Hecke actions on certain strongly modular genera of lattices

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ABSTRACT * We calculate the action of some Hecke operators on spaces of modular forms spanned by the Siegel theta-series of certain genera of strongly modular lattices closely related to the Leech lattice. Their eigenforms provide explicit examples of Siegel cusp forms.

1 Introduction

One of the most remarkable lattices in Euclidean space is the Leech lattice, the unique even unimodular lattice $\Gamma_1 \subset (\mathbb{R}^{24}, (,))$ of dimension 24 that does not contain vectors of square length 2. Here a lattice $\Lambda \subset (\mathbb{R}^n, (,))$ is called unimodular, if Λ equals its dual lattice

$$\Lambda^{\#} := \{x \in \mathbb{R}^n \mid (x, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$$

and even, if the quadratic form $x \mapsto (x, x)$ takes only even values on Λ. [13] studies spaces of Siegel modular forms generated by the Siegel theta-series of the 24 isometry classes of lattices in the genus of Γ_1. The present paper extends this investigation to further genera of lattices, closely related to Γ_1. A unified construction is given in [16]: Consider the Matthieu group $M_{23} \leq \mathrm{Aut}(\Gamma_1)$, where the automorphism group of a lattice $\Lambda \subset (\mathbb{R}^n, (,))$ is $\mathrm{Aut}(\Lambda) := \{g \in O(n) \mid \Lambda g = \Lambda\}$. Let $g \in M_{23}$ be an element of square-free order $l := |\langle g \rangle|$. Then

$$l \in \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\} =: \mathcal{N} = \{n \in \mathbb{N} \mid \sigma_1(n) := \sum_{d|n} d \text{ divides } 24\}$$

and for each $l \in \mathcal{N}$, there is an up to conjugacy unique cyclic subgroup $\langle g \rangle \leq M_{23}$ of order l. Let $\Gamma_l := \{\lambda \in \Gamma_1 \mid \lambda g = \lambda\}$ denote the fixed lattice of g. Then Γ_l is an extremal strongly modular lattice of level l and of dimension $2k_l$, where

$$k_l := 12\sigma_0(l)/\sigma_1(l)$$

and $\sigma_0(l)$ denotes the number of divisors of l. In particular Γ_1 is the Leech lattice, Γ_2 the 16-dimensional Barnes-Wall lattice and Γ_3 the Coxeter-Todd lattice of dimension 12.

Let Λ be an even lattice. The minimal $l \in \mathbb{N}$ for which $\sqrt{l}\Lambda^{\#}$ is even, is called the level of Λ. Then $l\Lambda^{\#} \subset \Lambda$. For an exact divisor d of l let

$$\Lambda^{\#,d} := \Lambda^{\#} \cap \frac{1}{d}\Lambda$$

denote the d-partial dual of Λ. A lattice Λ is called strongly l-modular, if Λ is isometric to $\sqrt{d}\Lambda^{\#,d}$ for all exact divisors d of the level l of Λ. If l is a prime, this coincides with the notion of modular lattices, which just means that the lattice is similar to its dual lattice. The Siegel theta-series

$$\Theta_{\Lambda}^{(m)}(Z) := \sum_{(\lambda_1,\ldots,\lambda_m)\in\Lambda^m} \exp(i\pi \mathrm{trace}((\lambda_i, \lambda_j)_{i,j} Z))$$

*Mathematics Subject Classification: 11F46, 11H06 (which is a holomorphic function on the Siegel halfspace $\mathcal{H}^{(m)} = \{Z \in \mathrm{Sym}_m(\mathbb{C}) \mid \Im(Z)$ positive definite $\}$) of a strongly l-modular lattice is a modular form for the l-th congruence subgroup $\Gamma_0^{(m)}(l)$ of $\mathrm{Sp}_{2m}(\mathbb{Z})$ (to a certain character) invariant under all Atkin-Lehner-involutions (cf. [1]). In particular for $m = 1$ and $l \in \mathcal{N}$ the relevant ring of modular forms is a polynomial ring in 2 generators as shown in [14], [15]. Explicit generators of this ring allow to bound the minimum of an n-dimensional strongly l-modular lattice Λ with $l \in \mathcal{N}$,

$$\min(\Lambda) := \min_{0 \neq \lambda \in \Lambda} (\lambda, \lambda) \leq 2 + 2\lfloor \frac{n}{2k_l} \rfloor.$$

Lattices Λ achieving this bound are called extremal. For all $l \in \mathcal{N}$ there is a unique extremal strongly l-modular lattices of dimension $2k_l$ and this is the lattice Γ_l described above. All the genera are presented in the nice survey article [17].

