Eva Zerz SS 2006

Algebraic Geometry – Tutorial 1

To be handed in till: not applicable, To be discussed on: Tuesday, April 11

1. Let K be a field, m, n positive integers, $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ and

$$f : K^n \to K^m, \quad a \mapsto f(a) := (f_1(a), \dots, f_m(a)).$$

   Is f Zariski continuous?

   Describe all Zariski open subsets of K (what happens if K is finite?).

   Let K be infinite. Show that $K^n \setminus \mathcal{V}(f)$ is infinite for each $0 \neq f \in K[x_1, \dots, x_n]$, in particular, $\mathcal{V}(f) \neq K^n$. Conclude: Any two non-empty Zariski open sets must intersect. (Thus, the Zariski topology is not Hausdorff.) On the other hand, the Zariski topology has the following weaker separation property: For $a \neq b \in K^n$, there exists an open set U with $a \in U$, but $b \notin U$.

2. Plot $\mathcal{V}(f) \subset \mathbb{R}^2$:

   (a) $f = x^4 - 50x^2 + 2x^2y^2 + 49 - 14y^2 + y^4$
   (b) $f = (x^2 + y^2)^3 - x^2y^2$
   (c) $f = x^2 + x^3 - y^2$
   (d) $f = x^2(1 + x) - y^2(1 - x)$.
   (e) Try to find a parametrization of these curves, i.e., a (continuous/differentiable/smooth) map from an interval in $\mathbb{R}$ to $\mathbb{R}^2$ whose image equals $\mathcal{V}(f)$!

3. The following algebraic curves are given in terms of parametric representations, where $t \in \mathbb{R}$ is the parameter (except for (b), where $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $t \in (\frac{\pi}{2}, \frac{3\pi}{2})$), and $a, b > 0$ are fixed real numbers. Compute an implicit representation[1]:

   (a) $x(t) = (\cos(t) + a)\cos(t)$, $y(t) = (\cos(t) + a)\sin(t)$
   (b) $x(t) = a + b\cos(t)$, $y(t) = a\tan(t) + b\sin(t)$
   (c) $x(t) = t^2 - a, y(t) = t(a - t^2)$
   (d) $x(t) = \frac{a(t^2-1)}{t^2+1}$, $y(t) = \frac{at(t^2-1)}{t^2+1}$
   (e) $x(t) = \frac{at^2}{t^2+1}$, $y(t) = \frac{at^3}{t^2+1}$.

[1] by trial and error, or by looking up the terms conchoid, Pascal snail, strophoid, cissoid; later, we'll do this systematically.

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Algebraic Geometry – Tutorial 2

To be handed in till: Thursday, April 13 (noon)
To be discussed on: Tuesday, April 18

Let K be a field, n a positive integer,

$$\mathcal{J}(V) := \{f \in K[x_1, \ldots, x_n] \mid f(a) = 0 \ \forall a \in V\}$$

for $V \subseteq K^n$, and

$$\mathcal{V}(F) := \{a \in K^n \mid f(a) = 0 \ \forall f \in F\}$$

for $F \subseteq K[x_1, \ldots, x_n]$.

4. Prove the following:
   (a) $\mathcal{J}(\emptyset) = \langle 1 \rangle$ and, if $|K| = \infty$, $\mathcal{J}(K^n) = \langle 0 \rangle$
   (b) $\mathrm{Rad}(\mathcal{J}(V)) = \mathcal{J}(V)$
   (c) $\mathcal{J}(V_1 \cup V_2) = \mathcal{J}(V_1) \cap \mathcal{J}(V_2)$.
   (d) $\mathcal{V}$ and $\mathcal{J}$ are inclusion-reversing and for all V, F, we have $V \subseteq \mathcal{V}\mathcal{J}(V)$, $F \subseteq \mathcal{J}\mathcal{V}(F)$.
   (e) $\mathcal{V} \circ \mathcal{J} \circ \mathcal{V} = \mathcal{V}$ and $\mathcal{J} \circ \mathcal{V} \circ \mathcal{J} = \mathcal{J}$.

