6 & 3 & 1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
1 & 1 & 0 & 3 & 6 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -2 & 0 & 0 & 1 & 0 & -1 & 2 & 1 & 1 & 1 & -1 & 2 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 & 1 \\
2 & 1 & 2 & 1 & 1 & 6 & 1 & 2 & -1 & 1 & 1 & 0 & -1 & -2 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & 2 & 2 & 0 & -1 & 3 & 1 & 1 & -1 & -2 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & -1 & 1 & 6 & 3 & 1 & 1 & -1 & 2 & -1 & 0 & 2 & -2 & 2 & 0 & 0 & -2 & -1 & 0 & 1 & 1 & 1 & 0 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & -1 & 0 & -2 & 1 & 1 & 2 & -2 \\
3 & 0 & 1 & 0 & 0 & 2 & 3 & 6 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 2 & -2 & 1 & -2 & 0 & 0 & 0 & 2 & 3 & 0 & 1 & 2 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -2 & 1 & 1 & 1 & -1 & 1 & 1 & 6 & 1 & 1 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 0 & 1 \\
1 & 2 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 6 & 1 & 2 & -1 & -1 & -1 & 1 & 2 & 0 & -1 & -2 & 3 & 1 & 2 & 1 & 0 & 2 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 & 1 & -1 & 0 & 1 & 1 & 6 & 3 & 3 & 2 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & 1 \\
-1 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & 2 & 2 & 3 & 6 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 3 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & 1 & 1 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 3 & -1 & 3 & 2 & 6 & 2 & 0 & 2 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 0 & 1 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & -1 & 2 \\
0 & 0 & -1 & 0 & -1 & -2 & 0 & -2 & 1 & -1 & 2 & 2 & 2 & 6 & 1 & 1 & -2 & 1 & 0 & 3 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 0 & -1 & 0 & 1 & -2 & 0 & 1 & 0 & 1 \\
-2 & -1 & 0 & 0 & -2 & 0 & 2 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 6 & -1 & 2 & 2 & 3 & 1 & -1 & 1 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 2 & 0 \\
2 & 0 & 2 & 0 & 0 & 1 & -2 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & -1 & 6 & 1 & 1 & 1 & 2 & 3 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & -1 & 1 & 0 & -2 & 2 & 1 & 6 & 1 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & -1 \\
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0 & -2 & 0 & -1 & 0 & -1 & 0 & 1 & 2 & -1 & 0 & 1 & 1 & 0 & 3 & 1 & 2 & 1 & 6 & 0 & 0 & 3 & 0 & 2 & 3 & 1 & 2 & 1 & 1 & -1 & 2 & 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 2 & -1 & -1 & -2 & -2 & 1 & -2 & 1 & 0 & 3 & 3 & 1 & 2 & 0 & 2 & 0 & 6 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \\
0 & 3 & 0 & 1 & 2 & 0 & -1 & 0 & 0 & 3 & -1 & 1 & 0 & 0 & -1 & 3 & 2 & 1 & 0 & 1 & 6 & 2 & 2 & 0 & -1 & 3 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 & 0 \\
1 & 0 & -2 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 0 & 2 & 6 & 1 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & 0 \\
0 & 2 & 0 & 1 & 1 & 2 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 0 & 1 & 2 & 1 & 6 & 2 & 0 & 2 & 2 & -1 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
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1 & -2 & 1 & 0 & -1 & 0 & 1 & 3 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 2 & 2 & 0 & 3 & 0 & -1 & 1 & 0 & 3 & 6 & 2 & 2 & 2 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
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0 & -1 & 0 & -1 & -1 & 1 & -1 & 0 & 1 & -1 & 2 & 1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 6 & 3 & 3 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 3 \\
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0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 3 & 0 & 3 & 6 & 2 & 2 & 1 & 1 & 3 & 2 & 2 & 2 \\
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0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 2 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 3 & 2 & 2 & 2 & 3 & 6 & 3 & 2 & 3 & 2 & 1 & 3 \\
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0 & 1 & 2 & 0 & 2 & 0 & -2 & 0 & 1 & 1 & -1 & -2 & -1 & -2 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 6 & 1 & 1 & 0 & 2 \\
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0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & -1 & 1 & 2 & 3 & 2 & 2 & 2 & 3 & 1 & 3 & 6 & 1 & 1 \\
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-1 & 1 & -2 & 1 & 1 & -1 & -2 & -1 & 1 & -1 & 1 & 0 & 2 & 1 & 0 & 0 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 2 & 2 & 2 & 2 & 3 & 1 & 2 & 2 & 1 & 1 & 6
\end{pmatrix}$$

$$G_2^{20} = \left(\begin{array}{cccccccccccccccccccccccccccccccccccccccc}
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0 & 6 & 0 & 1 & 2 & 1 & 1 & 0 & -1 & 2 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & -1 & 1 & 3 & 0 & 2 & -1 & -2 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \\
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1 & 1 & 1 & 6 & 3 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & 2 & -1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
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1 & 1 & 2 & 1 & 1 & 6 & 2 & 2 & -1 & 0 & 1 & 0 & -1 & -2 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & 2 & 1 & 0 & -1 & 3 & 0 & 1 & -1 & -2 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
1 & 1 & 1 & 1 & 0 & 2 & 6 & 3 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & -2 & 2 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 2 & -1 \\
3 & 0 & 2 & 1 & 0 & 2 & 3 & 6 & 0 & 0 & -1 & -1 & 0 & -2 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 2 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 6 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 3 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 6 & 2 & 2 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 3 & 2 & 1 & 1 & 0 & 2 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 \\
-1 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 2 & 6 & 3 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\
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0 & 1 & -1 & 0 & 0 & -2 & 0 & -2 & 1 & 0 & 2 & 2 & 2 & 6 & 1 & 1 & -1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 0 & 2 & -1 & 1 & 0 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
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0 & -1 & 3 & 2 & 2 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 6 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\
0 & -1 & 0 & -1 & 1 & -1 & -1 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 6 & 0 & 1 & 3 & 0 & 2 & 2 & 1 & 2 & 1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 1 & 2 & 0 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & 3 & 3 & 1 & 1 & 1 & 2 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\
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-1 & 2 & -1 & 0 & 1 & 2 & 1 & -1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 6 & 1 & -1 & 2 & 2 & -1 & 0 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 1 & 6 & 3 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\
1 & -2 & 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 2 & 0 & -1 & 1 & -1 & 3 & 6 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 6 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 3 & 6 & 3 & 0 & 2 & 2 & 3 & 2 & 1 & 2 & 2 & 2 \\
0 & -1 & 0 & -1 & -1 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 3 & 3 & 6 & 3 & 1 & 2 & 3 & 2 & 2 & 3 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 3 & 0 & 3 & 6 & 2 & 2 & 1 & 2 & 3 & 2 & 2 & 2 \\
-1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & 2 & 1 & 2 & 6 & 2 & 3 & 2 & 3 & 2 & 2 & 2 \\
0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 2 & 2 & 2 & 1 & 2 \\
1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 & 3 & 3 & 1 & 3 & 2 & 6 & 2 & 3 & 3 & 2 & 2 \\
0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 2 & 1 & 1 & 2 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 2 & 6 & 3 & 2 & 2 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & -1 & 1 & 2 & 3 & 2 & 2 & 2 & 3 & 1 & 3 & 6 & 1 & 1 \\
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0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & 2 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 6
\end{array}\right)$$

$$G_1^{24} = \begin{pmatrix}
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3 & 6 & 1 & 1 & 2 & 1 & -1 & 1 & 0 & 2 & 0 & -1 & 0 & -1 & 2 & -2 & -3 & 2 & -2 & -1 & 3 & 1 & 1 & 2 & 0 & 1 & 0 & -1 & 0 & -3 & -2 & 1 & -3 & 0 & 1 & -2 & 0 & 0 & -1 & -1 & -2 & -1 & -3 & -1 & -1 & -1 & -1 & -1 \\
3 & 1 & 6 & -3 & 2 & 0 & 1 & 1 & -2 & 2 & -2 & 0 & 2 & 1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & -1 & -2 & -2 & -2 & 1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -3 & -3 & -3 & -1 & 0 & 0 & -1 & -3 & -1 & 0 & 1 & -1 & -2 \\
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1 & 2 & 2 & -2 & 6 & 3 & -1 & 3 & 0 & 3 & 0 & 1 & 2 & 1 & -1 & -2 & -1 & 3 & 1 & -1 & 2 & 0 & 3 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 1 & 2 & -1 & -1 & 3 & -1 & 2 & -2 & -3 & -2 & 1 & -1 & 0 & 1 & -3 & 1 & 0 & -2 \\
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-1 & -1 & 1 & -2 & -1 & 1 & 6 & 1 & -3 & -1 & 0 & 0 & -1 & 3 & -2 & 2 & -1 & -2 & 0 & 3 & 0 & 0 & 0 & -2 & -2 & 0 & 2 & -3 & 1 & 1 & 2 & -1 & 2 & 1 & -1 & -2 & -2 & 0 & 1 & -1 & 1 & 2 & -1 & 1 & -2 & -2 & 2 & 1 \\
0 & 1 & 1 & -1 & 3 & 0 & 1 & 6 & 0 & 3 & 2 & -1 & 0 & 2 & 0 & 1 & -2 & 2 & 1 & -1 & 0 & 0 & 0 & -2 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -2 & 1 & 0 & 2 & -2 & -1 & 0 & 2 & 2 & -1 & 2 & -3 & 1 & -1 & -1 \\
-2 & 0 & -2 & 3 & 0 & 0 & -3 & 0 & 6 & 2 & 2 & -1 & -2 & 0 & 3 & 0 & 2 & 0 & -1 & -3 & -1 & -1 & 1 & 1 & 4 & 0 & -2 & 1 & 2 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & 3 & 1 & -3 & 0 & -2 & -1 & 2 & 0 & -1 & 1 & 0 & 0 \\
1 & 2 & 2 & 0 & 3 & 0 & -1 & 3 & 2 & 6 & 0 & -1 & 1 & 1 & 1 & -1 & 0 & 2 & 0 & -2 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 1 & 2 & -1 & -1 & 0 & -1 & -1 & 2 & -1 & 0 & -2 & -3 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & -1 & 0 \\
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-2 & -2 & -1 & -1 & -2 & -1 & 2 & 1 & 0 & -1 & 1 & -1 & -2 & 1 & 1 & 6 & 0 & 1 & -1 & 0 & -2 & 2 & 0 & -1 & -1 & -1 & 2 & -2 & -1 & 3 & 0 & -1 & 2 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\
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0 & -2 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 3 & 3 & 1 & -2 & -1 & -1 & -1 & 6 & 0 & 1 & -2 & 0 & -2 & 1 & 0 & -1 & 2 & -1 & -1 & 2 & -1 & 0 & 0 & -1 & 2 & -1 & -2 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & -3 & -1 & 1 & 3 & -1 & -3 & -2 & -3 & -1 & 0 & 1 & -2 & 0 & -1 & -1 & 0 & 6 & 0 & 1 & -1 & 0 & -2 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & -2 & -1 & -3 & -1 & 2 & 1 & 1 & 1 & 1 & -2 & -1 & 0 & 0 & 0 \\
3 & 3 & 1 & 0 & 2 & 1 & 0 & 0 & -1 & 0 & 1 & 2 & 1 & -1 & 0 & -2 & -2 & 0 & 1 & 0 & 6 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & -2 & -3 & 0 & 1 & -2 & 2 & 0 & -1 & -1 & -1 & 1 & -1 & -1 & -2 & -2 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 2 & -2 & 0 & -1 & 0 & 0 & -1 & 1 & -3 & -4 & 0 & -1 & 1 & 2 & 1 & 1 & -2 & 1 & -2 & 8 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 & -3 & 2 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -2 & 1 & 0 & -1 & 1 \\
-1 & 1 & 1 & -1 & 3 & 3 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 1 & 1 & 0 & -1 & 2 & 1 & 6 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & -2 & 2 & 0 & -3 & -3 & -2 & -2 & 0 & 0 & -3 & 0 & 2 & -1 \\
1 & 2 & -1 & 0 & 1 & 2 & -2 & -2 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 2 & -2 & 0 & 2 & -1 & 2 & 6 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & -2 & -3 & 0 & -1 & 0 & 0 & 0 & -1 \\
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-2 & 1 & -2 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & 1 & -2 & -1 & 0 & 0 & -1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 6 & -1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & 2 & 2 & 1 & 0 & -1 & 1 & 2 & 0 & 2 & -1 & -1 & 0 & 0 \\
0 & 0 & -2 & -1 & -1 & 1 & 2 & -1 & -2 & -1 & -1 & 1 & 0 & 0 & -1 & 2 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & -2 & -1 & 6 & -1 & -2 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & 1 & 2 \\
1 & -1 & 1 & 0 & 1 & -1 & -3 & -1 & 1 & 1 & -1 & 1 & 2 & -2 & 1 & -2 & 2 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 6 & -1 & 0 & -1 & 1 & 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & -2 & 1 & 0 & 2 & 1 & -1 & 0 \\
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2 & 1 & 0 & -2 & 2 & 2 & -1 & -1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & -1 & 2 & -1 & 1 & 1 & 0 & 0 & 2 & -2 & 0 & 1 & 1 & -2 & 0 & 0 & 6 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -2 & 1 & -1 & 0 & 0 & 0 & 0 \\
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0 & 0 & -1 & 2 & -1 & 1 & 1 & -2 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -2 & 0 & -2 & 0 & 1 & 2 & -3 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 6 & -2 & -1 & -2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 \\
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1 & -1 & 0 & 1 & -2 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -3 & -1 & -2 & -1 & -1 & 0 & 1 & -1 & -2 & -1 & -3 & 0 & -2 & 1 & -2 & 0 & 2 & 6 & 2 & 2 & 1 & -2 & 1 & 1 & -2 & 0 \\
0 & -2 & 0 & -1 & 1 & 0 & 1 & 2 & -2 & -1 & 0 & 1 & 0 & 1 & -2 & 0 & -1 & 0 & 2 & 1 & -1 & -1 & -2 & -2 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & -1 & 1 & 0 & -1 & 2 & 2 & 6 & 2 & 2 & 1 & -1 & 1 & -1 & 0 \\
-2 & -1 & -1 & 1 & -1 & -1 & 2 & 2 & -1 & 0 & 1 & -2 & -1 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & -2 & 0 & -2 & -3 & -1 & 2 & -1 & -2 & 2 & 0 & 0 & -2 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & 2 & 2 & 6 & 0 & 1 & -1 & -1 & 0 & 1 \\
-2 & -3 & -3 & 0 & 0 & 2 & -1 & -1 & 2 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & -2 & -1 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & -1 & 1 & 2 & 1 & -1 & 1 & 2 & 0 & 8 & -2 & -2 & 1 & 1 & 1 \\
-3 & -1 & -1 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 4 & 2 & 0 & 1 & 0 & 0 & 0 & -1 & 3 & -2 & 0 & -2 & 0 & -1 & 2 & 2 & -1 & 0 & 1 & 0 & 2 & -1 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & -2 & 1 & 1 & -2 & 8 & 0 & -2 & 1 & 0 \\
1 & -1 & 0 & 1 & -3 & -4 & -2 & -3 & -1 & -1 & -1 & 0 & 1 & -3 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 1 & -3 & 0 & -2 & -1 & 0 & 2 & -2 & 0 & -2 & 0 & -1 & -1 & 0 & 2 & -2 & 0 & 2 & 1 & -1 & -1 & -2 & 0 & 8 & 0 & -1 & 1 \\
2 & -1 & 1 & -2 & 1 & -1 & -2 & 1 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -3 & 0 & 1 & 1 & -1 & 1 & -2 & 0 & 6 & -3 & -1 \\
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-1 & -1 & -2 & 1 & -2 & -2 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 & -1 & 1 & -2 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & 6
\end{pmatrix}$$