In this paper we investigate the spaces of Siegel modular forms generated by the Siegel theta-series of the lattices in the genus $\mathcal{G}(\Gamma_l)$ for $l \in \mathcal{N}$ using similar methods as for the case $l = 1$ which is treated in [13]. The vector space $\mathcal{V} := \mathcal{V}(\mathcal{G})$ of all complex formal linear combinations of the isometry classes of lattices in any genus $\mathcal{G}$ forms a finite dimensional commutative $\mathbb{C}$-algebra with positive definite Hermitian scalar product. Taking theta-series defines linear operators $\Theta^{(m)}$ from $\mathcal{V}$ into a certain space of modular forms and hence a filtration of $\mathcal{V}$ by the kernels of these operators. This filtration behaves nicely under the multiplication and is invariant under all Hecke-operators. With the Kneser neighbouring process we construct a family of commuting self-adjoint linear operators on $\mathcal{V}$. Their common eigenvectors provide explicit examples of Siegel cusp forms.

The genera $\mathcal{G}(\Gamma_l)$ ($l \in \mathcal{N}$) share the following properties:

Corollary 1.1 *Let $l \in \mathcal{N}$ and let p be the smallest prime not dividing l. The mapping $\Theta^{(k_l)}$ is injective on $\mathcal{V}(\mathcal{G}(\Gamma_l))$. For $l \neq 7$, the construction described in [5] (see Paragraph 2.3) gives a non-zero cusp form $\mathrm{BFW}(\Gamma_l, p) = \Theta^{(k_l)}(\mathrm{Per}(\Gamma_l, p))$. The eigenvalue of the Kneser operator K_2 at the eigenvector $\mathrm{Per}(\Gamma_l, p)$ is the negative of the number of pairs of minimal vectors in Γ_l which is also the minimal eigenvalue of K_2.*

Remark 1.2 *In Section 3 we also list the eigenvalues of some of the operators $T(q)$ defined in Subsection 2.4. These eigenvalues suggest that for even values of k_l, the cusp form $\mathrm{BFW}(\Gamma_l, p)$ is a generalized Duke-Imamoglu-Ikeda lift (see [8]) of the elliptic cusp form of minimal weight k_l.*

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2 Methods

The general method has already been explained in [13] (see also [19],[20], [21] and [2] for similar strategies).

2.1 The algebra $\mathcal{V} = \mathcal{V}(\mathcal{G})$

Let $\mathcal{G}$ be a genus of lattices in the Euclidean space $(\mathbb{R}^{2k}, (,))$. Then $\mathcal{G}$ is the disjoint union of finitely many isometry classes

$$\mathcal{G} = [\Lambda_1] \cup \ldots \cup [\Lambda_h].$$

Let $\mathcal{V} := \mathcal{V}(\mathcal{G}) \cong \mathbb{C}^h$ be the complex vector space with basis $([\Lambda_1], \ldots, [\Lambda_h])$. Let $\mathcal{V}_{\mathbb{Q}} = \langle [\Lambda_1], \ldots, [\Lambda_h] \rangle_{\mathbb{Q}} \cong \mathbb{Q}^h$ be the rational span of the basis.

The space $\mathcal{V}$ can be identified with the algebra $\mathcal{A}$ of complex functions on the double cosets $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathrm{Stab}_{G(\mathbb{A})}(\Lambda_{\mathbb{A}}) = \cup_{i=1}^h G(\mathbb{Q}) x_i \mathrm{Stab}_{G(\mathbb{A})}(\Lambda_{\mathbb{A}})$ where G is the integral form of the real orthogonal group $G(\mathbb{R}) = O_{2k}$ defined by Λ_1, $\mathbb{A}$ denotes the ring of rational adèles and $\Lambda_{\mathbb{A}}$ the adélic completion of Λ_1. If χ_i denotes the characteristic function mapping $G(\mathbb{Q}) x_j \mathrm{Stab}_{G(\mathbb{A})}(\Lambda_{\mathbb{A}})$ to δ_{ij} and $\Lambda_i = x_i \Lambda_1$ $(i = 1, \ldots, h)$ then the isomorphism maps $[\Lambda_i]$ to $|\mathrm{Aut}(\Lambda_i)| \chi_i$. The usual Petersson scalar product then translates into the Hermitian scalar product on $\mathcal{V}$ defined by

$$< [\Lambda_i], [\Lambda_j] > := \delta_{ij} |\mathrm{Aut}(\Lambda_i)|$$

and the multiplication of $\mathcal{A}$ defines a commutative and associative multiplication $\circ$ on $\mathcal{V}$ with

$$[\Lambda_i] \circ [\Lambda_j] := \#(\mathrm{Aut}(\Lambda_i)) \delta_{i,j} [\Lambda_i]$$