5. Let V, W be algebraic sets, and let I, J be ideals in $K[x_1, \ldots, x_n]$. Show that
   (a) $V \subseteq W \Leftrightarrow \mathcal{J}(V) \supseteq \mathcal{J}(W)$ and $V \subsetneq W \Leftrightarrow \mathcal{J}(V) \supsetneq \mathcal{J}(W)$
   (b) $\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cdot J) = \mathcal{V}(I \cap J)$
   (c) $\mathcal{V}(I) \cap \mathcal{V}(J) = \mathcal{V}(I + J) = \mathcal{V}(I \cup J)$.

6. Let I, J be ideals in $K[x_1, \ldots, x_n]$ and

   $$(I : J) := \{h \in K[x_1, \ldots, x_n] \mid hg \in I \ \forall g \in J\}.$$

   Prove the following:
   (a) $I \subseteq (I : J) \subseteq \mathcal{J}(\mathcal{V}(I) \setminus \mathcal{V}(J))$.
   (b) If V, W are algebraic sets, then $(\mathcal{J}(V) : \mathcal{J}(W)) = \mathcal{J}(V \setminus W)$.

Algebraic Geometry – Tutorial 3

To be handed in till: Monday, April 24 (noon), To be discussed on: Tuesday, April 25

7. Let $\overline{V} = \mathcal{V}\mathcal{J}(V)$ denote the Zariski closure of $V \subseteq K^n$. Let I, J be ideals in $K[x_1, \ldots, x_n]$.

   (a) Conclude from Exercise 6a:

   $$\mathcal{V}(I) \supseteq \mathcal{V}(I : J) \supseteq \overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \supseteq \mathcal{V}(I) \setminus \mathcal{V}(J).$$

   In particular, let $J = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$ for some $a \in V := \mathcal{V}(I)$. Then there are two possibilities: either $\overline{V \setminus \{a\}} = V$ or $\overline{V \setminus \{a\}} = V \setminus \{a\}$. Give a geometric interpretation of both cases.

   (b) If K is algebraically closed and I is radical, then we have

   $$\mathcal{V}(I : J) = \overline{\mathcal{V}(I) \setminus \mathcal{V}(J)}.$$

   Prove this in two different ways: first, by directly verifying the inclusion that is missing in view of 7a, and second, by plugging $V = \mathcal{V}(I)$, $W = \mathcal{V}(J)$ into the equation from Exercise 6b and by showing

   $$(I : \mathrm{Rad}(J)) \subseteq (I : J) \subseteq (\mathrm{Rad}(I) : J) = (\mathrm{Rad}(I) : \mathrm{Rad}(J)). \qquad (1)$$

   (c) Conclude: If K is algebraically closed (but I not necessarily radical), we have

   $$\mathcal{V}(I : J^\infty) = \overline{\mathcal{V}(I) \setminus \mathcal{V}(J)},$$

   where

   $$(I : J^\infty) = \{h \in K[x_1, \ldots, x_n] \mid \exists l \in \mathbb{N} : hg \in I \ \forall g \in J^l\} = \bigcup_{l \in \mathbb{N}} (I : J^l)$$

   is the so-called *saturation* of I with respect to J.

   (d) Compute the saturation, all ideals appearing in (1), and their radicals for

   $$I = \langle xy^2, y^3z^2\rangle, \ J = \langle y^2\rangle \subseteq K[x, y, z].$$

Hints: In 7c, it suffices to show that $\mathrm{Rad}(I : J^\infty) = (\mathrm{Rad}(I) : J)$. Since $K[x_1, \ldots, x_n]$ is Noetherian, J is finitely generated, and

$$I \subseteq (I : J) \subseteq (I : J^2) \subseteq \ldots \qquad (2)$$

becomes stationary, i.e., there exists k with $(I : J^\infty) = (I : J^k)$. The first equality in (2) already yields stationarity. If $I \cap \langle g\rangle = \langle h_1 g, \ldots, h_m g\rangle$, then $I : \langle g\rangle = \langle h_1, \ldots, h_m\rangle$.