$$G_2^{24} = \begin{pmatrix}
8 & 3 & 4 & -3 & 1 & -2 & -1 & -1 & -4 & -1 & -2 & 2 & 3 & -3 & -1 & -2 & -2 & 1 & 0 & 2 & 5 & 0 & 0 & 2 & -4 & -3 & 0 & 1 & -4 & -4 & 0 & 2 & -4 & 1 & -2 & -3 & -4 & -4 & 5 & 1 & 1 & -2 & -3 & -3 & 2 & 3 & -3 & -2 \\
3 & 8 & 1 & 2 & 3 & 1 & -2 & 1 & 1 & 2 & 1 & -1 & 0 & -2 & 3 & -3 & -3 & 2 & -3 & -2 & 4 & 2 & 3 & 4 & 2 & 2 & -1 & -1 & 0 & -4 & -2 & 0 & -3 & 0 & 1 & -2 & 1 & 0 & -1 & -3 & -4 & -1 & -5 & 1 & -2 & -2 & -1 & -2 \\
4 & 1 & 12 & -5 & 3 & -1 & 3 & 3 & -3 & 4 & -3 & -1 & 2 & 3 & -2 & 0 & 1 & -1 & 2 & 1 & 1 & 5 & 2 & -3 & -3 & -4 & -5 & 1 & 3 & -2 & 2 & -3 & 0 & -3 & 2 & -6 & -6 & -5 & -3 & 1 & 1 & 0 & -7 & 0 & 1 & 1 & -2 & -4 \\
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1 & 3 & 3 & -4 & 10 & 5 & -3 & 3 & 0 & 4 & -2 & 2 & 5 & 2 & -4 & -5 & 0 & 4 & 2 & -2 & 2 & 1 & 6 & 3 & 4 & 3 & -2 & 3 & 0 & -2 & 2 & 3 & -1 & -3 & 6 & -1 & 3 & -3 & -7 & -4 & 1 & -2 & -1 & 3 & -4 & 2 & -2 & -4 \\
-2 & 1 & -1 & -2 & 5 & 10 & 0 & -1 & 0 & 0 & -3 & 3 & 0 & 4 & -3 & -2 & 0 & 2 & -1 & 1 & 0 & -2 & 5 & 4 & 4 & 2 & 3 & 0 & 2 & 2 & 3 & 4 & 2 & 0 & 4 & -2 & 2 & 1 & -5 & -4 & 0 & -3 & 3 & 0 & -5 & 0 & 3 & -4 \\
-1 & -2 & 3 & -4 & -3 & 0 & 12 & 3 & -6 & -1 & 0 & -1 & -2 & 6 & -4 & 4 & -2 & -5 & 0 & 6 & -1 & 1 & -2 & -6 & -6 & -1 & 3 & -6 & 2 & 1 & 4 & -2 & 4 & 1 & -3 & -4 & -5 & 0 & 3 & -1 & 2 & 5 & -3 & 0 & -3 & -3 & 4 & 2 \\
-1 & 1 & 3 & -2 & 3 & -1 & 3 & 10 & 0 & 4 & 3 & -4 & -1 & 4 & 0 & 2 & -2 & 1 & 1 & -1 & -2 & 2 & -1 & -5 & -2 & 2 & -4 & -2 & 3 & -1 & 2 & -3 & 0 & -4 & 1 & -1 & 2 & -2 & -2 & 1 & 3 & 5 & -3 & 4 & -6 & 0 & -3 & -2 \\
-4 & 1 & -3 & 6 & 0 & 0 & -6 & 0 & 12 & 4 & 4 & -3 & -4 & 0 & 7 & 0 & 3 & 0 & -2 & -6 & -2 & -1 & 2 & 2 & 8 & 1 & -5 & 2 & 5 & 0 & -3 & -2 & 2 & -1 & 1 & 2 & 6 & 2 & -6 & 0 & -5 & -2 & 3 & 1 & -2 & 1 & -2 & 0 \\
-1 & 2 & 4 & 0 & 4 & 0 & -1 & 4 & 4 & 10 & -1 & -4 & 1 & 2 & 2 & -1 & 2 & 3 & -1 & -3 & -3 & 4 & 2 & -2 & 1 & -1 & -3 & 2 & 6 & 0 & -3 & -1 & 0 & -2 & 4 & -2 & -1 & -1 & -7 & 0 & -3 & 2 & -1 & 1 & -2 & -1 & -2 & 1 \\
-2 & 1 & -3 & 6 & -2 & -3 & 0 & 3 & 4 & -1 & 12 & 2 & -5 & 0 & 5 & 2 & -2 & -2 & 1 & -6 & 3 & -6 & -1 & 0 & 1 & 1 & -3 & -2 & 0 & -3 & 0 & -2 & 0 & 2 & -1 & 3 & 4 & 1 & 3 & -1 & -2 & 1 & -3 & 7 & -1 & -2 & 0 & 2 \\
2 & -1 & -1 & 1 & 2 & 3 & -1 & -4 & -3 & -4 & 2 & 14 & 6 & 1 & -6 & -3 & -1 & 0 & 7 & -3 & 6 & -10 & 4 & 3 & 4 & -3 & 3 & 2 & -6 & -1 & 5 & 2 & -1 & 3 & -1 & 0 & -2 & -2 & 1 & -4 & 1 & -6 & 0 & 4 & 1 & 0 & 4 & -1 \\
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-1 & 3 & -2 & 7 & -4 & -3 & -4 & 0 & 7 & 2 & 5 & -6 & -9 & -4 & 14 & 4 & 0 & 2 & -5 & -3 & -2 & 4 & -3 & 2 & 1 & 0 & -3 & 1 & 4 & 1 & -7 & -1 & -2 & 0 & -1 & 3 & 2 & 2 & 2 & 3 & -4 & 0 & 1 & 0 & 2 & -2 & -1 & 3 \\
-2 & -3 & 0 & -2 & -5 & -2 & 4 & 2 & 0 & -1 & 2 & -3 & -6 & 2 & 4 & 12 & 0 & 2 & -3 & 1 & -4 & 5 & -2 & -3 & -4 & -4 & 3 & -5 & -1 & 7 & -1 & -1 & 4 & -3 & 0 & -1 & 0 & -1 & 3 & 2 & 0 & 0 & 1 & -3 & -1 & 0 & 1 & 4 \\
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1 & 2 & -1 & -2 & 4 & 2 & -5 & 1 & 0 & 3 & -2 & 0 & 2 & -2 & 2 & 2 & -2 & 12 & -2 & -1 & -3 & 3 & 1 & 4 & -1 & -1 & 3 & 1 & -4 & 3 & -5 & 4 & -3 & -3 & 4 & 1 & 2 & -2 & -2 & 1 & 0 & -2 & 4 & -2 & -2 & -1 & -1 & 1 \\
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2 & -2 & 1 & -6 & -2 & 1 & 6 & -1 & -6 & -3 & -6 & -3 & 0 & 2 & -3 & 1 & -3 & -1 & 0 & 12 & -1 & 3 & -3 & -1 & -5 & 2 & 3 & -2 & -1 & 0 & 2 & 2 & 1 & 1 & -4 & -1 & -5 & -2 & 5 & 2 & 3 & 3 & 3 & -5 & -1 & 1 & 1 & 0 \\
5 & 4 & 1 & 1 & 2 & 0 & -1 & -2 & -2 & -3 & 3 & 6 & 3 & -3 & -2 & -4 & -2 & -3 & 3 & -1 & 12 & -5 & 5 & 4 & 2 & 0 & -1 & 1 & -5 & -6 & 3 & 0 & -3 & 4 & -1 & -1 & -1 & -2 & 3 & -3 & -3 & -4 & -5 & 2 & 1 & 1 & 0 & -3 \\
0 & 2 & 5 & -5 & 1 & -2 & 1 & 2 & -1 & 4 & -6 & -10 & -2 & -1 & 4 & 5 & 2 & 3 & -5 & 3 & -5 & 18 & 1 & -2 & -3 & 0 & -1 & -1 & 2 & 5 & -3 & 0 & 2 & -7 & 4 & -1 & 0 & -1 & -3 & 1 & -1 & 2 & -2 & -5 & 0 & 0 & -2 & 3 \\
0 & 3 & 2 & -2 & 6 & 5 & -2 & -1 & 2 & 2 & -1 & 4 & 4 & 2 & -3 & -2 & 2 & 1 & 1 & -3 & 5 & 1 & 12 & 5 & 7 & 0 & 0 & 2 & -1 & 1 & 4 & 0 & 3 & -1 & 4 & -3 & 3 & -2 & -6 & -7 & -5 & -5 & -2 & 0 & -4 & 1 & 1 & -2 \\
2 & 4 & -3 & 0 & 3 & 4 & -6 & -5 & 2 & -2 & 0 & 3 & -1 & -3 & 2 & -3 & 0 & 4 & -3 & -1 & 4 & -2 & 5 & 14 & 3 & 3 & 4 & 2 & -3 & -2 & -2 & 4 & 0 & -1 & 4 & 2 & 4 & 1 & 1 & -4 & -4 & -8 & 0 & 0 & 2 & 1 & -1 & -3 \\
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0 & -1 & -5 & -3 & -2 & 3 & 3 & -4 & -5 & -3 & -3 & 3 & 1 & -1 & -3 & 3 & -1 & 3 & -3 & 3 & -1 & -1 & 0 & 4 & -5 & -2 & 14 & -1 & -6 & 4 & -2 & 5 & 4 & 1 & 0 & 1 & 0 & 3 & 6 & -3 & 0 & -3 & 3 & -4 & 0 & -2 & 4 & 4 \\
1 & -1 & 1 & 1 & 3 & 0 & -6 & -2 & 2 & 2 & -2 & 2 & 4 & -4 & 1 & -5 & 3 & 1 & 4 & -2 & 1 & -1 & 2 & 2 & 3 & 0 & -1 & 12 & -1 & -1 & -2 & 2 & -1 & -1 & 0 & 5 & 1 & 0 & -1 & 1 & 1 & -3 & 3 & 1 & 3 & 1 & -2 & -1 \\
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0 & -2 & 2 & -4 & 2 & 3 & 4 & 2 & -3 & -3 & 0 & 5 & 4 & 6 & -7 & -1 & -1 & -5 & 6 & 2 & 3 & -3 & 4 & -2 & 2 & 1 & -2 & -2 & -1 & -1 & 12 & 0 & 5 & -1 & -1 & -2 & -1 & -3 & -1 & -5 & 2 & -1 & -2 & 3 & -3 & 3 & 2 & -4 \\
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1 & 0 & -3 & 5 & -3 & 0 & 1 & -4 & -1 & -2 & 2 & 3 & -1 & -1 & 0 & -3 & -1 & -3 & 0 & 1 & 4 & -7 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -3 & -1 & 0 & -5 & 12 & -5 & -2 & -5 & 1 & 4 & 1 & -1 & 3 & 2 & -1 & 1 & -1 & 2 & 4 \\
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-4 & 1 & -6 & 1 & 3 & 2 & -5 & 2 & 6 & -1 & 4 & -2 & -2 & -2 & 2 & 0 & 1 & 2 & -3 & -5 & -1 & 0 & 3 & 4 & 4 & 5 & 0 & 1 & -2 & 2 & -1 & 1 & 3 & -5 & 4 & 5 & 14 & 3 & -3 & -3 & -1 & -2 & 2 & 1 & -4 & 1 & -2 & -2 \\
-4 & 0 & -5 & 5 & -3 & 1 & 0 & -2 & 2 & -1 & 1 & -2 & -3 & -4 & 2 & -1 & 2 & -2 & -4 & -2 & -2 & -1 & -2 & 1 & -1 & 1 & 3 & 0 & 3 & 1 & -3 & 1 & 1 & 1 & 0 & 3 & 3 & 10 & 1 & 1 & -1 & 1 & 2 & 1 & 0 & -4 & 3 & 0 \\
5 & -1 & -3 & 0 & -7 & -5 & 3 & -2 & -6 & -7 & 3 & 1 & -2 & -5 & 2 & 3 & -4 & -2 & -1 & 5 & 3 & -3 & -6 & 1 & -10 & -1 & 6 & -1 & -6 & -2 & -1 & 3 & -1 & 4 & -7 & 3 & -3 & 1 & 18 & 3 & 3 & 2 & -1 & -2 & 3 & 1 & 0 & 3 \\
1 & -3 & 1 & 2 & -4 & -4 & -1 & 1 & 0 & 0 & -1 & -4 & -4 & -3 & 3 & 2 & 1 & 1 & -1 & 2 & -3 & 1 & -7 & -4 & -5 & -3 & -3 & 1 & 3 & 0 & -5 & -2 & -7 & 1 & -3 & 1 & -3 & 1 & 3 & 12 & 5 & 4 & 4 & -3 & 2 & 1 & -3 & 0 \\
1 & -4 & 1 & -3 & 1 & 0 & 2 & 3 & -5 & -3 & -2 & 1 & 0 & 1 & -4 & 0 & -1 & 0 & 3 & 3 & -3 & -1 & -5 & -4 & -3 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & -4 & -1 & -2 & 1 & -1 & -1 & 3 & 5 & 12 & 3 & 4 & 0 & -1 & 3 & -2 & -1 \\
-2 & -1 & 0 & 1 & -2 & -3 & 5 & 5 & -2 & 2 & 1 & -6 & -2 & 2 & 0 & 0 & -2 & -2 & -1 & 3 & -4 & 2 & -5 & -8 & -4 & 3 & -3 & -3 & 5 & -1 & -1 & -3 & -3 & 3 & -3 & 0 & -2 & 1 & 2 & 4 & 3 & 12 & 0 & 0 & -3 & -2 & -1 & 4 \\
-3 & -5 & -7 & 1 & -1 & 3 & -3 & -3 & 3 & -1 & -3 & 0 & -1 & -1 & 1 & 1 & 1 & 4 & 0 & 3 & -5 & -2 & -2 & 0 & 3 & 0 & 3 & 3 & 0 & 4 & -2 & 4 & -2 & 2 & -2 & 3 & 2 & 2 & -1 & 4 & 4 & 0 & 16 & -5 & -3 & 2 & 1 & 4 \\
-3 & 1 & 0 & 4 & 3 & 0 & 0 & 4 & 1 & 1 & 7 & 4 & 1 & 2 & 0 & -3 & -1 & -2 & 6 & -5 & 2 & -5 & 0 & 0 & 4 & 4 & -4 & 1 & 2 & -4 & 3 & -2 & 0 & -1 & 2 & 3 & 1 & 1 & -2 & -3 & 0 & 0 & -5 & 14 & 0 & -3 & 1 & -3 \\
2 & -2 & 1 & 2 & -4 & -5 & -3 & -6 & -2 & -2 & -1 & 1 & 0 & -5 & 2 & -1 & 2 & -2 & 1 & -1 & 1 & 0 & -4 & 2 & -3 & -2 & 0 & 3 & -2 & -1 & -3 & 0 & -2 & 1 & 0 & 3 & -4 & 0 & 3 & 2 & -1 & -3 & -3 & 0 & 14 & 0 & 0 & 1 \\
3 & -2 & 1 & -4 & 2 & 0 & -3 & 0 & 1 & -1 & -2 & 0 & 1 & 1 & -2 & 0 & 2 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & -2 & 1 & 0 & -1 & 3 & 2 & 0 & -1 & 1 & -1 & 1 & -4 & 1 & 1 & 3 & -2 & 2 & -3 & 0 & 10 & -5 & -2 \\
-3 & -1 & -2 & 2 & -2 & 3 & 4 & -3 & -2 & -2 & 0 & 4 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & -2 & 1 & -1 & 0 & -2 & 4 & -2 & -1 & 3 & 2 & 1 & 3 & 2 & -2 & 0 & -2 & 3 & 0 & -3 & -2 & -1 & 1 & 1 & 0 & -5 & 10 & 1 \\
-2 & -2 & -4 & 2 & -4 & -4 & 2 & -2 & 0 & 1 & 2 & -1 & -2 & 0 & 3 & 4 & 0 & 1 & -1 & 0 & -3 & 3 & -2 & -3 & -1 & -1 & 4 & -1 & -1 & 3 & -4 & 1 & 0 & 4 & -2 & 2 & -2 & 0 & 3 & 0 & -1 & 4 & 4 & -3 & 1 & -2 & 1 & 14
\end{pmatrix}$$

