Invariant Theory, Finiteness and Degree Bounds

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June 10, 2025

Notation 1.

- $\Gamma \leq \mathrm{GL}(\mathbb{C}^n)$ denotes a finite matrix (sub)group.
- $\mathbf{x} = (x_1, \cdots, x_n)$ denotes n variables.
- $\mathbb{C}[\mathbf{x}]$ is defined as the ring of polynomials of complex coefficients in n variables.
- we define
$$\mathbb{C}[\mathbf{x}]^\Gamma = \{p \in \mathbb{C}[\mathbf{x}] : \sigma \cdot p = p, \forall \sigma \in \Gamma\}$$
the **invariant subring** of $\mathbb{C}[\mathbf{x}]$ under the group operation of Γ.

Proposition 1. For every finite group $\Gamma \leq \mathrm{GL}(\mathbb{C}^n)$, the invariant ring $\mathbb{C}[\mathbf{x}]^\Gamma$ has n algebraically independent variables.

Definition 1. Given a finite matrix group $\Gamma \leq \mathrm{GL}(\mathbb{C}^n)$. The map

$$* : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]^\Gamma, p \mapsto p^* = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \sigma \cdot p \tag{1}$$

is called **Reynold operator**.

Some basic properties of Reynold operator:

Proposition 2.

1. $*$ is $\mathbb{C}$-linear, that is, $\forall f, g \in \mathbb{C}[\mathbf{x}], \forall \lambda, \kappa \in \mathbb{C} : (\lambda f + \kappa g)^* = \lambda f^* + \kappa g^*$.
2. $*|_{\mathbb{C}[\mathbf{x}]^\Gamma} = \mathrm{id}$, i.e. $\forall I \in \mathbb{C}[\mathbf{x}]^\Gamma : I^* = I$.
3. $*$ is a $\mathbb{C}[\mathbf{x}]^\Gamma$-module homomorphism, i.e $\forall p \in \mathbb{C}[\mathbf{x}], I \in \mathbb{C}[\mathbf{x}]^\Gamma : (pI)^* = p^* I$.

Theorem 1 (Hilbert's Finiteness Theorem)**.** Every invariant ring $\mathbb{C}[\mathbf{x}]^\Gamma$ is finitely generated, i.e. there is a set of invariants $\{I_1, \cdots, I_m\}$ generating $\mathbb{C}[\mathbf{x}]^\Gamma$.

Theorem 2 (Noether's degree bound)**.** The invariant ring $\mathbb{C}[\mathbf{x}]^\Gamma$ of a finite matrix group has an algebra basis of at most $\binom{n+|\Gamma|}{n}$ invariants with a degree not exceeding the group order $|\Gamma|$.