Algebraic Geometry – Tutorial 4

To be handed in till: Friday, April 28 (noon), To be discussed on: Tuesday, May 2

8. Let $I \neq 0$ be an ideal in $K[x_1, \ldots, x_n]$. Prove that the following are equivalent:

   (a) For each $1 \leq i \leq n$, there exists
   $$0 \neq g_i \in I \cap K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n].$$

   (b) I is not contained in any proper principal ideal of $K[x_1, \ldots, x_n]$.

   (c) Any finite set of non-zero generators of I consists of coprime polynomials.

   (d) There exists a finite generating set of I that consists of non-zero coprime polynomials.

   An algebraic set defined by a single, non-constant polynomial is called a *hypersurface*. If K is algebraically closed, then the four conditions from above are also equivalent to

   (e) $\mathcal{V}(I)$ does not contain an algebraic hypersurface.

   Useful background material ("divisibility theory reloaded"): $K[x_1, \ldots, x_n]$ is a unique factorization domain, i.e., any $f \in K[x_1, \ldots, x_n] \setminus K$ can be written as a product of prime polynomials. Thus, we have a well-defined concept of greatest common divisor (gcd). Two non-zero polynomials f, g are called coprime if $\gcd(f, g) = 1$, which is equivalent to $\langle f \rangle \cap \langle g \rangle = \langle fg \rangle$.

   A stronger coprimeness notion is obtained by requiring that $\langle f, g \rangle = K[x_1, \ldots, x_n]$. This is sometimes called *zero coprimeness* (guess why!). For $n = 1$ (principal ideal domain), coprimeness and zero coprimeness are equivalent.

   The ring $R := K(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)[x_i]$ is a localization of the polynomial ring, and therefore, coprime polynomials are still coprime when considered as elements of the principal ideal domain R.

9. Lagrange interpolation over the field with two elements: Let $K = \mathbb{Z}/2\mathbb{Z}$. Show that for any function $f : K^n \to K$, there exists a polynomial $p \in K[x_1, \ldots, x_n]$ with $p(a) = f(a)$ for all $a \in K^n$.

   Hint: It suffices to consider $p = \sum_{\nu \in \{0,1\}^n} p_\nu x^\nu$. This yields a system of 2^n linear equations for 2^n unknowns ...

Algebraic Geometry – Tutorial 5

To be handed in till: Monday, May 8 (noon), To be discussed on: Wednesday, May 10

10. (a) Let $\leq$ be an admissible order on $\mathbb{N}^n$. Show that every descending chain $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots$ in $\mathbb{N}^n$ must become stationary (i.e., $\leq$ is Artinian). Conclude that every non-empty subset of $\mathbb{N}^n$ contains a least element (i.e., $\leq$ is a well-order).

    (b) Let $f_\mu \in K[x_1, \ldots, x_n]$, $\mu \in \mathbb{N}^n$, be such that $\deg(f_\mu) = \mu$, where the degree is defined with respect to an admissible order. Show that $\{f_\mu \mid \mu \in \mathbb{N}^n\}$ is a K-basis of $K[x_1, \ldots, x_n]$.

    (c) Let $f \in \mathbb{Q}[x, x^{-1}]$, that is, $f = \sum_{i \in \mathbb{Z}} c_i x^i$ for some coefficients $c_i \in \mathbb{Q}$ which are almost all zero. Define the degree of f with respect to the natural order of $\mathbb{Z}$. Consider $f_m := x^m + x^{-m^2}$ for $m \in \mathbb{Z}$. Show that $\deg(f_m) = m$, but $\{f_m \mid m \in \mathbb{Z}\}$ is not a $\mathbb{Q}$-basis of $\mathbb{Q}[x, x^{-1}]$.

11. (a) Let N be a subset of $\mathbb{N}^n$ with

$$\nu \in N,\ \mu \in \mathbb{N}^n \quad \Rightarrow \quad \nu + \mu \in N.$$

    Prove that $M := \min_{\mathrm{cw}} N$, the set of component-wise minimal elements of N, is finite and $N = M + \mathbb{N}^n$.