Appendix D

Data for $\mathbb{Q}[\sqrt{3}]$

Extremal Hilbert modular forms

k	Extremal Hilbert modular form in $\overline{M}_k^+(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]))$
1	$1 + 12q_1^1q_2^1 + 12q_1^2q_2^1 + 12q_1^2q_2^2 + 12q_1^2q_2^3 + 12q_1^3q_2^3 + 12q_1^4q_2^2 + 24q_1^4q_2^3 + 12q_1^4q_2^4$ $+24q_1^4q_2^5 + 12q_1^4q_2^6 + 24q_1^5q_2^3 + 24q_1^5q_2^5 + 24q_1^5q_2^7$
2	$1 + 24q_1^1q_2^1 + 24q_1^2q_2^1 + 168q_1^2q_2^2 + 24q_1^2q_2^3 + 288q_1^3q_2^2 + 312q_1^3q_2^3+$ $288q_1^3q_2^4 + 168q_1^4q_2^2 + 336q_1^4q_2^3 + 744q_1^4q_2^4 + 336q_1^4q_2^5 + 168q_1^4q_2^6+$ $336q_1^5q_2^3 + 864q_1^5q_2^4 + 624q_1^5q_2^5 + 864q_1^5q_2^6 + 336q_1^5q_2^7$
3	$1 + 72q_1^2q_2^1 + 612q_1^2q_2^2 + 72q_1^2q_2^3 + 864q_1^3q_2^2 + 2304q_1^3q_2^3 + 864q_1^3q_2^4 + 324q_1^4q_2^2$ $+6480q_1^4q_2^3 + 6804q_1^4q_2^4 + 6480q_1^4q_2^5 + 324q_1^4q_2^6 + 5760q_1^5q_2^3 + 12960q_1^5q_2^4$ $+23040q_1^5q_2^5 + 12960q_1^5q_2^6 + 5760q_1^5q_2^7$
4	$1 + 720q_1^2q_2^2 + 2880q_1^3q_2^2 + 7680q_1^3q_2^3 + 2880q_1^3q_2^4 + 720q_1^4q_2^2 + 23040q_1^4q_2^3$ $+49680q_1^4q_2^4 + 23040q_1^4q_2^5 + 720q_1^4q_2^6 + 23040q_1^5q_2^3 + 123840q_1^5q_2^4$ $+161280q_1^5q_2^5 + 123840q_1^5q_2^6 + 23040q_1^5q_2^7$
5	$1 + 540q_1^2q_2^2 + 2880q_1^3q_2^2 + 15360q_1^3q_2^3 + 2880q_1^3q_2^4 + 540q_1^4q_2^2 + 69120q_1^4q_2^3$ $+147420q_1^4q_2^4 + 69120q_1^4q_2^5 + 540q_1^4q_2^6 + 69120q_1^5q_2^3 + 527040q_1^5q_2^4 + 940032q_1^5q_2^5$ $+527040q_1^5q_2^6 + 69120q_1^5q_2^7$
6	$1 + 3024q_1^3q_2^2 + 20160q_1^3q_2^3 + 3024q_1^3q_2^4 + 108864q_1^4q_2^3 + 312984q_1^4q_2^4 + 108864q_1^4q_2^5$ $+108864q_1^5q_2^3 + 1605744q_1^5q_2^4 + 2939328q_1^5q_2^5 + 1605744q_1^5q_2^6 + 108864q_1^5q_2^7$
7	$1 + 17472q_1^3q_2^3 + 1512q_1^4q_2^2 + 157248q_1^4q_2^3 + 424116q_1^4q_2^4 + 157248q_1^4q_2^5 + 1512q_1^4q_2^6$ $+108864q_1^5q_2^3 + 3048192q_1^5q_2^4 + 7076160q_1^5q_2^5 + 3048192q_1^5q_2^6 + 108864q_1^5q_2^7$
8	$1 + 9600q_1^3q_2^3 + 114624q_1^4q_2^2 - 337536q_1^4q_2^3 + 1184544q_1^4q_2^4 - 337536q_1^4q_2^5$ $+114624q_1^4q_2^6 - 2630016q_1^5q_2^3 + 15704064q_1^5q_2^4 - 5152896q_1^5q_2^5$ $+15704064q_1^5q_2^6 - 2630016q_1^5q_2^7$
9	$1 + 555228/5q_1^3q_2^3 - 420552/5q_1^4q_2^2 + 470124/5q_1^4q_2^3 - 6861348/5q_1^4q_2^4 + 470124/5q_1^4q_2^5$

	$-420552/5q_1^4q_2^6+7036308/5q_1^5q_2^3+19058976/5q_1^5q_2^4+146870172/5q_1^5q_2^5$ $+19058976/5q_1^5q_2^6+7036308/5q_1^5q_2^7$
10	$1+239880q_1^3q_2^3+298800q_1^4q_2^2+959400q_1^4q_2^3-3595320q_1^4q_2^4+959400q_1^4q_2^5$ $+298800q_1^4q_2^6-4902120q_1^5q_2^3-40538880q_1^5q_2^4+104569704q_1^5q_2^5-40538880q_1^5q_2^6$ $-4902120q_1^5q_2^7$
11	$1+300168q_1^4q_2^2-2013792q_1^4q_2^3+3601092q_1^4q_2^4-2013792q_1^4q_2^5+300168q_1^4q_2^6$ $-9172416q_1^5q_2^3+50160000q_1^5q_2^4-62815104q_1^5q_2^5+50160000q_1^5q_2^6-9172416q_1^5q_2^7$
12	$1+1312116/7q_1^3q_2^3-330912q_1^4q_2^2-3751524q_1^4q_2^3-764352\ 0/7q_1^4q_2^4-3751524q_1^4q_2^5$ $-330912q_1^4q_2^6-18745596q_1^5q_2\ ^3+847814400/7q_1^5q_2^4+823280004/7q_1^5q_2^5$ $+847814400/7q_1^5q_2^6-18745596q_1^5q_2^7$

k	Extremal Hilbert modular form in $M_k^{+,2+\sqrt{3}}(\mathrm{SL}_2(\mathbb{Z}[\sqrt{3}]))$
2	$1+72q_1^1q_2^2+96q_1^1q_2^3+72q_1^1q_2^4+96q_1^2q_2^3+360q_1^2q_2^4+288q_1^2q_2^5+672q_1^2q_2^6+288q_1^2q_2^7$ $+360q_1^2q_2^8+96q_1^2q_2^9+72q_1^3q_2^4+288q_1^3q_2^5+936q_1^3q_2^6+576q_1^3q_2^7+1008q_1^3q_2^8+960q_1^3q_2^9$ $+1008q_1^3q_2^{10}+576q_1^3q_2^{11}+936q_1^3q_2^{12}+288q_1^3q_2^{13}+72q_1^3q_2^{14}$
4	$1+480q_1^1q_2^3+480q_1^2q_2^3+6480q_1^2q_2^4+12960q_1^2q_2^5+22080q_1^2q_2^6+12960q_1^2q_2^7$ $+6480q_1^2q_2^8+480q_1^2q_2^9+12960q_1^3q_2^5+69120q_1^3q_2^6+129600q_1^3q_2^7+207360q_1^3q_2^8$ $+212160q_1^3q_2^9+207360q_1^3q_2^{10}+129600q_1^3q_2^{11}+69120q_1^3q_2^{12}+12960q_1^3q_2^{13}$
6	$1+10584q_1^2q_2^4+48384q_1^2q_2^5+78624q_1^2q_2^6+48384q_1^2q_2^7+10584q_1^2q_2^8+48384q_1^3q_2^5$ $+580608q_1^3q_2^6+1935360q_1^3q_2^7+3677184q_1^3q_2^8+4290048q_1^3q_2^9+3677184q_1^3q_2^{10}$ $+1935360q_1^3q_2^{11}+580608q_1^3q_2^{12}+48384q_1^3q_2^{13}$
8	$1+51840q_1^2q_2^5+43200q_1^2q_2^6+51840q_1^2q_2^7+51840q_1^3q_2^5+1152000q_1^3q_2^6+6220800q_1^3q_2^7$ $+15344640q_1^3q_2^8+19219200q_1^3q_2^9+15344640q_1^3q_2^{10}+6220800q_1^3q_2^{11}+1152000q_1^3q_2^{12}$ $+51840q_1^3q_2^{13}$
10	$1+39600q_1^2q_2^6+918720q_1^3q_2^6+6842880q_1^3q_2^7+21384000q_1^3q_2^8+29568000q_1^3q_2^9$ $+21384000q_1^3q_2^{10}+6842880q_1^3q_2^{11}+918720q_1^3q_2^{12}$
12	$1+150565824/989q_1^3q_2^6+3762153216/989q_1^3q_2^7+11589583680/989q_1^3q_2^8$ $+20834818560/989q_1^3q_2^9+11589583680/989q_1^3q_2^{10}$ $+3762153216/989q_1^3q_2^{11}+150565824/989q_1^3q_2^{12}$

14	$1 + 659307687696/1486873 q_1^2 q_2^6 - 2419438147576128/786555817 q_1^3 {q_2}^6$
	$+299534511209472/786555817 q_1^3 q_2^7 + 26285312780806464/786555817 q_1^3 q_2^8$
	$+30896464744384512/786555817 q_1^3 q_2^9 + 26285312780806464/786555817 q_1^3 q_2^{10}$
	$+299534511209472/786555817 q_1^3 q_2^{11} - 2419438147576128/786555817 q_1^3 {q_2}^{12}$
16	$1 + 1238518656/3233 q_1^2 q_2^6 + 281151066014208/39335911 q_1^3 q_2^6$
	$+591393083489280/39335911 q_1^3 q_2^7 + 1124801909153280/39335911 q_1^3 q_2^8$
	$+2617854216112128/39335911 q_1^3 q_2^9 + 1124801909153280/39335911 q_1^3 q_2^{10} +$
	$591393083489280/39335911 q_1^3 q_2^{11} + 281151066014208/39335911 q_1^3 q_2^{12}$
18	$1 + 46692143876859408/83485655063 q_1^2 q_2^6$
	$-366804972086439292 4592/44163911528327 q_1^3 q_2^6$
	$-3214167115080216491 9712/44163911528327 q_1^3 q_2^7$
	$-10279954162809226341 0000/44163911528327 q_1^3 q_2^8$
	$-15126159818017906100 2944/44163911528327 q_1^3 q_2^9$
	$-10279954162809226341 0000/44163911528327 q_1^3 q_2^{10}$
	$-3214167115080216491 9712/44163911528327 q_1^3 q_2^{11}$
	$-366804972086439292 4592/44163911528327 q_1^3 q_2^{12}$
20	$1 + 7135653802139447703019 20/102796424303780101 3 q_1^2 q_2^6$
	$+37038929550202235651752 3680/337759679855277475 7 q_1^3 q_2^6$
	$+21510355017040316810697 597120/2364317758986942329 9 q_1^3 q_2^7$
	$+69711458121540665510648 824800/2364317758986942329 9 q_1^3 q_2^8$
	$+10336752207164687445439 8280320/2364317758986942329 9 q_1^3 q_2^9$
	$+69711458121540665510648 824800/2364317758986942329 9 q_1^3 q_2^{10}$
	$+21510355017040316810697 597120/2364317758986942329 9 q_1^3 q_2^{11}$
	$+37038929550202235651752 3680/337759679855277475 7 q_1^3 q_2^{12}$

Extremal Type (ii) Lattices

Dimension 2:
The only Type (ii) lattice is G_2.
Gram matrix:

$$\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$$

Dimension 4:
There are $G_2 \perp G_2$ and F_4.
Gram matrix of F_4:

$$\begin{pmatrix} 2 & 1 & 1+\sqrt{3} & 1+\sqrt{3} \\ 1 & 2 & 1+\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} & 4+2\sqrt{3} & 2+\sqrt{3} \\ 1+\sqrt{3} & 1+\sqrt{3} & 2+\sqrt{3} & 4+2\sqrt{3} \end{pmatrix},$$

Dimension 6:
There is one extremla lattice. Its first trace lattice is K_{12}.
Gram matrix:

$$\begin{pmatrix} 4-2\sqrt{3} & 0 & 2-1\sqrt{3} & 2-1\sqrt{3} & -2\sqrt{3} & -464+268\sqrt{3} \\ 0 & 4-2\sqrt{3} & -2+1\sqrt{3} & 0 & -2+1\sqrt{3} & 2-1\sqrt{3} \\ 2-1\sqrt{3} & -2+1\sqrt{3} & 4 & 2 & -12-9\sqrt{3} & -250+144\sqrt{3} \\ 2-1\sqrt{3} & 0 & 2 & 4 & -14-9\sqrt{3} & -248+144\sqrt{3} \\ -2\sqrt{3} & -2+1\sqrt{3} & -12-9\sqrt{3} & -14-9\sqrt{3} & 252+128\sqrt{3} & 1381-812\sqrt{3} \\ -464+268\sqrt{3} & 2-1\sqrt{3} & -250+144\sqrt{3} & -248+144\sqrt{3} & 1381-812\sqrt{3} & 252938-146032\sqrt{3} \end{pmatrix}$$

Dimension 8:

There are three extremal lattices. Gram matrices:

$$\begin{pmatrix}
4 & 1\sqrt{3} & 2+2\sqrt{3} & -2+1\sqrt{3} & -3-2\sqrt{3} & -201+116\sqrt{3} & -86-49\sqrt{3} & -1\sqrt{3} \\
1\sqrt{3} & 4 & 2+1\sqrt{3} & 2 & 3+3\sqrt{3} & 195-113\sqrt{3} & -118-69\sqrt{3} & 0 \\
2+2\sqrt{3} & 2+1\sqrt{3} & 6+2\sqrt{3} & 0 & -4-2\sqrt{3} & -2+2\sqrt{3} & -204-118\sqrt{3} & -2-3\sqrt{3} \\
-2+1\sqrt{3} & 2 & 0 & 6-2\sqrt{3} & 6+4\sqrt{3} & 412-238\sqrt{3} & -32-20\sqrt{3} & -2+2\sqrt{3} \\
-3-2\sqrt{3} & 3+3\sqrt{3} & -4-2\sqrt{3} & 6+4\sqrt{3} & 52+20\sqrt{3} & -182+101\sqrt{3} & 63+39\sqrt{3} & 3+5\sqrt{3} \\
-201+116\sqrt{3} & 195-113\sqrt{3} & -2+2\sqrt{3} & 412-238\sqrt{3} & -182+101\sqrt{3} & 44094-25454\sqrt{3} & 18-157\sqrt{3} & -237+130\sqrt{3} \\
-86-49\sqrt{3} & -118-69\sqrt{3} & -204-118\sqrt{3} & -32-20\sqrt{3} & 63+39\sqrt{3} & 18-157\sqrt{3} & 15318+8842\sqrt{3} & 323+185\sqrt{3} \\
-1\sqrt{3} & 0 & -2-3\sqrt{3} & -2+2\sqrt{3} & 3+5\sqrt{3} & -237+130\sqrt{3} & 323+185\sqrt{3} & 24+6\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2\sqrt{3} & -2+1\sqrt{3} & 2+1\sqrt{3} & 1 & -283-164\sqrt{3} & -4003-2311\sqrt{3} & -42914-24777\sqrt{3} \\
2\sqrt{3} & 4 & 2-1\sqrt{3} & 2+1\sqrt{3} & 1+1\sqrt{3} & -346-201\sqrt{3} & -4867-2809\sqrt{3} & -52184-30129\sqrt{3} \\
-2+1\sqrt{3} & 2-1\sqrt{3} & 6-2\sqrt{3} & 1\sqrt{3} & -2+8\sqrt{3} & -42-27\sqrt{3} & -952-550\sqrt{3} & -9984-5765\sqrt{3} \\
2+1\sqrt{3} & 2+1\sqrt{3} & 1\sqrt{3} & 6+2\sqrt{3} & 2\sqrt{3} & -631-365\sqrt{3} & -9281-5358\sqrt{3} & -99232-57291\sqrt{3} \\
1 & 1+1\sqrt{3} & -2+8\sqrt{3} & 2\sqrt{3} & 106-6\sqrt{3} & 760+427\sqrt{3} & 8449+4900\sqrt{3} & 91940+53072\sqrt{3} \\
-283-164\sqrt{3} & -346-201\sqrt{3} & -42-27\sqrt{3} & -631-365\sqrt{3} & 760+427\sqrt{3} & 129144+74494\sqrt{3} & 1806103+1042763\sqrt{3} & 19358544+11176641\sqrt{3} \\
-4003-2311\sqrt{3} & -4867-2809\sqrt{3} & -952-550\sqrt{3} & -9281-5358\sqrt{3} & 8449+4900\sqrt{3} & 1806103+1042763\sqrt{3} & 25416772+14674348\sqrt{3} & 272338701+157234828\sqrt{3} \\
-42914-24777\sqrt{3} & -52184-30129\sqrt{3} & -9984-5765\sqrt{3} & -99232-57291\sqrt{3} & 91940+53072\sqrt{3} & 19358544+11176641\sqrt{3} & 272338701+157234828\sqrt{3} & 2918145212+1684791916\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & -2-1\sqrt{3} & -2+1\sqrt{3} & -6+4\sqrt{3} & -178+104\sqrt{3} & -51+55\sqrt{3} & -344-204\sqrt{3} & 25245-14575\sqrt{3} \\
-2-1\sqrt{3} & 6+2\sqrt{3} & 1\sqrt{3} & 10-7\sqrt{3} & -9+4\sqrt{3} & -11+10\sqrt{3} & 280+166\sqrt{3} & 932-537\sqrt{3} \\
-2+1\sqrt{3} & 1\sqrt{3} & 6-2\sqrt{3} & -2+1\sqrt{3} & 359-207\sqrt{3} & -3+3\sqrt{3} & -78-48\sqrt{3} & -53003+30602\sqrt{3} \\
-6+4\sqrt{3} & 10-7\sqrt{3} & -2+1\sqrt{3} & 146-56\sqrt{3} & 636-381\sqrt{3} & 18-37\sqrt{3} & -226-148\sqrt{3} & -96475+55698\sqrt{3} \\
-178+104\sqrt{3} & -9+4\sqrt{3} & 359-207\sqrt{3} & 636-381\sqrt{3} & 34120-19662\sqrt{3} & -277+366\sqrt{3} & 562-945\sqrt{3} & -5062979+2923098\sqrt{3} \\
-51+55\sqrt{3} & -11+10\sqrt{3} & -3+3\sqrt{3} & 18-37\sqrt{3} & -277+366\sqrt{3} & 10226-3090\sqrt{3} & -8303-3307\sqrt{3} & 179298-103403\sqrt{3} \\
-344-204\sqrt{3} & 280+166\sqrt{3} & -78-48\sqrt{3} & -226-148\sqrt{3} & 562-945\sqrt{3} & -8303-3307\sqrt{3} & 78712+44328\sqrt{3} & -179478+103642\sqrt{3} \\
25245-14575\sqrt{3} & 932-537\sqrt{3} & -53003+30602\sqrt{3} & -96475+55698\sqrt{3} & -5062979+2923098\sqrt{3} & 179298-103403\sqrt{3} & -179478+103642\sqrt{3} & 753916638-435273932\sqrt{3}
\end{pmatrix}$$