(see for instance [3, Section 1.1]). Note that the Hermitian form $\langle , \rangle$ is associative, i.e.

$$\langle v_1 \circ v_2, v_3 \rangle = \langle v_1, v_2 \circ v_3 \rangle \text{ for all } v_1, v_2, v_3 \in \mathcal{V}.$$

2.2 The two basic filtrations of $\mathcal{V}$

For simplicity we now assume that $\mathcal{G}$ consists of even lattices. Let l be the level of the lattices in $\mathcal{G}$. Taking the degree-n Siegel theta-series $\Theta^{(n)}_{\Lambda_i}$ $(n = 0, 1, 2, \ldots)$ of the lattices Λ_i $(i = 1, \ldots, h)$ then defines a linear map

$$\Theta^{(n)} : \mathcal{V} \to M_{n,k}(l) \text{ by } \Theta^{(n)}(\sum_{i=1}^h c_i [\Lambda_i]) := \sum_{i=1}^h c_i \Theta^{(n)}_{\Lambda_i}$$

with values in a space of modular forms of degree n and weight k for the group $\Gamma_0^{(n)}(l)$ (see [1]).

For $n = 0, \ldots, 2k$ let $\mathcal{V}_n := \ker(\Theta^{(n)})$ be the kernel of this linear map. Then we get the filtration

$$\mathcal{V} =: \mathcal{V}_{-1} \supseteq \mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \ldots \supseteq \mathcal{V}_{2k} = \{0\}$$

where $\mathcal{V}_0 = \{ v = \sum_{i=1}^h c_i [\Lambda_i] \mid \sum_{i=1}^h c_i = 0 \}$ is of codimension 1 in $\mathcal{V}$.

Clearly $\Theta^{(n)}(\mathcal{V}_{n-1})$ is the kernel of the Siegel Φ-operator mapping $\Theta^{(n)}(\mathcal{V})$ onto $\Theta^{(n-1)}(\mathcal{V})$. For square-free level one even has

Theorem 2.1 *(see [4, Theorem 8.1]) If l is square-free, then $\Theta^{(n)}(\mathcal{V}_{n-1})$ is the space of cusp forms in $\Theta^{(n)}(\mathcal{V})$.*

Let $\mathcal{W}_n := \mathcal{V}_n^{\perp}$ be the orthogonal complement of $\mathcal{V}_n$. We then have the ascending filtration

$$0 = \mathcal{W}_{-1} \subseteq \mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \ldots \subseteq \mathcal{W}_{2k} = \mathcal{V}.$$

By [13, Proposition 2.3, Corollary 2.4] one has the following lemma:

Lemma 2.2

$$\mathcal{W}_n \circ \mathcal{W}_m \subset \mathcal{W}_{n+m} \textit{ for all } m, n \in \{-1, \ldots, 2k\}$$

and

$$\mathcal{W}_n \circ \mathcal{V}_m \subset \mathcal{V}_{m-n} \textit{ for all } m > n \in \{-1, \ldots, 2k\}.$$

Since theta-series have rational coefficients, both filtrations are rational, i.e. $\mathcal{V}_n = \mathbb{C} \otimes (\mathcal{V}_n \cap \mathcal{V}_{\mathbb{Q}})$ and $\mathcal{W}_n = \mathbb{C} \otimes (\mathcal{W}_n \cap \mathcal{V}_{\mathbb{Q}})$, hence the same statements hold when $\mathcal{V}$ is replaced by $\mathcal{V}_{\mathbb{Q}}$.