    (b) Let $0 \notin F$ be a finite, non-empty subset of $K[x_1, \ldots, x_n]$ and let an admissible order on $\mathbb{N}^n$ be given. Show that the following are equivalent:

    i. F is a Gröbner basis of $\langle F \rangle$.

    ii. For all $\mu \in \min_{\mathrm{cw}} \deg(\langle F \rangle)$, there exists $f \in F$ with $\deg(f) = \mu$.

Algebraic Geometry – Tutorial 6

To be handed in till: Monday, May 15 (noon)
To be discussed on: Wednesday, May 17

12. Let $F, F', G \dots$ denote finite, non-empty subsets of $K[x_1, \dots, x_n] \setminus \{0\}$.

    (a) Let I be a non-zero ideal in $K[x_1, \dots, x_n]$. Show that if $F \subset I$ is such that $\deg(F) + \mathbb{N}^n = \deg(I)$, then $I = \langle F \rangle$, and thus, F is a Gröbner basis of I.

    (b) Show that for every F there exists F' such that F' is inter-reduced and $\langle F \rangle = \langle F' \rangle$.
    Do this in two different ways: Firstly, design a constructive inter-reduction procedure (whose termination will be due to a Noetherian argument, as usual). Secondly, use that $I := \langle F \rangle$ has a Gröbner basis, and prove the following statement: If G is a GB of $I = \langle G \rangle$, and $g \in G$ is such that $\deg(g) \in \deg(G \setminus \{g\}) + \mathbb{N}^n$, then $G \setminus \{g\}$ is still a GB of I.

    (c) Show that F is an inter-reduced Gröbner basis of $\langle F \rangle$ if and only if

      i. $\forall \mu \in M := \min_{\text{cw}} \deg(\langle F \rangle)\ \exists! f \in F$: $\deg(f) = \mu$, and

      ii. $\deg(F) \subseteq M$.

    (In other words: $\deg : F \to M$, $f \mapsto \deg(f)$ is well-defined and bijective.)

    (d) Let F be a Gröbner basis of $\langle F \rangle$. Suppose that all elements of F are monic, that is, $\text{lc}(f) = 1$ for all $f \in F$. Recall that by definition

    $$F \text{ inter-reduced} \quad \Leftrightarrow \quad \forall f \in F : \deg(f) \notin \deg(F \setminus \{f\}) + \mathbb{N}^n.$$

    Show that

    $$F \text{ reduced} \quad \Leftrightarrow \quad \forall f \in F : f \in \bigoplus_{\mu \notin \deg(F \setminus \{f\}) + \mathbb{N}^n} K x^\mu,$$

    that is, not only the degree of f, but every κ with $c_\kappa \neq 0$ in $f = \sum c_\kappa x^\kappa$ satisfies $\kappa \notin \deg(F \setminus \{f\}) + \mathbb{N}^n$.

13. Compute the reduced Gröbner basis of $I = \langle x^3 + xy, x^2y - y^3 \rangle \subseteq \mathbb{Q}[x, y]$ with respect to the lexicographic order. Verify your result using the Maple[1] commands
```
> with(Groebner);
> F:={x^3+x*y,x^2*y-y^3};
> gbasis(F,plex(x,y));
```
(or similarly, depending on the version you use).
Compute $V = \mathcal{V}(I)$ and $\dim_{\mathbb{Q}} \mathbb{Q}[x, y]/I$. How would you define the "multiplicity" of an element of V?

[1] Feel free to use another computer algebra system, e.g., Gap.

Algebraic Geometry – Tutorial 7

To be handed in till: Monday, May 22 (noon)
To be discussed on: Tuesday, May 23

14. Let K be algebraically closed.

 (a) Let I be an ideal in $K[x_1,\ldots,x_n]$. Let $1 \le j \le n$ and let $\pi : K^n \to K^{n-j+1}$, $(a_1,\ldots,a_n) \mapsto (a_j,\ldots,a_n)$ denote the projection onto the last $n-j+1$ components. Show that (by a slight abuse of notation, the letter $\mathcal{V}$ is used both with respect to K^n and K^{n-j+1})
$$\pi(\mathcal{V}(I)) \subseteq \mathcal{V}(I \cap K[x_j,\ldots,x_n])$$
and that (considering $\mathcal{J}(V)$ as a subset of $K[x_j,\ldots,x_n]$ for $V \subseteq K^{n-j+1}$)
$$\mathcal{J}(\pi(\mathcal{V}(I))) \subseteq \mathcal{J}\mathcal{V}(I).$$

 (b) Conclude that
$$\overline{\pi(\mathcal{V}(I))} = \mathcal{V}(I \cap K[x_j,\ldots,x_n]),$$
where $\overline{V}$ is the Zariski closure of V in K^{n-j+1}.