Dimension 10:
There are 21 extremal lattices. Gram matrices:

$$\begin{pmatrix}
4 & 2 & 2 & 1\sqrt{3} & 0 & 2+1\sqrt{3} & -3+1\sqrt{3} & -2383-1377\sqrt{3} & 64-37\sqrt{3} & -205-122\sqrt{3} \\
2 & 4 & 2 & -1\sqrt{3} & 1\sqrt{3} & 2+1\sqrt{3} & -1+1\sqrt{3} & -2385-1378\sqrt{3} & 64-37\sqrt{3} & -201-120\sqrt{3} \\
2 & 2 & 4 & 0 & -1\sqrt{3} & 2+2\sqrt{3} & -1 & -2389-1380\sqrt{3} & 64-38\sqrt{3} & -197-118\sqrt{3} \\
1\sqrt{3} & -1\sqrt{3} & 0 & 4 & -2 & 0 & -1\sqrt{3} & 2+1\sqrt{3} & 0 & -4-2\sqrt{3} \\
0 & 1\sqrt{3} & -1\sqrt{3} & -2 & 4 & -2 & 2 & 4+2\sqrt{3} & 2 & -4-2\sqrt{3} \\
2+1\sqrt{3} & 2+1\sqrt{3} & 2+2\sqrt{3} & 0 & -2 & 6+2\sqrt{3} & 2-1\sqrt{3} & -4658-2690\sqrt{3} & -295+170\sqrt{3} & -404-241\sqrt{3} \\
-3+1\sqrt{3} & -1+1\sqrt{3} & -1 & -1\sqrt{3} & 2 & 2-1\sqrt{3} & 90-40\sqrt{3} & -2741-1587\sqrt{3} & -4062+2348\sqrt{3} & -495-305\sqrt{3} \\
-2383-1377\sqrt{3} & -2385-1378\sqrt{3} & -2389-1380\sqrt{3} & 2+1\sqrt{3} & 4+2\sqrt{3} & -4658-2690\sqrt{3} & -2741-1587\sqrt{3} & 5829242+3365428\sqrt{3} & -1507-49\sqrt{3} & 580738+335583\sqrt{3} \\
64-37\sqrt{3} & 64-37\sqrt{3} & 64-38\sqrt{3} & 0 & 2 & -295+170\sqrt{3} & -4062+2348\sqrt{3} & -1507-49\sqrt{3} & 257938-148916\sqrt{3} & -1830+927\sqrt{3} \\
-205-122\sqrt{3} & -201-120\sqrt{3} & -197-118\sqrt{3} & -4-2\sqrt{3} & -4-2\sqrt{3} & -404-241\sqrt{3} & -495-305\sqrt{3} & 580738+335583\sqrt{3} & -1830+927\sqrt{3} & 66128+36932\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2\sqrt{3} & 2 & 1\sqrt{3} & 2 & 1\sqrt{3} & -1\sqrt{3} & 5+2\sqrt{3} & -7-3\sqrt{3} & -2 \\
2\sqrt{3} & 4 & 1\sqrt{3} & 2 & 1\sqrt{3} & 2 & -2 & 7+5\sqrt{3} & -9-5\sqrt{3} & 1 \\
2 & 1\sqrt{3} & 4 & 2\sqrt{3} & 2 & 1\sqrt{3} & -1\sqrt{3} & 3+2\sqrt{3} & -5-2\sqrt{3} & -2 \\
1\sqrt{3} & 2 & 2\sqrt{3} & 4 & 1\sqrt{3} & 2 & -2 & 7+4\sqrt{3} & -7-4\sqrt{3} & 1 \\
2 & 1\sqrt{3} & 2 & 1\sqrt{3} & 4 & 2\sqrt{3} & -1\sqrt{3} & 3+2\sqrt{3} & -5-2\sqrt{3} & -2 \\
1\sqrt{3} & 2 & 1\sqrt{3} & 2 & 2\sqrt{3} & 4 & -2 & 7+4\sqrt{3} & -7-4\sqrt{3} & 1 \\
-1\sqrt{3} & -2 & -1\sqrt{3} & -2 & -1\sqrt{3} & -2 & 10-4\sqrt{3} & -12-6\sqrt{3} & 6+4\sqrt{3} & 8+3\sqrt{3} \\
5+2\sqrt{3} & 7+5\sqrt{3} & 3+2\sqrt{3} & 7+4\sqrt{3} & 3+2\sqrt{3} & 7+4\sqrt{3} & -12-6\sqrt{3} & 140+72\sqrt{3} & 29+18\sqrt{3} & -35-23\sqrt{3} \\
-7-3\sqrt{3} & -9-5\sqrt{3} & -5-2\sqrt{3} & -7-4\sqrt{3} & -5-2\sqrt{3} & -7-4\sqrt{3} & 6+4\sqrt{3} & 29+18\sqrt{3} & 384+218\sqrt{3} & -17-11\sqrt{3} \\
-2 & 1 & -2 & 1 & -2 & 1 & 8+3\sqrt{3} & -35-23\sqrt{3} & -17-11\sqrt{3} & 78+42\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2\sqrt{3} & 2 & 1\sqrt{3} & 2-1\sqrt{3} & -3-2\sqrt{3} & -1+1\sqrt{3} & -2373+1369\sqrt{3} & 1502+868\sqrt{3} & -120664+69665\sqrt{3} \\
2\sqrt{3} & 4 & 1\sqrt{3} & 2 & -2+1\sqrt{3} & -4-2\sqrt{3} & 1\sqrt{3} & 2242-1295\sqrt{3} & 2510+1449\sqrt{3} & 114078-65864\sqrt{3} \\
2 & 1\sqrt{3} & 4 & 2\sqrt{3} & 2-1\sqrt{3} & -3-2\sqrt{3} & -1+1\sqrt{3} & -2377+1372\sqrt{3} & 1498+866\sqrt{3} & -120664+69665\sqrt{3} \\
1\sqrt{3} & 2 & 2\sqrt{3} & 4 & -2+1\sqrt{3} & -4-2\sqrt{3} & 1\sqrt{3} & 2248-1297\sqrt{3} & 2506+1447\sqrt{3} & 114078-65864\sqrt{3} \\
2-1\sqrt{3} & -2+1\sqrt{3} & 2-1\sqrt{3} & -2+1\sqrt{3} & 6-2\sqrt{3} & -1-1\sqrt{3} & -7+5\sqrt{3} & -4632+2674\sqrt{3} & -1314-761\sqrt{3} & -234795+135559\sqrt{3} \\
-3-2\sqrt{3} & -4-2\sqrt{3} & -3-2\sqrt{3} & -4-2\sqrt{3} & -1-1\sqrt{3} & 32+18\sqrt{3} & -5-10\sqrt{3} & 114-63\sqrt{3} & -10508-6067\sqrt{3} & 5720-3298\sqrt{3} \\
-1+1\sqrt{3} & 1\sqrt{3} & -1+1\sqrt{3} & 1\sqrt{3} & -7+5\sqrt{3} & -5-10\sqrt{3} & 174-72\sqrt{3} & 16167-9324\sqrt{3} & 7658+4433\sqrt{3} & 819954-473405\sqrt{3} \\
-2373+1369\sqrt{3} & 2242-1295\sqrt{3} & -2377+1372\sqrt{3} & 2248-1297\sqrt{3} & -4632+2674\sqrt{3} & 114-63\sqrt{3} & 16167-9324\sqrt{3} & 5793018-3344556\sqrt{3} & 121+1714\sqrt{3} & 294052774-169771433\sqrt{3} \\
1502+868\sqrt{3} & 2510+1449\sqrt{3} & 1498+866\sqrt{3} & 2506+1447\sqrt{3} & -1314-761\sqrt{3} & -10508-6067\sqrt{3} & 7658+4433\sqrt{3} & 121+1714\sqrt{3} & 8141830+4700672\sqrt{3} & -74984+39566\sqrt{3} \\
-120664+69665\sqrt{3} & 114078-65864\sqrt{3} & -120664+69665\sqrt{3} & 114078-65864\sqrt{3} & -234795+135559\sqrt{3} & 5720-3298\sqrt{3} & 819954-473405\sqrt{3} & 294052774-169771433\sqrt{3} & -74984+39566\sqrt{3} & 14926342420-8617727804\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2\sqrt{3} & 2 & 1\sqrt{3} & 2-1\sqrt{3} & -3-2\sqrt{3} & -1+1\sqrt{3} & -2373+1369\sqrt{3} & 1502+868\sqrt{3} & -120664+69665\sqrt{3} \\
2\sqrt{3} & 4 & 1\sqrt{3} & 2 & -2+1\sqrt{3} & -4-2\sqrt{3} & 1\sqrt{3} & 2242-1295\sqrt{3} & 2510+1449\sqrt{3} & 114078-65864\sqrt{3} \\
2 & 1\sqrt{3} & 4 & 2\sqrt{3} & 2-1\sqrt{3} & -3-2\sqrt{3} & -1+1\sqrt{3} & -2377+1372\sqrt{3} & 1498+866\sqrt{3} & -120664+69665\sqrt{3} \\
1\sqrt{3} & 2 & 2\sqrt{3} & 4 & -2+1\sqrt{3} & -4-2\sqrt{3} & 1\sqrt{3} & 2248-1297\sqrt{3} & 2506+1447\sqrt{3} & 114078-65864\sqrt{3} \\
2-1\sqrt{3} & -2+1\sqrt{3} & 2-1\sqrt{3} & -2+1\sqrt{3} & 6-2\sqrt{3} & -1-1\sqrt{3} & -7+5\sqrt{3} & -4632+2674\sqrt{3} & -1314-761\sqrt{3} & -234795+135559\sqrt{3} \\
-3-2\sqrt{3} & -4-2\sqrt{3} & -3-2\sqrt{3} & -4-2\sqrt{3} & -1-1\sqrt{3} & 32+18\sqrt{3} & -5-10\sqrt{3} & 114-63\sqrt{3} & -10508-6067\sqrt{3} & 5720-3298\sqrt{3} \\
-1+1\sqrt{3} & 1\sqrt{3} & -1+1\sqrt{3} & 1\sqrt{3} & -7+5\sqrt{3} & -5-10\sqrt{3} & 174-72\sqrt{3} & 16167-9324\sqrt{3} & 7658+4433\sqrt{3} & 819954-473405\sqrt{3} \\
-2373+1369\sqrt{3} & 2242-1295\sqrt{3} & -2377+1372\sqrt{3} & 2248-1297\sqrt{3} & -4632+2674\sqrt{3} & 114-63\sqrt{3} & 16167-9324\sqrt{3} & 5793018-3344556\sqrt{3} & 121+1714\sqrt{3} & 294052774-169771433\sqrt{3} \\
1502+868\sqrt{3} & 2510+1449\sqrt{3} & 1498+866\sqrt{3} & 2506+1447\sqrt{3} & -1314-761\sqrt{3} & -10508-6067\sqrt{3} & 7658+4433\sqrt{3} & 121+1714\sqrt{3} & 8141830+4700672\sqrt{3} & -74984+39566\sqrt{3} \\
-120664+69665\sqrt{3} & 114078-65864\sqrt{3} & -120664+69665\sqrt{3} & 114078-65864\sqrt{3} & -234795+135559\sqrt{3} & 5720-3298\sqrt{3} & 819954-473405\sqrt{3} & 294052774-169771433\sqrt{3} & -74984+39566\sqrt{3} & 14926342420-8617727804\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & -2 & 0 & 4-2\sqrt{3} & -1\sqrt{3} & 1 & -4-2\sqrt{3} & -3-2\sqrt{3} & -8-4\sqrt{3} & -4-2\sqrt{3} \\
-2 & 6+2\sqrt{3} & -2-1\sqrt{3} & -2+1\sqrt{3} & -2-1\sqrt{3} & 8+5\sqrt{3} & -12-8\sqrt{3} & -1 & 246+142\sqrt{3} & 1074+620\sqrt{3} \\
0 & -2-1\sqrt{3} & 6-2\sqrt{3} & 2-1\sqrt{3} & 2 & -4 & 10+6\sqrt{3} & 1+1\sqrt{3} & -125-74\sqrt{3} & -569-329\sqrt{3} \\
4-2\sqrt{3} & -2+1\sqrt{3} & 2-1\sqrt{3} & 12-6\sqrt{3} & 0 & -2 & 2+2\sqrt{3} & 4-1\sqrt{3} & -8-1\sqrt{3} & -55-35\sqrt{3} \\
-1\sqrt{3} & -2-1\sqrt{3} & 2 & 0 & 6 & -7-5\sqrt{3} & 6+2\sqrt{3} & 2\sqrt{3} & -105-60\sqrt{3} & -504-290\sqrt{3} \\
1 & 8+5\sqrt{3} & -4 & -2 & -7-5\sqrt{3} & 156+88\sqrt{3} & 29+16\sqrt{3} & -76-45\sqrt{3} & 39+23\sqrt{3} & 228+132\sqrt{3} \\
-4-2\sqrt{3} & -12-8\sqrt{3} & 10+6\sqrt{3} & 2+2\sqrt{3} & 6+2\sqrt{3} & 29+16\sqrt{3} & 144+78\sqrt{3} & 3 & -1238-718\sqrt{3} & -5790-3342\sqrt{3} \\
-3-2\sqrt{3} & -1 & 1+1\sqrt{3} & 4-1\sqrt{3} & 2\sqrt{3} & -76-45\sqrt{3} & 3 & 66+24\sqrt{3} & 51+30\sqrt{3} & 147+79\sqrt{3} \\
-8-4\sqrt{3} & 246+142\sqrt{3} & -125-74\sqrt{3} & -8-1\sqrt{3} & -105-60\sqrt{3} & 39+23\sqrt{3} & -1238-718\sqrt{3} & 51+30\sqrt{3} & 16348+9400\sqrt{3} & 72751+42018\sqrt{3} \\
-4-2\sqrt{3} & 1074+620\sqrt{3} & -569-329\sqrt{3} & -55-35\sqrt{3} & -504-290\sqrt{3} & 228+132\sqrt{3} & -5790-3342\sqrt{3} & 147+79\sqrt{3} & 72751+42018\sqrt{3} & 328842+189846\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & \sqrt{3} & -1 & 2 & 2-2\sqrt{3} & 7+5\sqrt{3} & -681+393\sqrt{3} & 128300-74073\sqrt{3} & -2784412+1607580\sqrt{3} & -3501-2022\sqrt{3} \\
\sqrt{3} & 4 & 1-1\sqrt{3} & 2+2\sqrt{3} & -2+1\sqrt{3} & 6+5\sqrt{3} & 672-388\sqrt{3} & -127117+73391\sqrt{3} & 2758978-1592897\sqrt{3} & -1332-770\sqrt{3} \\
-1 & 1-1\sqrt{3} & 4 & 0 & -1 & 1\sqrt{3} & 582-335\sqrt{3} & -107887+62289\sqrt{3} & 2341618-1351933\sqrt{3} & 2046+1181\sqrt{3} \\
2 & 2+2\sqrt{3} & 0 & 8+2\sqrt{3} & 2 & 19+14\sqrt{3} & 8-6\sqrt{3} & 1151-664\sqrt{3} & -25537+14742\sqrt{3} & -1839-1063\sqrt{3} \\
2-2\sqrt{3} & -2+1\sqrt{3} & -1 & 2 & 8-2\sqrt{3} & 4 & -1353+780\sqrt{3} & 255454-147487\sqrt{3} & -5543604+3200601\sqrt{3} & 869+501\sqrt{3} \\
7+5\sqrt{3} & 6+5\sqrt{3} & 1\sqrt{3} & 19+14\sqrt{3} & 4 & 170+76\sqrt{3} & -1630+913\sqrt{3} & 298586-172351\sqrt{3} & -6478082+3740114\sqrt{3} & -20398-11771\sqrt{3} \\
-681+393\sqrt{3} & 672-388\sqrt{3} & 582-335\sqrt{3} & 8-6\sqrt{3} & -1353+780\sqrt{3} & -1630+913\sqrt{3} & 499236-288198\sqrt{3} & -93399788+53924391\sqrt{3} & 2026878836-1170219039\sqrt{3} & 1823+780\sqrt{3} \\
128300-74073\sqrt{3} & -127117+73391\sqrt{3} & -107887+62289\sqrt{3} & 1151-664\sqrt{3} & 255454-147487\sqrt{3} & 298586-172351\sqrt{3} & -93399788+53924391\sqrt{3} & 17480521466-10092383734\sqrt{3} & -379347210054+219016213854\sqrt{3} & -49639+21356\sqrt{3} \\
-2784412+1607580\sqrt{3} & 2758978-1592897\sqrt{3} & 2341618-1351933\sqrt{3} & -25537+14742\sqrt{3} & -5543604+3200601\sqrt{3} & -6478082+3740114\sqrt{3} & 2026878836-1170219039\sqrt{3} & -379347210054+219016213854\sqrt{3} & 8232266446972-4752901249186\sqrt{3} & 941417-541773\sqrt{3} \\
-3501-2022\sqrt{3} & -1332-770\sqrt{3} & 2046+1181\sqrt{3} & -1839-1063\sqrt{3} & 869+501\sqrt{3} & -20398-11771\sqrt{3} & 1823+780\sqrt{3} & -49639+21356\sqrt{3} & 941417-541773\sqrt{3} & 8104916+4679368\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 1\sqrt{3} & -1\sqrt{3} & 2-2\sqrt{3} & -15-8\sqrt{3} & -27+16\sqrt{3} & 163-95\sqrt{3} & 265+153\sqrt{3} & 15850-9151\sqrt{3} & 7861+4538\sqrt{3} \\
1\sqrt{3} & 4 & 1\sqrt{3} & -2+1\sqrt{3} & -10-7\sqrt{3} & 38-23\sqrt{3} & -227+132\sqrt{3} & 211+123\sqrt{3} & -21481+12401\sqrt{3} & 5437+3138\sqrt{3} \\
-1\sqrt{3} & 1\sqrt{3} & 4 & 2 & 6+4\sqrt{3} & 2-2\sqrt{3} & -40+23\sqrt{3} & -10-6\sqrt{3} & -3525+2036\sqrt{3} & -994-574\sqrt{3} \\
2-2\sqrt{3} & -2+1\sqrt{3} & 2 & 8-2\sqrt{3} & 3+3\sqrt{3} & -65+38\sqrt{3} & 381-221\sqrt{3} & -417-239\sqrt{3} & 36782-21235\sqrt{3} & -12778-7380\sqrt{3} \\
-15-8\sqrt{3} & -10-7\sqrt{3} & 6+4\sqrt{3} & 3+3\sqrt{3} & 288+160\sqrt{3} & -24-7\sqrt{3} & 24-36\sqrt{3} & -760-440\sqrt{3} & 3529-2053\sqrt{3} & -12284-7091\sqrt{3} \\
-27+16\sqrt{3} & 38-23\sqrt{3} & 2-2\sqrt{3} & -65+38\sqrt{3} & -24-7\sqrt{3} & 1420-810\sqrt{3} & -8657+4996\sqrt{3} & -50-10\sqrt{3} & -815860+471039\sqrt{3} & -1815-1097\sqrt{3} \\
163-95\sqrt{3} & -227+132\sqrt{3} & -40+23\sqrt{3} & 381-221\sqrt{3} & 24-36\sqrt{3} & -8657+4996\sqrt{3} & 53534-30896\sqrt{3} & -197-254\sqrt{3} & 5040702-2910245\sqrt{3} & -10903-5929\sqrt{3} \\
265+153\sqrt{3} & 211+123\sqrt{3} & -10-6\sqrt{3} & -417-239\sqrt{3} & -760-440\sqrt{3} & -50-10\sqrt{3} & -197-254\sqrt{3} & 136496+78786\sqrt{3} & 11015-6699\sqrt{3} & 4085534+2358790\sqrt{3} \\
15850-9151\sqrt{3} & -21481+12401\sqrt{3} & -3525+2036\sqrt{3} & 36782-21235\sqrt{3} & 3529-2053\sqrt{3} & -815860+471039\sqrt{3} & 5040702-2910245\sqrt{3} & 11015-6699\sqrt{3} & 474987706-274234256\sqrt{3} & -36801+10297\sqrt{3} \\
7861+4538\sqrt{3} & 5437+3138\sqrt{3} & -994-574\sqrt{3} & -12778-7380\sqrt{3} & -12284-7091\sqrt{3} & -1815-1097\sqrt{3} & -10903-5929\sqrt{3} & 4085534+2358790\sqrt{3} & -36801+10297\sqrt{3} & 123218920+71140300\sqrt{3}
\end{pmatrix}$$