2.3 The Borcherds-Freitag-Weissauer cusp form

The article [5] gives a quite general construction of a cusp form of degree k. Let Λ be a $2k$-dimensional even lattice and choose some prime p such that the quadratic space $(\Lambda/p\Lambda, Q_p)$ (where $Q_p(x) := \frac{1}{2}(x,x) + p\mathbb{Z}$) is isometric to the sum of k hyperbolic planes. Fix a totally isotropic subspace F of $\Lambda/p\Lambda$ of dimension k. For $\lambda := (\lambda_1, \ldots, \lambda_k) \in \Lambda^k$ we put $E(\lambda) := \langle \lambda_1, \ldots, \lambda_k \rangle + p\Lambda$ and $S(\lambda) := \frac{1}{p}((\lambda_i, \lambda_j)_{i,j}) \in \mathrm{Sym}_k(\mathbb{R})$. Define $\epsilon(E(\lambda)) = \epsilon(\lambda) := (-1)^{\dim(F \cap E(\lambda))}$ if $E(\lambda)$ is a k-dimensional totally isotropic subspace of $\Lambda/p\Lambda$ and $\epsilon(E(\lambda)) = \epsilon(\lambda) := 0$ otherwise.

Definition 2.3 $\mathrm{BFW}(\Lambda, p)(Z) := \sum_{\lambda \in \Lambda^k} \epsilon(\lambda) \exp(i\pi \mathrm{trace}(S(\lambda)Z))$.

By [5] the form $\mathrm{BFW}(\Lambda, p)$ is a linear combination of Siegel theta-series of lattices in the genus of Λ: For any k-dimensional totally isotropic subspace E of $\Lambda/p\Lambda$ let $\Gamma(E) := \langle E, p\Lambda \rangle$ be the full preimage of E. Dividing the scalar product by p, one obtains a lattice ${}^{1/p}\Gamma(E) := (\Gamma(E), \frac{1}{p}(,)) \in \mathcal{G}$. Then we define

$$\mathrm{Per}(\Lambda, p) := \sum_E \epsilon(E)[\,{}^{1/p}\Gamma(E)] \in \mathcal{V}$$

where the sum runs over all k-dimensional totally isotropic subspaces of $\Lambda/p\Lambda$. As ϵ is only defined up to a sign, also $\mathrm{Per}(\Lambda, p)$ is only well defined up to a factor ± 1. It is shown in [5, Theorem 2] that

$$\Theta^{(k)}(\mathrm{Per}(\Lambda, p)) = \mathrm{BFW}(\Lambda, p).$$

In analogy to the notation in [10] we call $\mathrm{Per}(\Lambda, p)$ the perestroika of Λ. Clearly $\mathrm{BFW}(\Lambda, p)$ is in the kernel of the Φ-operator and hence a cusp form, if the level of Λ is square-free by Theorem 2.1.

2.4 Hecke-actions

Strongly related to the Borcherds-Freitag-Weissauer construction are the Hecke operators $T(p)$ which define self-adjoint linear operators on $\mathcal{V}$ and whose action on theta series coincides with the one of $T(p)$ in [7, Theorem IV.5.10] and [22, Proposition 1.9] up to a scalar factor (depending on the degree of the theta series). Assume that the genus $\mathcal{G}$ consists of even $2k$-dimensional lattices of level l. For primes p not dividing l we define $T(p) : \mathcal{V} \to \mathcal{V}$ by

$$T(p)([\Lambda]) := \sum_E [\,{}^{1/p}\Gamma(E)]$$

where the sum runs over all k-dimensional totally isotropic subspaces of $(\Lambda/p\Lambda, Q_p)$. Note that $T(p)$ is 0 if $(\Lambda/p\Lambda, Q_p)$ is not isomorphic to the sum of k hyperbolic planes.

The following operators commute with the $T(p)$ and are usually easier to calculate using the Kneser neighbouring-method (see [9]): For a prime p define the linear operator K_p by

$$K_p([\Lambda]) := \sum_\Gamma [\Gamma], \text{ for all } \Lambda \in \mathcal{G}$$

where the sum runs over all lattices Γ in $\mathcal{G}$ such that the intersection $\Lambda \cap \Gamma$ has index p in Λ and in Γ. If p does not divide the level l [22, Proposition 1.10] shows that the operators K_p are essentially the Hecke operators $T^{(m-1)}(p^2)$ (up to a summand, which is a multiple of the identity and a scalar factor). Also if p divides l, the operators K_p are self-adjoint: For Λ and Γ in $\mathcal{G}$, the number $n(\Gamma, [\Lambda])$ of neighbours of Γ that are isometric to Λ equals the number of rational matrices $X \in \mathrm{GL}_{2k}(\mathbb{Z})\mathrm{diag}(p^{-1}, 1^{2k-1}, p)\mathrm{GL}_{2k}(\mathbb{Z})$ solving

