Dual strongly perfect lattices

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Strongly perfect Lattices

Def: $X \subset S^{n-1}(a)$ is called a **spherical t-design** if

$$\int_{S^{n-1}(a)} f(x)dx = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for all $f \in \mathcal{F}_{n,m} \subset \mathbb{R}[X_1, \dots, X_n]$ homogenous polynomilas with degree $m \leq t$.

Def: A n-dimensional lattice L is called **strongly perfect** if

$$\mathcal{S}(L) := \{l \in L | (l, l) = min(L)\} \subset S^{n-1}(\min(L))$$

is a spherical 4-design.

Characterisation

1. L is strongly perfect if and only if

$$D_2(\alpha) := \sum_{x \in \mathcal{S}(L)} (x, \alpha)^2 = \frac{|S(L)|}{n} \min(L)(\alpha, \alpha)$$

$$D_4(\alpha) := \sum_{x \in \mathcal{S}(L)} (x, \alpha)^4 = \frac{3|S(L)|}{n(n+2)} \min(L)^2(\alpha, \alpha)^2$$

for all $\alpha \in \mathbb{R}^n$.

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   for all $\alpha \in \mathbb{R}^n$.

2. L is strongly perfect if and only if $\sum_{x \in \mathcal{S}(L)} f(x) = 0$ for all harmonic polynomials $f \in \mathrm{Harm}_{n,m} := \{f \in \mathcal{F}_{n,m} | \Delta(f) = 0\}$ with degree $m \leq 4$.

Spherical designs

Theorem: Let $X = -X \subset S^{n-1}(a)$ be a finite set then

$$\sum_{x,y \in X} (x,y)^{2l} \geq \frac{1 \cdot 3 \cdot \dots \cdot (2l-1)}{n \cdot (n+2) \cdot \dots \cdot (n+2l-2)} a^{2l} |X|^2$$

for all $l \in \mathbb{N}_0$.

Remark: If X is a spherical $2l$ design equality holds above.

Dual strongly perfect lattices

Def.: A lattice L is called **dual strongly perfect** if L and its dual lattice L^* are strongly perfect.

Remark: Exept K'_{21}, all known strongly perfect lattices are dual strongly perfect.

Minimal type

Lemma: Let L be a strongly perfect lattice then

$$\min(L) \cdot \min(L^*) \geq \frac{n+2}{3}.$$

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Remark: If equaltity holds in the lemma above, then L is called a lattice of **minimal type** and $(x, \alpha) \in \{-1, 0, 1\}$ for all $x \in \mathcal{S}(L)$ and $\alpha \in \mathcal{S}(L^*)$.

Classifying lattices of minimal type

Theorem: Let L be a dual strongly perfect lattice of minimal type with $s := |S(L)|$ and $t := |S(L^*)|$ and $n > 1$ then:

$$
\begin{aligned}
P(b) :=& (s+t)^2 \left(\frac{15}{n(n+2)(n+4)} + \frac{3(2b - \frac{1}{4})}{n(n+2)} + \frac{(b^2 - \frac{b}{2})}{n} - \frac{b^2}{4} \right) \\
& - 2st \left(\frac{(10-n)}{12n(n+2)^2} (3 + b(n+2)^2) - \frac{n-1}{6n} b^2 \right) \\
& - \frac{3}{2}(s+t)(1+b^2) \leq 0
\end{aligned}
$$

for all $b \in \mathbb{R}$.

Known classification

All strongly perfect lattice up to dimension 12 are known (Nebe, Venkov):

dim	1	2	4	6	7	8	10	12
	$\mathbb{Z}$	$\mathbb{A}_2$	$\mathbb{D}_4$	$\mathbb{E}_6, \mathbb{E}_6^*$	$\mathbb{E}_7, \mathbb{E}_7^*$	$\mathbb{E}_8$	$K'_{10}, (K'_{10})^*$	K_{12}, K_{12}^*

In dimension 13 to 15 all dual strongly perfect lattices are known:

- dim. 13 and 15: no dual strongly perfect lattices exist.
- dim. 14: Q_{14} is the only dual strongly perfect lattice.(Nebe, Venkov)
- dim. 16: Barnes-Wall lattice Λ_{16}, O_{16}, O_{16}^* and N_{16} are known.

Theorem (Venkov,1984): Extremal unimodular even lattices are strongly perfect if $n \equiv 0$ or 8 mod 24.