$$
\begin{pmatrix}
4 & 2 & 2+1\sqrt{3} & -2+1\sqrt{3} & -1\sqrt{3} & 1\sqrt{3} & 4+3\sqrt{3} & 1 & 2113-1221\sqrt{3} & -924+533\sqrt{3} \\
2 & 4 & 2+1\sqrt{3} & -2+1\sqrt{3} & -1\sqrt{3} & -1\sqrt{3} & 6+4\sqrt{3} & 1 & 2113-1221\sqrt{3} & -922+533\sqrt{3} \\
2+1\sqrt{3} & 2+1\sqrt{3} & 6+2\sqrt{3} & 0 & -1 & 0 & 9+5\sqrt{3} & -5-6\sqrt{3} & 267-154\sqrt{3} & -99+56\sqrt{3} \\
-2+1\sqrt{3} & -2+1\sqrt{3} & 0 & 6-2\sqrt{3} & 1+1\sqrt{3} & 0 & 1-1\sqrt{3} & 3+4\sqrt{3} & -3981+2298\sqrt{3} & 1729-998\sqrt{3} \\
-1\sqrt{3} & -1\sqrt{3} & -1 & 1+1\sqrt{3} & 6 & 0 & -4-2\sqrt{3} & -2-2\sqrt{3} & 1750-1011\sqrt{3} & -780+452\sqrt{3} \\
1\sqrt{3} & -1\sqrt{3} & 0 & 0 & 0 & 4 & -2-1\sqrt{3} & 0 & 0 & -1\sqrt{3} \\
4+3\sqrt{3} & 6+4\sqrt{3} & 9+5\sqrt{3} & 1-1\sqrt{3} & -4-2\sqrt{3} & -2-1\sqrt{3} & 26+14\sqrt{3} & 1\sqrt{3} & -1298+748\sqrt{3} & 562-329\sqrt{3} \\
1 & 1 & -5-6\sqrt{3} & 3+4\sqrt{3} & -2-2\sqrt{3} & 0 & 1\sqrt{3} & 152+52\sqrt{3} & 4693-2709\sqrt{3} & -1925+1181\sqrt{3} \\
2113-1221\sqrt{3} & 2113-1221\sqrt{3} & 267-154\sqrt{3} & -3981+2298\sqrt{3} & 1750-1011\sqrt{3} & 0 & -1298+748\sqrt{3} & 4693-2709\sqrt{3} & 4443240-2565302\sqrt{3} & -1932989+1116009\sqrt{3} \\
-924+533\sqrt{3} & -922+533\sqrt{3} & -99+56\sqrt{3} & 1729-998\sqrt{3} & -780+452\sqrt{3} & -1\sqrt{3} & 562-329\sqrt{3} & -1925+1181\sqrt{3} & -1932989+1116009\sqrt{3} & 841446-485618\sqrt{3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & -2 & 0 & 0 & 1\sqrt{3} & -2\sqrt{3} & 0 & 0 & 4-2\sqrt{3} & -2 \\
-2 & 4 & 0 & -2 & 0 & 1\sqrt{3} & -2 & -2 & -8+5\sqrt{3} & 18+12\sqrt{3} \\
0 & 0 & 6+2\sqrt{3} & 0 & -1+1\sqrt{3} & 0 & -1 & 198+115\sqrt{3} & -130+74\sqrt{3} & -31-22\sqrt{3} \\
0 & -2 & 0 & 6-2\sqrt{3} & 1-1\sqrt{3} & 0 & 1+1\sqrt{3} & 30+17\sqrt{3} & 1488-859\sqrt{3} & 15+12\sqrt{3} \\
1\sqrt{3} & 0 & -1+1\sqrt{3} & 1-1\sqrt{3} & 6 & -2 & -2-1\sqrt{3} & 39+22\sqrt{3} & 716-415\sqrt{3} & 123+87\sqrt{3} \\
-2\sqrt{3} & 1\sqrt{3} & 0 & 0 & -2 & 4 & 0 & 0 & 4-2\sqrt{3} & 1\sqrt{3} \\
0 & -2 & -1 & 1+1\sqrt{3} & -2-1\sqrt{3} & 0 & 6 & -37-22\sqrt{3} & -659+381\sqrt{3} & -76-54\sqrt{3} \\
0 & -2 & 198+115\sqrt{3} & 30+17\sqrt{3} & 39+22\sqrt{3} & 0 & -37-22\sqrt{3} & 14066+8118\sqrt{3} & 346+391\sqrt{3} & 2852+1606\sqrt{3} \\
4-2\sqrt{3} & -8+5\sqrt{3} & -130+74\sqrt{3} & 1488-859\sqrt{3} & 716-415\sqrt{3} & 4-2\sqrt{3} & -659+381\sqrt{3} & 346+391\sqrt{3} & 634414-366122\sqrt{3} & -7473+4193\sqrt{3} \\
-2 & 18+12\sqrt{3} & -31-22\sqrt{3} & 15+12\sqrt{3} & 123+87\sqrt{3} & 1\sqrt{3} & -76-54\sqrt{3} & 2852+1606\sqrt{3} & -7473+4193\sqrt{3} & 12610+5924\sqrt{3}
\end{pmatrix}
$$