$$I(\Gamma, \Lambda) : \quad X F_\Gamma X^{tr} = F_\Lambda$$

(where F_Γ and F_Λ denote fixed Gram matrices of Γ respectively Λ) divided by the order of the automorphism group of Λ (since one only counts lattices, X and gX have to be identified for all $g \in \mathrm{GL}_{2k}(\mathbb{Z})$ with $gF_\Lambda g^{tr} = F_\Lambda$). Mapping X to X^{-1} gives a bijection between the set of solutions of $I(\Gamma, \Lambda)$ and $I(\Lambda, \Gamma)$. Therefore

$$n(\Gamma, [\Lambda])|\mathrm{Aut}(\Lambda)| = n(\Lambda, [\Gamma])|\mathrm{Aut}(\Gamma)|.$$

Hence the linear operators K_p and $T(p)$ generate a commutative subalgebra

$$\mathcal{H} := \langle T(q), K_p \mid q, p \text{ primes }, q \not| l \rangle \leq \mathrm{End}^s(\mathcal{V})$$

of the space of self-adjoint endomorphisms of $\mathcal{V}$ and $\mathcal{V}$ has an orthogonal basis $(d_1, \ldots, d_h)$, consisting of common eigenvectors of $\mathcal{H}$.

For each $1 \leq i \leq h$ we define $v(i) \in \{-1, \ldots, 2k-1\}$ by $d_i \in \mathcal{V}_{v(i)}, \quad d_i \notin \mathcal{V}_{v(i)+1}$. Analogously let $w(i) \in \{0, \ldots, 2k\}$ be defined by $d_i \in \mathcal{W}_{w(i)}, \quad d_i \notin \mathcal{W}_{w(i)-1}$.

Lemma 2.4 *([13, Lemma 2.5]) Let $1 \leq i \leq h$ and assume that d_i generates a full eigenspace of $\mathcal{H}$. Then $w(i) = v(i) + 1$.*

If the genus $\mathcal{G}$ is strongly modular of level l, by which we mean that $\sqrt{d}\Lambda^{\#,d} \in \mathcal{G}$ for all $\Lambda \in \mathcal{G}$ and all exact divisors d of l, then the Atkin-Lehner involutions

$$W_d : [\Lambda] \mapsto [\sqrt{d}\Lambda^{\#,d}]$$

for exact divisors d of l define further self-adjoint linear operators on $\mathcal{V}$. In this case let

$$\hat{\mathcal{H}} := \langle \mathcal{H}, W_d \mid d \text{ exact divisor of } l \rangle.$$

If all lattices in $\mathcal{G}$ are strongly modular, then $W_d = 1$ for all d and $\hat{\mathcal{H}} = \mathcal{H}$ is commutative.

Again, the Hecke action is rational on $\mathcal{V}_\mathbb{Q}$ hence the $\mathbb{Q}$-algebras $\mathcal{H}_\mathbb{Q}$ and $\hat{\mathcal{H}}_\mathbb{Q}$ spanned by the K_p respectively the K_p and W_d act on $\mathcal{V}_\mathbb{Q}$.

Remark 2.5 *For $v \in \{-1, 0, \ldots, 2k-1\}$ let*

$$\mathcal{D}_v := \langle d_i \mid v(i) = v \rangle.$$

If all eigenspaces of $\mathcal{H}$ are 1-dimensional, the decomposition $\mathcal{V} = \oplus_{v=-1}^{2k-1} \mathcal{D}_v$ is preserved under any semi-simple algebra $\mathcal{A}$ with $\mathcal{H} \leq \mathcal{A} \leq \mathrm{End}(V)$ that respects the filtration.