 (c) Let $f_1,\ldots,f_n \in K[t_1,\ldots,t_m]$. Consider $I = \langle x_1 - f_1,\ldots,x_n - f_n\rangle \subseteq K[t_1,\ldots,t_m,x_1,\ldots,x_n]$. Conclude from the previous part that
$$\mathcal{V}(I \cap K[x_1,\ldots x_n]) = \overline{\mathrm{im}(f)},$$
where $\overline{V}$ denotes the Zariski closure of V in K^n, and $f : K^m \to K^n$ is the map defined by $f(t) = (f_1(t),\ldots,f_n(t))$.

 (d) Let $f_1 = t^2 - a, f_2 = t(a-t^2) \in K[t,a]$. Compute $\overline{\mathrm{im}(f)}$ and explain the connection with Exercise 3c.

 (e) Although the theory from above is not directly applicable to the other algebraic curves from Exercise 3, there exist similar methods: for instance, let $I = \langle x - (c+a)c, y - (c+a)s, c^2 + s^2 - 1\rangle \subseteq \mathbb{R}[c,s,a,x,y]$. Compute (e.g., with MAPLE) a Gröbner basis of $I \cap \mathbb{R}[a,x,y]$ and convince yourself that the result is what you'd expect. Apply analogous methods to the remaining algebraic curves from Exercise 3.

15. Let $I = \langle f_1,\ldots,f_k\rangle$ and $J = \langle g_1,\ldots,g_l\rangle$ be ideals in $K[x] = K[x_1,\ldots,x_n]$. Define
$$L := \langle tf_1,\ldots,tf_k,(1-t)g_1,\ldots,(1-t)g_l\rangle \subseteq K[x,t].$$
Show that $I \cap J = L \cap K[x]$. Describe a procedure to compute a generating set for $I \cap J$ from the given generating sets of I and J.

Algebraic Geometry – Tutorial 8

To be handed in till: Monday, May 29 (noon), To be discussed on: Tuesday, May 30

16. Let $R \neq \{0\}$ be a commutative ring (with unity). Prove the following:

    (a) If $I_1, \ldots, I_k$ are pairwise zero-coprime ideals in R, i.e., if $I_i + I_j = R$ for all $i \neq j$ (such I_i are also called *comaximal*), then we have
    $$I_1 \cdot \ldots \cdot I_k = I_1 \cap \ldots \cap I_k.$$

    (b) If $\mathfrak{m} \neq \mathfrak{n}$ are maximal ideals in R, then we have $\mathfrak{m}^d + \mathfrak{n}^d = R$ for all $d \in \mathbb{N}$.