$$\begin{pmatrix}
4 & -\sqrt{3} & -2+1\sqrt{3} & -2+1\sqrt{3} & -1\sqrt{3} & 4+2\sqrt{3} & 8-11\sqrt{3} & 722-418\sqrt{3} & -165896+95781\sqrt{3} & -2973-1718\sqrt{3} \\
-\sqrt{3} & 6-2\sqrt{3} & 2-1\sqrt{3} & -2+1\sqrt{3} & 3+2\sqrt{3} & 1-1\sqrt{3} & 3-2\sqrt{3} & 3939-2274\sqrt{3} & -945893+546111\sqrt{3} & 2276+1315\sqrt{3} \\
-2+1\sqrt{3} & 2-1\sqrt{3} & 6-2\sqrt{3} & 0 & 2+3\sqrt{3} & 4+1\sqrt{3} & -12+14\sqrt{3} & 2175-1256\sqrt{3} & -532574+307482\sqrt{3} & 597+343\sqrt{3} \\
-2+1\sqrt{3} & -2+1\sqrt{3} & 0 & 6-2\sqrt{3} & -2+1\sqrt{3} & 1 & -14+15\sqrt{3} & -1577+911\sqrt{3} & 372480-215051\sqrt{3} & -351-204\sqrt{3} \\
-1\sqrt{3} & 3+2\sqrt{3} & 2+3\sqrt{3} & -2+1\sqrt{3} & 42+6\sqrt{3} & 28+25\sqrt{3} & -39+40\sqrt{3} & -8627+4992\sqrt{3} & 2057776-1188068\sqrt{3} & 3374+1955\sqrt{3} \\
4+2\sqrt{3} & 1-1\sqrt{3} & 4+1\sqrt{3} & 1 & 28+25\sqrt{3} & 126+66\sqrt{3} & 18-8\sqrt{3} & 5278-3056\sqrt{3} & -1261386+728248\sqrt{3} & -17334-10011\sqrt{3} \\
8-11\sqrt{3} & 3-2\sqrt{3} & -12+14\sqrt{3} & -14+15\sqrt{3} & -39+40\sqrt{3} & 18-8\sqrt{3} & 812-238\sqrt{3} & 39698-22851\sqrt{3} & -9340493+5392779\sqrt{3} & 2608+1290\sqrt{3} \\
722-418\sqrt{3} & 3939-2274\sqrt{3} & 2175-1256\sqrt{3} & -1577+911\sqrt{3} & -8627+4992\sqrt{3} & 5278-3056\sqrt{3} & 39698-22851\sqrt{3} & 8171084-4717470\sqrt{3} & -1950741602+1126261160\sqrt{3} & 19553+928\sqrt{3} \\
-165896+95781\sqrt{3} & -945893+546111\sqrt{3} & -532574+307482\sqrt{3} & 372480-215051\sqrt{3} & 2057776-1188068\sqrt{3} & -1261386+728248\sqrt{3} & -9340493+5392779\sqrt{3} & -1950741602+1126261160\sqrt{3} & 465770182838-268912540394\sqrt{3} & -2122016+1216864\sqrt{3} \\
-2973-1718\sqrt{3} & 2276+1315\sqrt{3} & 597+343\sqrt{3} & -351-204\sqrt{3} & 3374+1955\sqrt{3} & -17334-10011\sqrt{3} & 2608+1290\sqrt{3} & 19553+928\sqrt{3} & -2122016+1216864\sqrt{3} & 9210186+5315430\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & -1\sqrt{3} & -2+1\sqrt{3} & -2 & -4-3\sqrt{3} & -10+6\sqrt{3} & -18+11\sqrt{3} & -6-3\sqrt{3} & 5406-3122\sqrt{3} & -105-61\sqrt{3} \\
-1\sqrt{3} & 6+2\sqrt{3} & 2+1\sqrt{3} & 1\sqrt{3} & 1+2\sqrt{3} & -28+16\sqrt{3} & -6+4\sqrt{3} & -158-96\sqrt{3} & 11525-6652\sqrt{3} & -383-221\sqrt{3} \\
-2+1\sqrt{3} & 2+1\sqrt{3} & 6-2\sqrt{3} & -2+1\sqrt{3} & -2+1\sqrt{3} & 172-99\sqrt{3} & -60+36\sqrt{3} & -81-48\sqrt{3} & -53827+31077\sqrt{3} & -66-40\sqrt{3} \\
-2 & 1\sqrt{3} & -2+1\sqrt{3} & 8-2\sqrt{3} & 5+3\sqrt{3} & -376+218\sqrt{3} & 93-55\sqrt{3} & 63+41\sqrt{3} & 117669-67936\sqrt{3} & -384-224\sqrt{3} \\
-4-3\sqrt{3} & 1+2\sqrt{3} & -2+1\sqrt{3} & 5+3\sqrt{3} & 82+22\sqrt{3} & 411-243\sqrt{3} & -282+153\sqrt{3} & -315-202\sqrt{3} & -132988+76780\sqrt{3} & 3976+2302\sqrt{3} \\
-10+6\sqrt{3} & -28+16\sqrt{3} & 172-99\sqrt{3} & -376+218\sqrt{3} & 411-243\sqrt{3} & 37678-21722\sqrt{3} & -10853+6330\sqrt{3} & 737+252\sqrt{3} & -11997988+6927028\sqrt{3} & -1781-737\sqrt{3} \\
-18+11\sqrt{3} & -6+4\sqrt{3} & -60+36\sqrt{3} & 93-55\sqrt{3} & -282+153\sqrt{3} & -10853+6330\sqrt{3} & 7746-4334\sqrt{3} & 239+1039\sqrt{3} & 3356920-1938148\sqrt{3} & -2087-1157\sqrt{3} \\
-6-3\sqrt{3} & -158-96\sqrt{3} & -81-48\sqrt{3} & 63+41\sqrt{3} & -315-202\sqrt{3} & 737+252\sqrt{3} & 239+1039\sqrt{3} & 24518+13474\sqrt{3} & -16998+9476\sqrt{3} & -61916-35720\sqrt{3} \\
5406-3122\sqrt{3} & 11525-6652\sqrt{3} & -53827+31077\sqrt{3} & 117669-67936\sqrt{3} & -132988+76780\sqrt{3} & -11997988+6927028\sqrt{3} & 3356920-1938148\sqrt{3} & -16998+9476\sqrt{3} & 3831275294-2211987800\sqrt{3} & 82963-47646\sqrt{3} \\
-105-61\sqrt{3} & -383-221\sqrt{3} & -66-40\sqrt{3} & -384-224\sqrt{3} & 3976+2302\sqrt{3} & -1781-737\sqrt{3} & -2087-1157\sqrt{3} & -61916-35720\sqrt{3} & 82963-47646\sqrt{3} & 592380+341976\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2 & -1-1\sqrt{3} & 1\sqrt{3} & -3+1\sqrt{3} & 2-4\sqrt{3} & 119+69\sqrt{3} & -11726-6769\sqrt{3} & 13430-7763\sqrt{3} & 130949-75604\sqrt{3} \\
2 & 4 & -1-1\sqrt{3} & -1\sqrt{3} & -3+2\sqrt{3} & 4-4\sqrt{3} & 123+71\sqrt{3} & -11724-6768\sqrt{3} & 13434-7765\sqrt{3} & 130945-75602\sqrt{3} \\
-1-1\sqrt{3} & -1-1\sqrt{3} & 6+2\sqrt{3} & 0 & 1+1\sqrt{3} & -3+8\sqrt{3} & -234-136\sqrt{3} & 25503+14724\sqrt{3} & 6745-3901\sqrt{3} & 29984-17310\sqrt{3} \\
1\sqrt{3} & -1\sqrt{3} & 0 & 4 & -2 & -1\sqrt{3} & -4-2\sqrt{3} & -2-1\sqrt{3} & 4-2\sqrt{3} & -4+2\sqrt{3} \\
-3+1\sqrt{3} & -3+2\sqrt{3} & 1+1\sqrt{3} & -2 & 10-4\sqrt{3} & -3+8\sqrt{3} & -98-56\sqrt{3} & 10023+5784\sqrt{3} & -35287+20396\sqrt{3} & -358197+206805\sqrt{3} \\
2-4\sqrt{3} & 4-4\sqrt{3} & -3+8\sqrt{3} & -1\sqrt{3} & -3+8\sqrt{3} & 250-48\sqrt{3} & -931-479\sqrt{3} & 77785+44905\sqrt{3} & -32521+18767\sqrt{3} & -80280+46370\sqrt{3} \\
119+69\sqrt{3} & 123+71\sqrt{3} & -234-136\sqrt{3} & -4-2\sqrt{3} & -98-56\sqrt{3} & -931-479\sqrt{3} & 19376+11068\sqrt{3} & -1879513-1085142\sqrt{3} & 15048-11069\sqrt{3} & 294916-170043\sqrt{3} \\
-11726-6769\sqrt{3} & -11724-6768\sqrt{3} & 25503+14724\sqrt{3} & -2-1\sqrt{3} & 10023+5784\sqrt{3} & 77785+44905\sqrt{3} & -1879513-1085142\sqrt{3} & 187094206+108018842\sqrt{3} & 84716+61528\sqrt{3} & -134391+84752\sqrt{3} \\
13430-7763\sqrt{3} & 13434-7765\sqrt{3} & 6745-3901\sqrt{3} & 4-2\sqrt{3} & -35287+20396\sqrt{3} & -32521+18767\sqrt{3} & 15048-11069\sqrt{3} & 84716+61528\sqrt{3} & 195321132-112643556\sqrt{3} & 1822086463-1052019392\sqrt{3} \\
130949-75604\sqrt{3} & 130945-75602\sqrt{3} & 29984-17310\sqrt{3} & -4+2\sqrt{3} & -358197+206805\sqrt{3} & -80280+46370\sqrt{3} & 294916-170043\sqrt{3} & -134391+84752\sqrt{3} & 1822086463-1052019392\sqrt{3} & 18322274388-10578358756\sqrt{3}
\end{pmatrix}$$

$$
\begin{pmatrix}
4 & 2 & 1-1\sqrt{3} & 1+2\sqrt{3} & -2-1\sqrt{3} & 0 & 0 & -2+2\sqrt{3} & 77-46\sqrt{3} & 625-360\sqrt{3} \\
2 & 4 & -1\sqrt{3} & 1+1\sqrt{3} & -2-1\sqrt{3} & 0 & 1+1\sqrt{3} & -1+2\sqrt{3} & 55-32\sqrt{3} & 626-361\sqrt{3} \\
1-1\sqrt{3} & -1\sqrt{3} & 4 & -1 & 0 & -1 & 0 & 1\sqrt{3} & 76-44\sqrt{3} & 724-419\sqrt{3} \\
1+2\sqrt{3} & 1+1\sqrt{3} & -1 & 6+2\sqrt{3} & -3-2\sqrt{3} & 0 & -1 & 4-3\sqrt{3} & -30+16\sqrt{3} & -107+62\sqrt{3} \\
-2-1\sqrt{3} & -2-1\sqrt{3} & 0 & -3-2\sqrt{3} & 6+2\sqrt{3} & -1\sqrt{3} & -1 & 3-3\sqrt{3} & 10-7\sqrt{3} & 67-38\sqrt{3} \\
0 & 0 & -1 & 0 & -1\sqrt{3} & 4 & -1 & 1-2\sqrt{3} & 14-9\sqrt{3} & 24-13\sqrt{3} \\
0 & 1+1\sqrt{3} & 0 & -1 & -1 & -1 & 4 & -9+7\sqrt{3} & -37+21\sqrt{3} & -549+318\sqrt{3} \\
-2+2\sqrt{3} & -1+2\sqrt{3} & 1\sqrt{3} & 4-3\sqrt{3} & 3-3\sqrt{3} & 1-2\sqrt{3} & -9+7\sqrt{3} & 94-38\sqrt{3} & 174-98\sqrt{3} & 2685-1556\sqrt{3} \\
77-46\sqrt{3} & 55-32\sqrt{3} & 76-44\sqrt{3} & -30+16\sqrt{3} & 10-7\sqrt{3} & 14-9\sqrt{3} & -37+21\sqrt{3} & 174-98\sqrt{3} & 4918-2814\sqrt{3} & 46680-26961\sqrt{3} \\
625-360\sqrt{3} & 626-361\sqrt{3} & 724-419\sqrt{3} & -107+62\sqrt{3} & 67-38\sqrt{3} & 24-13\sqrt{3} & -549+318\sqrt{3} & 2685-1556\sqrt{3} & 46680-26961\sqrt{3} & 499484-288368\sqrt{3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & 2 & 0 & 1\sqrt{3} & 2+1\sqrt{3} & 1+1\sqrt{3} & -3083+1781\sqrt{3} & 37458+21626\sqrt{3} & 287-191\sqrt{3} & -164242+94826\sqrt{3} \\
2 & 4 & 0 & -1\sqrt{3} & 2+2\sqrt{3} & -1 & -3083+1780\sqrt{3} & 37450+21622\sqrt{3} & 287-191\sqrt{3} & -164242+94825\sqrt{3} \\
0 & 0 & 4 & 0 & 0 & 0 & -9+5\sqrt{3} & 50+29\sqrt{3} & 17-10\sqrt{3} & -452+260\sqrt{3} \\
1\sqrt{3} & -1\sqrt{3} & 0 & 4 & -2 & 2+1\sqrt{3} & 2 & 8+4\sqrt{3} & 0 & 2 \\
2+1\sqrt{3} & 2+2\sqrt{3} & 0 & -2 & 6+2\sqrt{3} & 9+9\sqrt{3} & 45-24\sqrt{3} & 74689+43121\sqrt{3} & 169-110\sqrt{3} & 1736-1000\sqrt{3} \\
1+1\sqrt{3} & -1 & 0 & 2+1\sqrt{3} & 9+9\sqrt{3} & 500+156\sqrt{3} & -14890+8571\sqrt{3} & 463230+267407\sqrt{3} & 3656-2442\sqrt{3} & -797826+460598\sqrt{3} \\
-3083+1781\sqrt{3} & -3083+1780\sqrt{3} & -9+5\sqrt{3} & 2 & 45-24\sqrt{3} & -14890+8571\sqrt{3} & 10224422-5903030\sqrt{3} & -4828+6197\sqrt{3} & -1789488+1034209\sqrt{3} & 546701203-315638084\sqrt{3} \\
37458+21626\sqrt{3} & 37450+21622\sqrt{3} & 50+29\sqrt{3} & 8+4\sqrt{3} & 74689+43121\sqrt{3} & 463230+267407\sqrt{3} & -4828+6197\sqrt{3} & 1489655060+860052666\sqrt{3} & -604038-350012\sqrt{3} & -415487+241721\sqrt{3} \\
287-191\sqrt{3} & 287-191\sqrt{3} & 17-10\sqrt{3} & 0 & 169-110\sqrt{3} & 3656-2442\sqrt{3} & -1789488+1034209\sqrt{3} & -604038-350012\sqrt{3} & 478336-239700\sqrt{3} & -96065163+55462220\sqrt{3} \\
-164242+94826\sqrt{3} & -164242+94825\sqrt{3} & -452+260\sqrt{3} & 2 & 1736-1000\sqrt{3} & -797826+460598\sqrt{3} & 546701203-315638084\sqrt{3} & -415487+241721\sqrt{3} & -96065163+55462220\sqrt{3} & 29233186896-16877788202\sqrt{3}
\end{pmatrix}
$$

$$\begin{pmatrix}
4 & 2 & 1\sqrt{3} & 0 & 3-1\sqrt{3} & 0 & 12+7\sqrt{3} & 4710-2720\sqrt{3} & 20719+11961\sqrt{3} & -25808+14901\sqrt{3} \\
2 & 4 & -1\sqrt{3} & 1\sqrt{3} & 3-1\sqrt{3} & 2 & 10+7\sqrt{3} & 4710-2719\sqrt{3} & 20711+11957\sqrt{3} & -25808+14901\sqrt{3} \\
1\sqrt{3} & -1\sqrt{3} & 4 & -2 & 0 & -1\sqrt{3} & 1\sqrt{3} & -2 & 8+4\sqrt{3} & 0 \\
0 & 1\sqrt{3} & -2 & 4 & 2 & -1 & 3+1\sqrt{3} & -32+18\sqrt{3} & -251-144\sqrt{3} & 110-63\sqrt{3} \\
3-1\sqrt{3} & 3-1\sqrt{3} & 0 & 2 & 10-4\sqrt{3} & 4-1\sqrt{3} & 7+4\sqrt{3} & 15158-8751\sqrt{3} & 20260+11699\sqrt{3} & -83179+48023\sqrt{3} \\
0 & 2 & -1\sqrt{3} & -1 & 4-1\sqrt{3} & 8+2\sqrt{3} & 13+8\sqrt{3} & 6277-3627\sqrt{3} & 53351+30802\sqrt{3} & -34453+19892\sqrt{3} \\
12+7\sqrt{3} & 10+7\sqrt{3} & 1\sqrt{3} & 3+1\sqrt{3} & 7+4\sqrt{3} & 13+8\sqrt{3} & 278+138\sqrt{3} & 3831-2101\sqrt{3} & 382571+220884\sqrt{3} & -20453+11761\sqrt{3} \\
4710-2720\sqrt{3} & 4710-2719\sqrt{3} & -2 & -32+18\sqrt{3} & 15158-8751\sqrt{3} & 6277-3627\sqrt{3} & 3831-2101\sqrt{3} & 30937492-17861598\sqrt{3} & 67705+40777\sqrt{3} & -169676471+97962717\sqrt{3} \\
20719+11961\sqrt{3} & 20711+11957\sqrt{3} & 8+4\sqrt{3} & -251-144\sqrt{3} & 20260+11699\sqrt{3} & 53351+30802\sqrt{3} & 382571+220884\sqrt{3} & 67705+40777\sqrt{3} & 834417934+481751364\sqrt{3} & -26493-24445\sqrt{3} \\
-25808+14901\sqrt{3} & -25808+14901\sqrt{3} & 0 & 110-63\sqrt{3} & -83179+48023\sqrt{3} & -34453+19892\sqrt{3} & -20453+11761\sqrt{3} & -169676471+97962717\sqrt{3} & -26493-24445\sqrt{3} & 930597330-537280540\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2 & -2 & -2+1\sqrt{3} & -1\sqrt{3} & -2\sqrt{3} & -46-27\sqrt{3} & -8+5\sqrt{3} & -187+108\sqrt{3} & -1497+865\sqrt{3} \\
2 & 4 & -2-1\sqrt{3} & -2+2\sqrt{3} & 0 & 0 & -44-25\sqrt{3} & 1\sqrt{3} & -181+104\sqrt{3} & -1484+857\sqrt{3} \\
-2 & -2-1\sqrt{3} & 6+2\sqrt{3} & 1 & 2+1\sqrt{3} & -1+1\sqrt{3} & 76+44\sqrt{3} & -21+11\sqrt{3} & -282+164\sqrt{3} & -2597+1501\sqrt{3} \\
-2+1\sqrt{3} & -2+2\sqrt{3} & 1 & 6-2\sqrt{3} & 1+1\sqrt{3} & 1\sqrt{3} & 5+1\sqrt{3} & -20+13\sqrt{3} & 23-14\sqrt{3} & -51+30\sqrt{3} \\
-1\sqrt{3} & 0 & 2+1\sqrt{3} & 1+1\sqrt{3} & 8 & 3-1\sqrt{3} & 32+19\sqrt{3} & -23+12\sqrt{3} & -461+268\sqrt{3} & -3955+2283\sqrt{3} \\
-2\sqrt{3} & 0 & -1+1\sqrt{3} & 1\sqrt{3} & 3-1\sqrt{3} & 8 & 45+26\sqrt{3} & 18-10\sqrt{3} & 163-94\sqrt{3} & 1635-945\sqrt{3} \\
-46-27\sqrt{3} & -44-25\sqrt{3} & 76+44\sqrt{3} & 5+1\sqrt{3} & 32+19\sqrt{3} & 45+26\sqrt{3} & 1950+1118\sqrt{3} & -195+12\sqrt{3} & -1184+673\sqrt{3} & -10554+6110\sqrt{3} \\
-8+5\sqrt{3} & 1\sqrt{3} & -21+11\sqrt{3} & -20+13\sqrt{3} & -23+12\sqrt{3} & 18-10\sqrt{3} & -195+12\sqrt{3} & 2708-1450\sqrt{3} & 32281-18617\sqrt{3} & 288503-166576\sqrt{3} \\
-187+108\sqrt{3} & -181+104\sqrt{3} & -282+164\sqrt{3} & 23-14\sqrt{3} & -461+268\sqrt{3} & 163-94\sqrt{3} & -1184+673\sqrt{3} & 32281-18617\sqrt{3} & 427848-246982\sqrt{3} & 3805069-2196872\sqrt{3} \\
-1497+865\sqrt{3} & -1484+857\sqrt{3} & -2597+1501\sqrt{3} & -51+30\sqrt{3} & -3955+2283\sqrt{3} & 1635-945\sqrt{3} & -10554+6110\sqrt{3} & 288503-166576\sqrt{3} & 3805069-2196872\sqrt{3} & 33866200-19552652\sqrt{3}
\end{pmatrix}$$