3 Results

The explicit calculations are performed in MAGMA ([12]). Fix $l \in \mathcal{N}$, let $\mathcal{G} := \mathcal{G}(\Gamma_l)$, $\mathcal{V} = \mathcal{V}(\mathcal{G})$ and denote by $\Lambda_1 := \Gamma_l, \Lambda_2, \ldots, \Lambda_h$ representatives of the isometry classes of lattices in $\mathcal{G}$. We find that in all cases $\mathcal{H} = \langle K_2, K_3\rangle \cong \mathbb{C}^h$ is a maximal commutative subalgebra of $\mathrm{End}(\mathcal{V})$. Therefore the common eigenspaces are of dimension one and it is straightforward to calculate an explicit orthogonal basis $(d_1, \ldots, d_h)$ of $\mathcal{V}$ consisting of eigenvectors of $\mathcal{H}$. In particular $v(i) = w(i) - 1$ for all $i = 1, \ldots, h$ by Lemma 2.4. Here we choose $d_1 := \sum_{i=1}^{h} |Aut(\Lambda_i)|^{-1}[\Lambda_i] \in \mathcal{V}_0 - \mathcal{V}_1$ to be the unit element of $\mathcal{V}$ and (for $l \neq 7$) $d_h = \mathrm{Per}(\Gamma_l, p) \in \mathcal{V}_{k_l - 1}$, where p is the smallest prime not dividing l. We then determine some Fourier-coefficients of the series $\Theta^{(n)}(d_i)$ $(n = 0, 1, \ldots, k_l)$ to get upper bounds on $v(i)$. In all cases the degree-k_l Siegel theta-series of the lattices are linearly independent hence $\mathcal{V}_{k_l} = \{0\}$. Moreover $\mathcal{V}_{k_l - 1} = \langle d_h\rangle$ if $l \neq 7$. We also know that $w(1) = 0$ and we may choose d_2 such that $w(2) = 1$. By Lemma 2.2 and 2.4 the product $d_j \circ d_i$ lies in $\mathcal{W}_{w(i)+w(j)}$. If the coefficient of d_h in the product is non-zero, this yields lower bounds on the sum $w(i) + w(j)$ which often yield sharp lower bound for $w(i)$ and $w(j)$. The method is illustrated in [13, Section 3.2] and an example is given in Paragraph 3.1.

3.1 The genus of the Barnes-Wall lattice in dimension 16.

The lattices in this genus are given in [18]. The class number is $h = 24$ and we find

$$\langle K_2, K_3\rangle = \mathcal{H}_{\mathbb{Q}} \cong \mathbb{Q}^{13} \oplus F_1 \oplus F_2 \oplus F_3$$

where the totally real number fields $F_i \cong \mathbb{Q}[x]/(f_i(x))$ are given by

$$\begin{aligned} f_1 = &\ x^3 - 11496x^2 + 41722560x - 47249837568 \\ f_2 = &\ x^3 - 1704x^2 + 400320x + 173836800 \\ f_3 = &\ x^5 - 11544x^4 + 42868800x^3 - 53956108800x^2 + 1813238784000x \\ &+20094119608320000 \end{aligned}$$

and $\langle K_2, K_3, W_2\rangle = \hat{\mathcal{H}}_{\mathbb{Q}} \cong \mathbb{Q}^{13} \oplus \mathrm{Mat}_3(\mathbb{Q}) \oplus \mathrm{Mat}_3(\mathbb{Q}) \oplus \mathrm{Mat}_5(\mathbb{Q})$. Let α_i, β_i and γ_j $(i = 1, \ldots 3, j = 1, \ldots, 5)$ denote the complex roots of the polynomials f_1, f_2 respectively f_3. Let ϵ_i $(i = 1, \ldots, 3)$ denote the primitive idempotents of $\mathcal{H}_{\mathbb{Q}}$ with $\mathcal{H}_{\mathbb{Q}}\epsilon_i \cong F_i$.

Since the image of $\mathcal{V}_{\mathbb{Q}}$ under $\Theta^{(n)}$ has rational Fourier-coefficients, the functions v and w are constant on the eigenspaces $E_i = \mathcal{V}\epsilon_i$ $(i = 1, 2, 3)$. We therefore give their values in one line in the following tabular:

Theorem 3.1 *The functions v and and the eigenvalues of ev_2 and ev_3 of K_2 respectively K_3 on $(d_1, \ldots, d_{24})$ are as follows:*

i	$v(i)$	ev_2	ev_3		i	$v(i)$	ev_2	ev_3
1	-1	34560	7176640		15	3, 4	1320	8640
2	0	16200	2389440		E_2	4	β_j	31680
3	1	8760	792000		19	3, 4, 5	1080	-45120
4	1	7128	804288		20	3, 4, 5	312	4032
E_1	2	α_j	266688		21	5	-216	8640
8	3	2664	90048		22	5	-216	20928
9	3	1320	77760		23	6	-936	13248
E_3	3	γ_j	100800		24	7	-2160	39360

For the dimensions of $\mathcal{D}_v$ one finds

v	-1	0	1	2	3	4	5	6	7
$\dim(\mathcal{D}_v)$	1	1	2	3	7-10	3-5	2-4	1	1

Proof. By explicit calculations of the Fourier-coefficients the values given in the table are upper bounds for the $v(i)$. By Lemma 2.4 they also provide upper bounds on the $w(i) = v(i) + 1$.