17. Let K be algebraically closed and let I be a zero-dimensional ideal in $K[x] = K[x_1, \ldots, x_n]$. Then $\mathcal{V}(I)$ is a finite set, and, according to the proof of Theorem 1.15, we have $|\mathcal{V}(I)| = \dim_K K[x]/\mathrm{Rad}(I)$. We know from Lemma 1.14 that also $\dim_K K[x]/I < \infty$, but in general, we only have $\dim_K K[x]/\mathrm{Rad}(I) \leq \dim_K K[x]/I$. Analogously to the case $n = 1$, $\dim_K K[x]/I$ will be interpreted as the number of zeros of I *counted with multiplicities*. The question is how this overall multiplicity should be distributed to the individual zeros. For this, we shall prove the following result: We have
    $$K[x]/I \cong \bigoplus_{a \in \mathcal{V}(I)} (K[x]/I)_{\mathfrak{m}_a},$$
    where $\mathfrak{m}_a$ is the maximal ideal belonging to $a \in K^n$, and $(K[x]/I)_{\mathfrak{m}_a} \cong K[x]_{\mathfrak{m}_a}/I_{\mathfrak{m}_a}$ is the localization of $K[x]/I$ at $\mathfrak{m}_a$. Then one defines the multiplicity of a by
    $$\mu(a) := \dim_K (K[x]/I)_{\mathfrak{m}_a}.$$
    For the proof, let $\mathcal{V}(I) = \{a_1, \ldots, a_k\}$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the corresponding maximal ideals. Consider
    $$\varphi : K[x] \to K[x]_{\mathfrak{m}_1}/I_{\mathfrak{m}_1} \times \ldots \times K[x]_{\mathfrak{m}_k}/I_{\mathfrak{m}_k}, \quad r \mapsto ([\tfrac{r}{1}], \ldots, [\tfrac{r}{1}]).$$
    Clearly, $I \subseteq \ker(\varphi)$. In view of the homomorphism theorem, it therefore suffices to prove (i) the converse inclusion and (ii) the surjectivity of φ. For this, the following auxiliary results (and the exercise from above) should be helpful:

    (a) There exists $d \in \mathbb{N}$ such that $\bigcap_{i=1}^k \mathfrak{m}_i^d \subseteq I$.

    (b) By Lagrange interpolation, there exist $\varepsilon_i \in K[x]$ with $\varepsilon_i(a_i) = 1$ and $\varepsilon_i(a_j) = 0$ for $i \neq j$. Define $e_i := 1 - (1 - \varepsilon_i^d)^d$, where d is as in (a). Then we have the following identities modulo I: $\sum_{i=1}^k e_i \equiv 1$, $e_i e_j \equiv 0$ for $i \neq j$, and $e_i^2 \equiv e_i$. Moreover, $e_i - 1 \in I_{\mathfrak{m}_i}$ and $e_i \in I_{\mathfrak{m}_j}$ for $i \neq j$.

    (c) For $g \notin \mathfrak{m}_i$, there exists $h \in K[x]$ such that $hg \equiv e_i$ modulo I. (Hint: Set $\tilde{g} := 1 - \frac{g}{g(a_i)}$ and consider $1 + \tilde{g} + \ldots + \tilde{g}^{d-1}$.)

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Algebraic Geometry – Tutorial 9

To be handed in till: Monday, June 12 (noon), To be discussed on: Tuesday, June 13

18. Let $f : V \to W$ be a morphism of algebraic sets and let $\varphi := K[f]$ be the induced K-algebra homomorphism. Show that

$$f^{-1}(\mathcal{V}_W(F)) = \mathcal{V}_V(\varphi(F))$$

for $F \subseteq K[W]$.

19. Let $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$. Then $f : K^n \to K^n$, $a \mapsto (f_1(a), \ldots, f_n(a))$ is a morphism of algebraic sets.

    (a) Show that f is an isomorphism of algebraic sets if and only if there exist $g_1, \ldots, g_n \in K[y_1, \ldots, y_n]$ with

$$\langle y_1 - f_1, \ldots, y_n - f_n \rangle = \langle x_1 - g_1, \ldots, x_n - g_n \rangle.$$

    (b) How can this condition be tested using Gröbner bases?

20. Consider the radical ideal

$$I = \langle xz - y^2, x - yz \rangle \subset \mathbb{C}[x, y, z].$$

Compute the irreducible components of $V = \mathcal{V}(I)$. Is V connected?

Hint: You may use the following result from commutative algebra: If I is radical and $p, q \notin I$ are such that $pq \in I$, then we have $I = I_1 \cap I_2$, where $I_1 := (I : p)$ and $I_2 := I + \langle p \rangle$.

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Algebraic Geometry – Tutorial 10

Abgabe bis: Monday, June 19 (noon), To be discussed on: Tuesday, June 20

21. Let R be a ring and let $\emptyset \neq P$ be a closed subset of $\operatorname{Spec}(R)$, that is, $P = \mathcal{V}(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$ for an ideal I in R. Show that P is irreducible if and only if $\mathcal{J}(P) = \bigcap_{\mathfrak{p}\in P} \mathfrak{p}$ is prime.