$$\begin{pmatrix}
4 & 2 & -1\sqrt{3} & -1+1\sqrt{3} & 3+2\sqrt{3} & -2\sqrt{3} & 46+27\sqrt{3} & 114-65\sqrt{3} & 44-25\sqrt{3} & -12757+7366\sqrt{3} \\
2 & 4 & -1\sqrt{3} & -1+2\sqrt{3} & 3+1\sqrt{3} & 2+3\sqrt{3} & 28+17\sqrt{3} & 92-54\sqrt{3} & 70-41\sqrt{3} & -12785+7382\sqrt{3} \\
-1\sqrt{3} & -1\sqrt{3} & 4 & -2+1\sqrt{3} & -2-2\sqrt{3} & 1-1\sqrt{3} & -78-45\sqrt{3} & 83-49\sqrt{3} & 345-199\sqrt{3} & -23786+13732\sqrt{3} \\
-1+1\sqrt{3} & -1+2\sqrt{3} & -2+1\sqrt{3} & 6-2\sqrt{3} & -1+1\sqrt{3} & 2+4\sqrt{3} & -37-22\sqrt{3} & -243+140\sqrt{3} & -650+375\sqrt{3} & 52299-30195\sqrt{3} \\
3+2\sqrt{3} & 3+1\sqrt{3} & -2-2\sqrt{3} & -1+1\sqrt{3} & 10+4\sqrt{3} & -1\sqrt{3} & 227+131\sqrt{3} & 46-27\sqrt{3} & 286-165\sqrt{3} & -17485+10098\sqrt{3} \\
-2\sqrt{3} & 2+3\sqrt{3} & 1-1\sqrt{3} & 2+4\sqrt{3} & -1\sqrt{3} & 94+24\sqrt{3} & -282-166\sqrt{3} & 396-258\sqrt{3} & 1354-806\sqrt{3} & -97727+56391\sqrt{3} \\
46+27\sqrt{3} & 28+17\sqrt{3} & -78-45\sqrt{3} & -37-22\sqrt{3} & 227+131\sqrt{3} & -282-166\sqrt{3} & 11640+6684\sqrt{3} & -292+83\sqrt{3} & 920-91\sqrt{3} & -3994+2600\sqrt{3} \\
114-65\sqrt{3} & 92-54\sqrt{3} & 83-49\sqrt{3} & -243+140\sqrt{3} & 46-27\sqrt{3} & 396-258\sqrt{3} & -292+83\sqrt{3} & 18578-10698\sqrt{3} & 36669-21186\sqrt{3} & -3352072+1935331\sqrt{3} \\
44-25\sqrt{3} & 70-41\sqrt{3} & 345-199\sqrt{3} & -650+375\sqrt{3} & 286-165\sqrt{3} & 1354-806\sqrt{3} & 920-91\sqrt{3} & 36669-21186\sqrt{3} & 177828-102582\sqrt{3} & -11297551+6522668\sqrt{3} \\
-12757+7366\sqrt{3} & -12785+7382\sqrt{3} & -23786+13732\sqrt{3} & 52299-30195\sqrt{3} & -17485+10098\sqrt{3} & -97727+56391\sqrt{3} & -3994+2600\sqrt{3} & -3352072+1935331\sqrt{3} & -11297551+6522668\sqrt{3} & 813707366-469794144\sqrt{3}
\end{pmatrix}$$

$$
\begin{pmatrix}
6+2\sqrt{3} & 1+1\sqrt{3} & 0 & 2+2\sqrt{3} & 1-1\sqrt{3} & 3+3\sqrt{3} & 108+63\sqrt{3} & 4018+2320\sqrt{3} & -15+45\sqrt{3} & 148420+85691\sqrt{3} \\
1+1\sqrt{3} & 6-2\sqrt{3} & -3+2\sqrt{3} & 1+1\sqrt{3} & -1+2\sqrt{3} & 2 & 8+6\sqrt{3} & 64+39\sqrt{3} & -41+105\sqrt{3} & 3626+2096\sqrt{3} \\
0 & -3+2\sqrt{3} & 6-2\sqrt{3} & 2-1\sqrt{3} & 3-1\sqrt{3} & -3-1\sqrt{3} & -6-6\sqrt{3} & -513-294\sqrt{3} & 6\sqrt{3} & -18723-10808\sqrt{3} \\
2+2\sqrt{3} & 1+1\sqrt{3} & 2-1\sqrt{3} & 6 & 3+1\sqrt{3} & 3+1\sqrt{3} & 52+29\sqrt{3} & 1727+998\sqrt{3} & -5+16\sqrt{3} & 64954+37502\sqrt{3} \\
1-1\sqrt{3} & -1+2\sqrt{3} & 3-1\sqrt{3} & 3+1\sqrt{3} & 10+2\sqrt{3} & -3 & -58-35\sqrt{3} & -2751-1590\sqrt{3} & -14+38\sqrt{3} & -98415-56821\sqrt{3} \\
3+3\sqrt{3} & 2 & -3-1\sqrt{3} & 3+1\sqrt{3} & -3 & 12+4\sqrt{3} & 106+63\sqrt{3} & 4437+2562\sqrt{3} & 6-15\sqrt{3} & 163389+94334\sqrt{3} \\
108+63\sqrt{3} & 8+6\sqrt{3} & -6-6\sqrt{3} & 52+29\sqrt{3} & -58-35\sqrt{3} & 106+63\sqrt{3} & 3392+1942\sqrt{3} & 133721+77211\sqrt{3} & -879-755\sqrt{3} & 4899229+2828578\sqrt{3} \\
4018+2320\sqrt{3} & 64+39\sqrt{3} & -513-294\sqrt{3} & 1727+998\sqrt{3} & -2751-1590\sqrt{3} & 4437+2562\sqrt{3} & 133721+77211\sqrt{3} & 5391190+3112350\sqrt{3} & -63522-36380\sqrt{3} & 197189562+113847336\sqrt{3} \\
-15+45\sqrt{3} & -41+105\sqrt{3} & 6\sqrt{3} & -5+16\sqrt{3} & -14+38\sqrt{3} & 6-15\sqrt{3} & -879-755\sqrt{3} & -63522-36380\sqrt{3} & 20894-2726\sqrt{3} & -2249960-1298907\sqrt{3} \\
148420+85691\sqrt{3} & 3626+2096\sqrt{3} & -18723-10808\sqrt{3} & 64954+37502\sqrt{3} & -98415-56821\sqrt{3} & 163389+94334\sqrt{3} & 4899229+2828578\sqrt{3} & 197189562+113847336\sqrt{3} & -2249960-1298907\sqrt{3} & 7214401582+4165236632\sqrt{3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & -1+1\sqrt{3} & -1 & -2 & -2+1\sqrt{3} & 49+29\sqrt{3} & -205+119\sqrt{3} & 6762-3904\sqrt{3} & 13531+7811\sqrt{3} & 46-33\sqrt{3} \\
-1+1\sqrt{3} & 6-2\sqrt{3} & -4+2\sqrt{3} & -1-1\sqrt{3} & 3-1\sqrt{3} & 6+3\sqrt{3} & 238-138\sqrt{3} & -7962+4597\sqrt{3} & 1075+623\sqrt{3} & 45-32\sqrt{3} \\
-1 & -4+2\sqrt{3} & 8-4\sqrt{3} & -1+1\sqrt{3} & -3+2\sqrt{3} & -6-2\sqrt{3} & -218+126\sqrt{3} & 7950-4590\sqrt{3} & -1081-623\sqrt{3} & -11+7\sqrt{3} \\
-2 & -1-1\sqrt{3} & -1+1\sqrt{3} & 6 & 2-3\sqrt{3} & -49-28\sqrt{3} & 205-119\sqrt{3} & -7060+4075\sqrt{3} & -13780-7956\sqrt{3} & -23+14\sqrt{3} \\
-2+1\sqrt{3} & 3-1\sqrt{3} & -3+2\sqrt{3} & 2-3\sqrt{3} & 8-2\sqrt{3} & 13+7\sqrt{3} & 363-211\sqrt{3} & -12497+7216\sqrt{3} & 3751+2168\sqrt{3} & -41+29\sqrt{3} \\
49+29\sqrt{3} & 6+3\sqrt{3} & -6-2\sqrt{3} & -49-28\sqrt{3} & 13+7\sqrt{3} & 3376+1936\sqrt{3} & 390-227\sqrt{3} & -12910+8140\sqrt{3} & 995795+574927\sqrt{3} & 504+705\sqrt{3} \\
-205+119\sqrt{3} & 238-138\sqrt{3} & -218+126\sqrt{3} & 205-119\sqrt{3} & 363-211\sqrt{3} & 390-227\sqrt{3} & 40310-23250\sqrt{3} & -1367944+789788\sqrt{3} & -784-319\sqrt{3} & -13991+8163\sqrt{3} \\
6762-3904\sqrt{3} & -7962+4597\sqrt{3} & 7950-4590\sqrt{3} & -7060+4075\sqrt{3} & -12497+7216\sqrt{3} & -12910+8140\sqrt{3} & -1367944+789788\sqrt{3} & 46571154-26887540\sqrt{3} & 189948+104796\sqrt{3} & 493663-284360\sqrt{3} \\
13531+7811\sqrt{3} & 1075+623\sqrt{3} & -1081-623\sqrt{3} & -13780-7956\sqrt{3} & 3751+2168\sqrt{3} & 995795+574927\sqrt{3} & -784-319\sqrt{3} & 189948+104796\sqrt{3} & 295915128+170846572\sqrt{3} & 278631+160680\sqrt{3} \\
46-33\sqrt{3} & 45-32\sqrt{3} & -11+7\sqrt{3} & -23+14\sqrt{3} & -41+29\sqrt{3} & 504+705\sqrt{3} & -13991+8163\sqrt{3} & 493663-284360\sqrt{3} & 278631+160680\sqrt{3} & 17214-8236\sqrt{3}
\end{pmatrix}
$$

$$
\begin{pmatrix}
4 & -2-1\sqrt{3} & -2-3\sqrt{3} & -3+11\sqrt{3} & 956+553\sqrt{3} & -48-28\sqrt{3} & -482+279\sqrt{3} & -1868+1079\sqrt{3} & 108207-62473\sqrt{3} & 405812+234296\sqrt{3} \\
-2-1\sqrt{3} & 6+2\sqrt{3} & -6-5\sqrt{3} & 3-9\sqrt{3} & -1887-1089\sqrt{3} & 550+317\sqrt{3} & -610+354\sqrt{3} & -898+521\sqrt{3} & 45336-26173\sqrt{3} & -713703-412056\sqrt{3} \\
-2-3\sqrt{3} & -6-5\sqrt{3} & 636+264\sqrt{3} & 77+93\sqrt{3} & 7238+4143\sqrt{3} & -11610-6698\sqrt{3} & 10486-5950\sqrt{3} & 3015-1714\sqrt{3} & 19270-10904\sqrt{3} & 507083+292732\sqrt{3} \\
-3+11\sqrt{3} & 3-9\sqrt{3} & 77+93\sqrt{3} & 354-36\sqrt{3} & 7818+4565\sqrt{3} & -2890-1595\sqrt{3} & -7415+4304\sqrt{3} & 9811-5650\sqrt{3} & -767713+443157\sqrt{3} & 2831923+1635028\sqrt{3} \\
956+553\sqrt{3} & -1887-1089\sqrt{3} & 7238+4143\sqrt{3} & 7818+4565\sqrt{3} & 960676+554606\sqrt{3} & -301328-174004\sqrt{3} & 4988-2617\sqrt{3} & -1075+295\sqrt{3} & 144080-87039\sqrt{3} & 357741066+206541903\sqrt{3} \\
-48-28\sqrt{3} & 550+317\sqrt{3} & -11610-6698\sqrt{3} & -2890-1595\sqrt{3} & -301328-174004\sqrt{3} & 281458+162420\sqrt{3} & -2565-325\sqrt{3} & -10564+5776\sqrt{3} & 632078-369164\sqrt{3} & -69105035-39897827\sqrt{3} \\
-482+279\sqrt{3} & -610+354\sqrt{3} & 10486-5950\sqrt{3} & -7415+4304\sqrt{3} & 4988-2617\sqrt{3} & -2565-325\sqrt{3} & 1980018-1143132\sqrt{3} & 1729737-998669\sqrt{3} & -70045140+40440580\sqrt{3} & -265239-157628\sqrt{3} \\
-1868+1079\sqrt{3} & -898+521\sqrt{3} & 3015-1714\sqrt{3} & 9811-5650\sqrt{3} & -1075+295\sqrt{3} & -10564+5776\sqrt{3} & 1729737-998669\sqrt{3} & 4219218-2435896\sqrt{3} & -233161795+134616038\sqrt{3} & -281369-165627\sqrt{3} \\
108207-62473\sqrt{3} & 45336-26173\sqrt{3} & 19270-10904\sqrt{3} & -767713+443157\sqrt{3} & 144080-87039\sqrt{3} & 632078-369164\sqrt{3} & -70045140+40440580\sqrt{3} & -233161795+134616038\sqrt{3} & 13399151238-7736001014\sqrt{3} & -2596580-1385876\sqrt{3} \\
405812+234296\sqrt{3} & -713703-412056\sqrt{3} & 507083+292732\sqrt{3} & 2831923+1635028\sqrt{3} & 357741066+206541903\sqrt{3} & -69105035-39897827\sqrt{3} & -265239-157628\sqrt{3} & -281369-165627\sqrt{3} & -2596580-1385876\sqrt{3} & 143391211330+82786951006\sqrt{3}
\end{pmatrix}
$$

$$\begin{pmatrix}
4 & 3+1\sqrt{3} & 1+1\sqrt{3} & -1\sqrt{3} & -3+1\sqrt{3} & -3+6\sqrt{3} & 8716+5033\sqrt{3} & 400754+231375\sqrt{3} & -1237-727\sqrt{3} & 525105-303170\sqrt{3} \\
3+1\sqrt{3} & 10+4\sqrt{3} & 3+1\sqrt{3} & 1\sqrt{3} & -4 & -2+16\sqrt{3} & 33622+19411\sqrt{3} & 1544589+891768\sqrt{3} & -4983-2930\sqrt{3} & -428213+247233\sqrt{3} \\
1+1\sqrt{3} & 3+1\sqrt{3} & 8 & -2-1\sqrt{3} & 1 & -3+10\sqrt{3} & 12583+7265\sqrt{3} & 577591+333471\sqrt{3} & -2309-1360\sqrt{3} & -1792405+1034844\sqrt{3} \\
-1\sqrt{3} & 1\sqrt{3} & -2-1\sqrt{3} & 6 & 2-1\sqrt{3} & 4-8\sqrt{3} & 2+2\sqrt{3} & 467+271\sqrt{3} & 633+373\sqrt{3} & 1298290-749567\sqrt{3} \\
-3+1\sqrt{3} & -4 & 1 & 2-1\sqrt{3} & 14-6\sqrt{3} & 4-11\sqrt{3} & -9008-5201\sqrt{3} & -413872-238950\sqrt{3} & 1573+925\sqrt{3} & 1552158-896138\sqrt{3} \\
-3+6\sqrt{3} & -2+16\sqrt{3} & -3+10\sqrt{3} & 4-8\sqrt{3} & 4-11\sqrt{3} & 360-40\sqrt{3} & 113174+65346\sqrt{3} & 5195430+2999602\sqrt{3} & -24570-14337\sqrt{3} & 9647029-5569703\sqrt{3} \\
8716+5033\sqrt{3} & 33622+19411\sqrt{3} & 12583+7265\sqrt{3} & 2+2\sqrt{3} & -9008-5201\sqrt{3} & 113174+65346\sqrt{3} & 166382306+96060836\sqrt{3} & 7641168156+4411630497\sqrt{3} & -28989021-16736158\sqrt{3} & 2565202-1456576\sqrt{3} \\
400754+231375\sqrt{3} & 1544589+891768\sqrt{3} & 577591+333471\sqrt{3} & 467+271\sqrt{3} & -413872-238950\sqrt{3} & 5195430+2999602\sqrt{3} & 7641168156+4411630497\sqrt{3} & 350923779488+202605938542\sqrt{3} & -1331027671-768468781\sqrt{3} & -400392+1355593\sqrt{3} \\
-1237-727\sqrt{3} & -4983-2930\sqrt{3} & -2309-1360\sqrt{3} & 633+373\sqrt{3} & 1573+925\sqrt{3} & -24570-14337\sqrt{3} & -28989021-16736158\sqrt{3} & -1331027671-768468781\sqrt{3} & 5564472+3153770\sqrt{3} & -30913+38104\sqrt{3} \\
525105-303170\sqrt{3} & -428213+247233\sqrt{3} & -1792405+1034844\sqrt{3} & 1298290-749567\sqrt{3} & 1552158-896138\sqrt{3} & 9647029-5569703\sqrt{3} & 2565202-1456576\sqrt{3} & -400392+1355593\sqrt{3} & -30913+38104\sqrt{3} & 2232834728564-1289127707414\sqrt{3}
\end{pmatrix}$$