We see that

$$d_i \circ d_j = A_{ij} d_{24} + \sum_{m=1}^{23} b_{ij}^m d_m$$

with a nonzero coefficient A_{ij} for the following pairs (i, j):

$$(23, 2),\ (22, 3),\ (21, 4),\ (E_1, E_2),\ (E_3, E_3),\ (8, 8),\ (9, 9)$$

(where (E_1, E_2) means that there is some vector in E_1 and some in E_2 such that this coefficient is non-zero, similarly (E_3, E_3)). Since $d_m \in \mathcal{W}_7$ for all $m \leq 23$ and $d_j \circ d_i \in \mathcal{W}_{w(i)+w(j)}$ the inequality $w(i) + w(j) \leq 7$ together with $A_{ij} \neq 0$ implies that $d_{24} \in \mathcal{W}_7$ which is a contradiction. Hence $w(i) + w(j) \geq 8$ for all pairs (i, j) above. This yields equality for all values $v(i)$ and $v(j)$ for these pairs. Similarly we get $3 \leq v(i)$ for $i = 15, 19, 20$ since $A_{i,i} \neq 0$ for these i. q.e.d.

Conjecture 3.2 $v(19) = 5$ *and* $v(20) = 5$.

Since $d_{15} \circ d_2 = \sum_{m=1}^{18} c_m d_m + A_1 d_{19} + A_2 d_{20}$ with $A_1 \neq 0 \neq A_2$, we get $w(15) + 1 \geq \max(w(19), w(20))$.

Remark 3.3 *If the conjecture is true, then* $v(15) = 4$ *and* $\dim(\mathcal{D}_3) = 7$, $\dim(\mathcal{D}_4) = 4$, *and* $\dim(\mathcal{D}_5) = 4$.

Using the formula in [11, Korollar 3] (resp. [22, Proposition 1.9]) we may calculate the eigenvalues of $T^{(m-1)}(3^2)$ from the one of K_3 and compare them with the ones given in [6, formula (7)]. The result suggests that $\Theta^{(2)}(d_4)$, $\Theta^{(4)}(v)$ (for some $v \in E_3$), $\Theta^{(6)}(d_{19})$ and $\Theta^{(8)}(d_{24})$ are generalized Duke-Imamoglu-Ikeda-lifts (cf. [8]) of the elliptic cusp forms $\delta_8 \theta_{D_4}^i$ (i=3,2,1,0) where $\delta_8 = \frac{1}{96}(\theta_{D_4}^4 - \theta_{\Gamma_3})$ is the cusp form of $\Gamma_0(2)$ of weight 8 and θ_{D_4} the theta series of the 4-dimensional 2-modular root lattice D_4. This would imply that $v(19) = 5$ and, with Lemma 2.2, $v(15) = 4$.

3.2 The genus of the Coxeter-Todd lattice in dimension 12.

For $l = 3$ one has $h = 10$, all lattices in this genus are modular, and $\mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^{10} = \hat{\mathcal{H}}_{\mathbb{Q}}$

Theorem 3.4 *There is some* $a \in \{0, 1\}$ *such that the function* v *and the eigenvalues* ev_2 *of* K_2 *and* e_2 *of* $T(2)$ *are as follows:*

i	$v(i)$	ev_2	e_2		i	$v(i)$	ev_2	e_2
1	-1	2079	151470		6	$3-a$	234	7560
2	0	1026	-27540		7	3	126	2376
3	1	594	17820		8	3	-36	432
4	1	432	3240		9	4	-144	-864
5	2	288	-5400		10	5	-378	1944

For the dimensions of $\mathcal{D}_v$ *one finds*

v	-1	0	1	2	3	4	5
$\dim(\mathcal{D}_v)$	1	1	2	$1+a$	$3-a$	1	1

We conjecture that $a = 0$ but cannot prove this using Lemma 2.2.