22. (a) Let (M, A) be an algebraic variety and let $U \subseteq M$ be an open set. Let I be an arbitrary set, and let $U_i \subseteq M$, $i \in I$, be basic open sets with

$$U = \bigcup_{i\in I} U_i.$$

Show that $\exists k \in \mathbb{N}$, $i_1, \ldots, i_k \in I$: $U = \bigcup_{j=1}^{k} U_{i_j}$.

(b) Let (M, A) be an irreducible algebraic variety, $M \neq \emptyset$, and $h \in K(M) = \operatorname{Quot}(A)$. Show that

$$D(h) = \bigcup \{D(\tfrac{1}{g}) \mid 0 \neq g \in A, gh \in A\}$$

and conclude from part (a): $\exists k \in \mathbb{N}$, $f_1, \ldots, f_k \in A$, $g_1, \ldots, g_k \in A \setminus \{0\}$ with $h = \frac{f_1}{g_1} = \ldots = \frac{f_k}{g_k}$ such that $D(h) = \bigcup_{j=1}^{k} D(\frac{1}{g_j})$.

(c) Let (M, A) be an irreducible algebraic variety, $M \neq \emptyset$, and $0 \neq g \in A$. Show that

$$A_g = \{h \in K(M) \mid M_g \subseteq D(h)\}.$$

23. Let (M, A) be an algebraic variety, let $U \subseteq M$ be an open set, $x \in U$, and let $h : U \to K$ be a map. Prove that the following are equivalent:

(a) $\exists f, g \in A$: $x \in D(\frac{1}{g}) \subseteq U$ and $h(y) = \frac{f(y)}{g(y)}$ for all $y \in D(\frac{1}{g})$;

(b) $\exists f, g \in A$: $\exists V$, open neighborhood of x in $U \cap D(\frac{1}{g})$, with $h(y) = \frac{f(y)}{g(y)}$ for all $y \in V$.

Moreover, if (a) is satisfied for all $x \in U$, then we have: $\exists k \in \mathbb{N}$, $f_1, \ldots, f_k$, $g_1, \ldots, g_k \in A$: $U = \bigcup_{j=1}^{k} D(\frac{1}{g_j})$ and $h(y) = \frac{f_j(y)}{g_j(y)}$ for all $y \in D(\frac{1}{g_j})$.

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Algebraic Geometry – Tutorial 11

To be handed in till: Monday, June 26 (noon), To be discussed on: Tuesday, June 27

24. (a) Let (M, A) be an irreducible algebraic variety, $M \neq \emptyset$, and $h \in K(M)$. Let A be a unique factorization domain (UFD). Show that $D(h)$ is a basic open set.

    (b) Consider $A = \mathbb{C}[x, y, z]/\langle x^2 - yz\rangle$. We have seen in the lecture that there exists $h \in \mathrm{Quot}(A)$ for which $D(h)$ is not a basic open set. Thus A cannot be a UFD. Prove this in the following alternative way: Consider the morphism $f : \mathbb{C}^2 \to V = \mathcal{V}(x^2 - yz) \subseteq \mathbb{C}^3$, $(s, t) \mapsto (st, s^2, t^2)$. Show that f is dominant. Thus $\varphi := \mathbb{C}[f]$ is a monomorphism. Use φ to determine the units of A and to show that A contains elements that are irreducible, but not prime.

25. (a) Let (M, A) be an algebraic variety. Consider pairs (h, U), where $U \subseteq M$ is open and dense, and $h \in \mathcal{O}(U)$. Show that

    $(h_1, U_1) \sim (h_2, U_2) \Leftrightarrow \exists W \subseteq U_1 \cap U_2, W$ open and dense in M: $h_1|_W = h_2|_W$

    is an equivalence relation. The set of equivalence classes $[(h, U)]$ is denoted by $R(M)$.