Extremal Fundamentally Invariant Type (iii) Lattices

Dimension 4:

The only fundamentally invariant Type (iii) lattice has the following Gram matrix:

$$\begin{pmatrix}
1-\sqrt{3}/3 & 1/2-\sqrt{3}/6 & 1/2-\sqrt{3}/6 & 1/2-\sqrt{3}/6 \\
1/2-\sqrt{3}/6 & 1/2-\sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/6 \\
1/2-\sqrt{3}/6 & \sqrt{3}/6 & 1-\sqrt{3}/3 & \sqrt{3}/6 \\
1/2-\sqrt{3}/6 & \sqrt{3}/6 & \sqrt{3}/6 & 1-\sqrt{3}/3
\end{pmatrix}$$

Dimension 8:
One extremal lattice has the Gram matrix

$$\begin{pmatrix}
470+778/3\sqrt{3} & 94+146/3\sqrt{3} & 10+79/6\sqrt{3} & 8195+14234/3\sqrt{3} & -17859-21173/2\sqrt{3} & 4158481/2-7204649/6\sqrt{3} & 698806251-806912483/2\sqrt{3} & -3676684105/2+3184102630/3\sqrt{3} \\
94+146/3\sqrt{3} & 366-574/3\sqrt{3} & 311/2-257/3\sqrt{3} & 1569+5561/6\sqrt{3} & -6353/2-11801/6\sqrt{3} & 19775991/2-17127358/3\sqrt{3} & 94372119247/2-163457305295/6\sqrt{3} & -248063750351/2+429659019073/6\sqrt{3} \\
10+79/6\sqrt{3} & 311/2-257/3\sqrt{3} & 12256-4907\sqrt{3} & -40527/2-28300/3\sqrt{3} & -94951/2-148621/6\sqrt{3} & 99127437/2-85851035/3\sqrt{3} & 274909499920-317438146723/2\sqrt{3} & -722532460297+2502925868033/6\sqrt{3} \\
8195+14234/3\sqrt{3} & 1569+5561/6\sqrt{3} & -40527/2-28300/3\sqrt{3} & 331189+572293/3\sqrt{3} & 73023+250235/6\sqrt{3} & -22340907/2+12928557/2\sqrt{3} & -101599212621/2+87987423421/3\sqrt{3} & 267030854433/2-462510985025/6\sqrt{3} \\
-17859-21173/2\sqrt{3} & -6353/2-11801/6\sqrt{3} & -94951/2-148621/6\sqrt{3} & 73023+250235/6\sqrt{3} & 17063699+28798726/3\sqrt{3} & 349751676-1223105785/6\sqrt{3} & 228280950665/2-197677279798/3\sqrt{3} & -600494079171/2+520037345221/3\sqrt{3} \\
4158481/2-7204649/6\sqrt{3} & 19775991/2-17127358/3\sqrt{3} & 99127437/2-85851035/3\sqrt{3} & -22340907/2+12928557/2\sqrt{3} & 349751676-1223105785/6\sqrt{3} & 1320597545583-2287225185400/3\sqrt{3} & 2724757444066401-1573139452811769\sqrt{3} & -7162376081586061+12405599270541908/3\sqrt{3} \\
698806251-806912483/2\sqrt{3} & 94372119247/2-163457305295/6\sqrt{3} & 274909499920-317438146723/2\sqrt{3} & -101599212621/2+87987423421/3\sqrt{3} & 228280950665/2-197677279798/3\sqrt{3} & 2724757444066401-1573139452811769\sqrt{3} & 12771977959476260581-7373904912982065107\sqrt{3} & -67140144724100405177/2+116290141889671688243/6\sqrt{3} \\
-3676684105/2+3184102630/3\sqrt{3} & -248063750351/2+429659019073/6\sqrt{3} & -722532460297+2502925868033/6\sqrt{3} & 267030854433/2-462510985025/6\sqrt{3} & -600494079171/2+520037345221/3\sqrt{3} & -7162376081586061+12405599270541908/3\sqrt{3} & -67140144724100405177/2+116290141889671688243/6\sqrt{3} & 88236119089052111986-50943147108311449129\sqrt{3}
\end{pmatrix}$$

Dimension 24
The extremal lattice is given by the following two Gram matricies of degree 24 over $\mathbb{Q}$. The first belongs to the Leech lattice, the second to the extremal strongly 6-modular lattice given in
http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/P24.7.html.
The $\sqrt{3}$ structure is given by the formula $(G_1)^{-1}G_2 = 3+\sqrt{3}$. Matrices G_1 and G_2:

$$\begin{pmatrix}
8 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 4 & 4 & 4 & 2 & 4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & -3 \\
4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & -1 \\
4 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & -1 \\
4 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & -1 \\
4 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 0 & -1 \\
4 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & -1 \\
4 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & -1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 0 & 0 & 1 \\
4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & -1 \\
4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 1 & 0 & -1 \\
4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 1 & -1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 4 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 4 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 4 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 1 \\
2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 4 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
4 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 4 & 2 & 2 & 2 & 1 & 1 & 1 & -1 \\
2 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 1 & 2 & 1 \\
2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 4 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 4 & 2 \\
-3 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 2 & 2 & 2 & 4
\end{pmatrix}
\begin{pmatrix}
24 & 11 & 12 & 11 & 12 & 12 & 13 & 7 & 13 & 12 & 13 & 7 & 12 & 5 & 6 & 9 & 11 & 6 & 6 & 7 & -1 & -3 & 0 & -9 \\
11 & 12 & 5 & 5 & 5 & 7 & 6 & 7 & 6 & 6 & 7 & 7 & 5 & 6 & 2 & 3 & 6 & 3 & 4 & 6 & 3 & -1 & 0 & -3 \\
12 & 5 & 12 & 6 & 6 & 6 & 8 & 7 & 5 & 6 & 7 & 6 & 6 & 2 & 6 & 3 & 6 & 5 & 3 & 4 & 2 & -2 & 0 & -3 \\
11 & 5 & 6 & 12 & 7 & 5 & 5 & 7 & 5 & 5 & 6 & 6 & 6 & 2 & 3 & 8 & 6 & 3 & 7 & 3 & 4 & -3 & 0 & -3 \\
12 & 5 & 6 & 7 & 10 & 6 & 6 & 6 & 6 & 6 & 7 & 4 & 6 & 4 & 5 & 7 & 4 & 4 & 5 & 4 & 1 & -3 & -1 & -4 \\
12 & 7 & 6 & 5 & 6 & 12 & 8 & 8 & 5 & 6 & 6 & 3 & 6 & 5 & 3 & 4 & 5 & 2 & 5 & 4 & -1 & -3 & -1 & -3 \\
13 & 6 & 8 & 5 & 6 & 8 & 12 & 7 & 6 & 7 & 7 & 4 & 7 & 3 & 6 & 4 & 6 & 4 & 3 & 6 & -1 & -2 & 0 & -3 \\
7 & 7 & 7 & 7 & 6 & 8 & 7 & 14 & 1 & 3 & 4 & 6 & 4 & 5 & 5 & 5 & 3 & 4 & 6 & 5 & 5 & -4 & -2 & 1 \\
13 & 6 & 5 & 5 & 6 & 5 & 6 & 1 & 12 & 7 & 7 & 5 & 6 & 4 & 4 & 6 & 6 & 5 & 4 & 5 & 0 & 0 & 2 & -6 \\
12 & 6 & 6 & 5 & 6 & 6 & 7 & 3 & 7 & 12 & 7 & 7 & 5 & 5 & 4 & 3 & 5 & 5 & 3 & 4 & -1 & 0 & 1 & -3 \\
13 & 7 & 7 & 6 & 7 & 6 & 7 & 4 & 7 & 7 & 14 & 8 & 7 & 4 & 7 & 5 & 5 & 3 & 6 & 4 & 1 & -1 & 3 & -3 \\
7 & 7 & 6 & 6 & 4 & 3 & 4 & 6 & 5 & 7 & 8 & 12 & 2 & 5 & 5 & 4 & 4 & 5 & 6 & 6 & 5 & 0 & 2 & 1 \\
12 & 5 & 6 & 6 & 6 & 6 & 7 & 4 & 6 & 5 & 7 & 2 & 12 & 4 & 6 & 7 & 5 & 3 & 3 & 2 & 2 & 1 & 3 & -3 \\
5 & 6 & 2 & 2 & 4 & 5 & 3 & 5 & 4 & 5 & 4 & 5 & 4 & 8 & 4 & 3 & 1 & 3 & 4 & 3 & 3 & 2 & 2 & 1 \\
6 & 2 & 6 & 3 & 5 & 3 & 6 & 5 & 4 & 4 & 7 & 5 & 6 & 4 & 10 & 4 & 1 & 4 & 4 & 3 & 3 & 0 & 4 & 1 \\
9 & 3 & 3 & 8 & 7 & 4 & 4 & 5 & 6 & 3 & 5 & 4 & 7 & 3 & 4 & 12 & 4 & 4 & 6 & 4 & 3 & -1 & 1 & -1 \\
11 & 6 & 6 & 6 & 4 & 5 & 6 & 3 & 6 & 5 & 5 & 4 & 5 & 1 & 1 & 4 & 12 & 5 & 5 & 7 & 3 & 1 & 2 & -3 \\
6 & 3 & 5 & 3 & 4 & 2 & 4 & 4 & 5 & 5 & 3 & 5 & 3 & 3 & 4 & 4 & 5 & 8 & 3 & 5 & 3 & 2 & 1 & 0 \\
6 & 4 & 3 & 7 & 5 & 5 & 3 & 6 & 4 & 3 & 6 & 6 & 3 & 4 & 4 & 6 & 5 & 3 & 10 & 4 & 5 & -1 & 4 & 1 \\
7 & 6 & 4 & 3 & 4 & 4 & 6 & 5 & 5 & 4 & 4 & 6 & 2 & 3 & 3 & 4 & 7 & 5 & 4 & 10 & 3 & 0 & 0 & 0 \\
-1 & 3 & 2 & 4 & 1 & -1 & -1 & 5 & 0 & -1 & 1 & 5 & 2 & 3 & 3 & 3 & 3 & 3 & 5 & 3 & 10 & 3 & 4 & 4 \\
-3 & -1 & -2 & -3 & -3 & -3 & -2 & -4 & 0 & 0 & -1 & 0 & 1 & 2 & 0 & -1 & 1 & 2 & -1 & 0 & 3 & 10 & 5 & 5 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & 2 & 1 & 3 & 2 & 3 & 2 & 4 & 1 & 2 & 1 & 4 & 0 & 4 & 5 & 10 & 4 \\
-9 & -3 & -3 & -3 & -4 & -3 & -3 & 1 & -6 & -3 & -3 & 1 & -3 & 1 & 1 & -1 & -3 & 0 & 1 & 0 & 4 & 5 & 4 & 10
\end{pmatrix}$$

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Subject Index

Notation Index

We use the following notations. Compare also Chapter "Notation".

F_4	the 2-modular lattice F_4, 97
$\mathcal{G}(L)$	Gram matrix of L, 14
$\Gamma,\ \Gamma(\mathfrak{n})$	(principal) congruence subgroup, 38
Γ_{72}	Extremal even unimodular lattice in dimension 72, found by Nebe in [Neb12], 89
$\mathrm{GL}_2^+(\mathbb{Z}_F)$	invertible matrices over the integer ring with totally positive determinant, 38
$\mathrm{GL}_2^+(F)$	invertible matrices with totally positive determinant, 38
Γ_F	the Hilbert modular group $\mathrm{SL}_2(\mathbb{Z}_F)$., 40
$\mathbb{H}$	the upper half plane of $\mathbb{C}$, 14
h_F	the class number of F, 13
h_F^+	the narrow class number of F, 13
$j(g,z)$	$= (\gamma^{(i)}z_i + \delta^{(i)})_{i=1,2}$ for $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2^+(F)$ and $z \in \mathbb{H}^r$, 39
k	weight $k \in \mathbb{Z}$, weight vector $k \in \mathbb{Z}^2$, or parallel weight vector $k = (k,k) \in \mathbb{Z}^2$, 39
K_{12}	the extremal 3-modular Coxeter-Todd lattice, 109
(Λ, Q), Λ	lattice over number field, 17
$(\Lambda, Q)^*,\ \Lambda^*$	Trace dual lattice, 19
$(\Lambda, Q)^\#,\ \Lambda^\#$	dual lattice of Λ, 19
$\Lambda(\alpha)$	layer of Λ at α, 18
Λ^{ε_0}	lattice Λ with quadratic form $\varepsilon_0 Q$, 26
Λ_{24}	the Leech lattice, 97
$\mathrm{Min}(\Lambda)$	minimal vectors of Λ, 19
$\min(\Lambda)$	minimum of Λ, 19
$M(\Gamma),\ M_{ev}(\Gamma)$	(even) Hilbert modular forms of level Γ and parallel weight, 39
$M^+(\Gamma),\ M_e^+v(\Gamma)$	space of (even) symmetric Hilbert modular forms of level Γ, 40
$M_k(\Gamma),\ M_k(\mathfrak{n})$	Hilbert modular forms $\mathbb{H} \times \mathbb{H} \to \mathbb{C}$ of weight k and level Γ or $\mathfrak{n}$, 39

$M_k^+(\Gamma)$	space of symmetric Hilbert modular forms of weight k and level Γ, 40
∇	$= (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})$, nabla, 70
$\mathcal{N}$	the norm of F over $\mathbb{Q}$, 13
P_{48n}	extremal unimodular lattice of dimension 48, cf. [Neb98a], 97
$\mathfrak{q}$	the quadratic form of the $\mathbb{Z}$-lattice L, 14
Q	quadratic form, 17
Q_{32}	Extremal 2-modular lattice of dimension 32, found by Quebbemann in [Que87b], 97
r	the degree $[F : \mathbb{Q}]$, 13
$\sigma_1, \ldots, \sigma_r$	embeddings $\sigma_1, \ldots, \sigma_r : F \hookrightarrow \mathbb{R}$, 13
$S_k(\Gamma)$, $\overline{S}_k(\Gamma)$	space of cusp forms of weight k and level Γ, 44
$\alpha \gg 0$	$\alpha \in F$ is totally positive, i.e. $\alpha^{(1)} > 0$ and $\alpha^{(2)} > 0$, 13
$\alpha \gtrless 0$	$\alpha \in F$ with $\alpha^{(1)} > 0$ and $\alpha^{(2)} < 0$, 13
Θ_L	the theta series of the lattice L, 15
$\Theta_{(\Lambda,Q)}$, Θ_Λ	theta series, 62
tr	the trace of F over $\mathbb{Q}$, 13
$\mathbb{Z}(F)$, $Z(\mathbb{Z}_F)$	Centers of $\mathrm{GL}_2^+(F)$ and $\mathrm{GL}_2^+(\mathbb{Z}_F)$, respectively., 38
$\mathbb{Z}_F$	ring of integers of F, 13
$\zeta_F(s)$	the Dedekind zeta function of F, 14
z^k	$= \prod_{i=1}^r z_i^{k_i}$ for $z \in \mathbb{C}^r$ and $k \in \mathbb{Z}^r$, 14