    (b) Show that

    $$(h_1, U_1) \sim (h_2, U_2) \quad \Leftrightarrow \quad h_1|_{U_1\cap U_2} = h_2|_{U_1\cap U_2}.$$

    The interesting direction of this equivalence is called *identity theorem*.
    Hint: $U := U_1 \cap U_2$, $h := h_1 - h_2 \in \mathcal{O}(U)$. $\mathcal{N}(h) := \{x \in U \mid h(x) = 0\}$ is closed in U (for each $x \in U \setminus \mathcal{N}(h)$ there exists an open neighborhood of x in $U \setminus \mathcal{N}(h)$) and it contains a set that is dense in M. Consider the closures with respect to U and note that $\overline{V}^U = U \cap \overline{V}$ for $V \subseteq U$.

26. Let (M, A) be an irreducible algebraic variety, $M \neq \emptyset$. Consider the map $\psi : K(M) \to R(M)$ that assigns to each representative $\frac{f}{g}$ ($f \in A, 0 \neq g \in A$) the equivalence class $[(\frac{f}{g}, D(\frac{1}{g}))]$, where $\frac{f}{g}$ is understood as a map $D(\frac{1}{g}) \to K$. Show that ψ is well-defined and bijective.

Algebraic Geometry – Tutorial 12

To be handed in till: Monday, July 3 (noon), To be discussed on: Tuesday, July 4

27. Let $V \subseteq K^n$ and $W \subseteq K^m$ be non-empty irreducible algebraic sets. We identify $K(V) = R(V)$ and $K(W) = R(W)$. Let $h_1, \ldots, h_m \in K(V)$ be such that

$$h : D(h) \to W, \quad v \mapsto h(v) := (h_1(v), \ldots, h_m(v))$$

is well-defined, i.e., $\mathrm{im}(h) \subseteq W$. Here, $D(h) := \bigcap_{i=1}^m D(h_i)$ is open and dense in V. We call h a rational map from V to W. The definition

$$K(h) : K(W) \to K(V), \quad g \mapsto g \circ h$$

does not necessarily make sense, since we may have $\mathrm{im}(h) \cap D(g) = \emptyset$.

(a) Give a simple example illustrating this phenomenon.

(b) Show that $K(h)$ from above is well-defined if h is dominant ($\overline{\mathrm{im}(h)} = W$). Conversely, every K-algebra homomorphism $\varphi : K(W) \to K(V)$ yields a dominant rational map h from V to W with $K(h) = \varphi$.

(c) Prove that the following are equivalent:

i. There exists $h : D(h) \to W$ as above and, analogously, $k : D(k) \to V$ such that $h \circ k$ and $k \circ h$ are defined on open and dense subsets of W and V, and equal to the respective identity maps.

ii. $K(W) \cong K(V)$.

Then one says that V and W are *birationally equivalent*. If $W = K^m$, then V is called a *rational variety*, and, if $m = 1$, a *rational curve*.

(d) Show that $V = \mathcal{V}(x^2 + y^2 - 1) \subset \mathbb{C}^2$ is a rational curve.
Hint: Stereographic projection.
Remark: $\mathcal{V}(x^n + y^n - 1) \subset \mathbb{C}^2$ is not a rational curve for $n \geq 3$.

28. Let L be an extension field of K, and let E be a finite subset of L. Show that E is a transcendence basis of L over K if and only if L is algebraic over $K(E)$, but not over any $K(E \setminus \{e\})$, $e \in E$.

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Algebraic Geometry – Tutorial 13

To be handed in till: Monday, July 10 (noon), To be discussed on: Tuesday, July 11

29. Prove Theorem 4.6 and Corollary 4.7 of the lecture.

30. Let $V = \mathcal{V}(x_1^2 + x_2^2 + x_3^2 - 1) \subset \mathbb{C}^3$ and $V_1 = V \cap \mathcal{V}(x_1x_2x_3)$. How do these objects look like in real space? Determine a finite morphism $f : V \to \mathbb{C}^2$ with $f(V_1) = \mathbb{C} \times \{0\}$.

    Remark: The ideal
    $$\langle x_1^2 + x_2^2 + x_3^2 - 1, x_1x_2x_3 \rangle$$
    can be written as an intersection of three prime ideals (which ones?), and thus it is radical.