The eigenvalues of $T(2)$ suggest that $\Theta^{(2)}(d_3)$, $\Theta^{(4)}(d_6)$ and $\Theta^{(6)}(d_{10})$ are generalized Duke-Imamoglu-Ikeda-lifts (cf. [8]) of the elliptic cusp forms $\delta_6\theta_{A_2}^2$, $\delta_6\theta_{A_2}$, respectively δ_6, where $\delta_6 = \frac{1}{36}(\theta_{A_2}^6 - \theta_{\Gamma_3})$ is the cusp form of $\Gamma_0(3)$ of weight 6 and θ_{A_2} the theta series of the hexagonal lattice A_2. This would imply $v(3) = 1$, $v(6) = 3$ and $v(10) = 5$ and hence $a = 0$.

3.3 The genus of the 5-modular lattices in dimension 8.

The class number of this genus is $h = 5$, all lattices in this genus are modular, and $\mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^5 = \hat{\mathcal{H}}_{\mathbb{Q}}$

Theorem 3.5 *For* $l = 5$ *one has* $\dim(\mathcal{D}_v) = 1$ *for* $v = -1, 0, 1, 2, 3$. *The function* v *and the eigenvalues* ev_2 *of* K_2 *and* e_p *of* $T(p)$ $(p = 2, 3)$ *are given in the following table:*

i	$v(i)$	ev_2	e_2	e_3		i	$v(i)$	ev_2	e_2	e_3
1	-1	135	270	2240		4	2	-8	-16	56
2	0	70	-120	160		5	3	-60	10	420
3	1	42	84	256						

3.4 The genus of the strongly 6-modular lattices in dimension 8.

The class number of $\mathcal{G}(\Gamma_6)$ is $h = 8$, the Hecke-algebras are $\hat{\mathcal{H}}_{\mathbb{Q}} = \langle K_2, W_2 \rangle \cong \mathbb{Q}^5 \oplus \mathrm{Mat}_3(\mathbb{Q})$ and $\mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^5 + \mathbb{Q}[x]/(f(x))$ where

$$f(x) = x^3 - 66x^2 - 216x + 31104.$$

Let $\delta_i \in \mathbb{R}$ $(i = 1, 2, 3)$ denote the roots of f.

Theorem 3.6 *Then the function* v *and the eigenvalues* ev_2 *of* K_2 *and* e_5 *of* $T(5)$ *are given in the following table:*

i	$v(i)$	ev_2	e_5		i	$v(i)$	ev_2	e_5
1	-1	144	39312		E	1	δ_j	1872
2	0	54	1872		7	2	-6	432
3	1	18	1008		8	3	-36	4752

Hence $\dim(\mathcal{D}_v) = 1$ *for* $v = -1, 0, 2, 3$ *and* $\dim(\mathcal{D}_1) = 4$.

3.5 The genus of the 7-modular lattices in dimension 6.

The class number is $h = 3$, all lattices are modular, and $\hat{\mathcal{H}}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^3$. In contrast to the other genera, the perestroika $\mathrm{Per}(\Gamma_7, 2)$ and hence also $\mathrm{BFW}(\Gamma_7, 2)$ vanishes due to the fact that the image of $\mathrm{Aut}(\Gamma_7)$ in $GO_6^+(2)$ is not contained in the derived subgroup $O_6^+(2)$. In fact, $\Theta^{(2)}$ is already injective. Since the discriminant of the space is not a square modulo 3 and 5, the Hecke operators $T(3)$ and $T(5)$ vanish.

Theorem 3.7 *We have* $v(i) = i - 2$ *for* $i = 1, 2, 3$ *and hence* $\dim(\mathcal{D}_v) = 1$ *for* $v = -1, 0, 1$. *The eigenvalues of* K_2 *are* 35, 19, *and* 5, *the ones of* $T(2)$ *are* 30, -18, *and* 10, *and* $T(11)$ *has eigenvalues* 2928, -144, *and* 248.

3.6 The genus of the strongly l-modular lattices in dimension 4 for $l = 11, 14, 15$.

For $l = 11, 14, 15$ the genus $\mathcal{G}(\Gamma_l)$ consists of 3 isometry classes and $\mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^3 = \hat{\mathcal{H}}_{\mathbb{Q}}$ since all lattices in the genus are strongly modular.

Theorem 3.8 *For $l = 11, 14, 15$ one has $\dim(\mathcal{D}_v) = 1$ for $v = -1, 0, 1$. The eigenvalues ev_2 of K_2 and e_p of $T(p)$ for primes $p \leq 7$ not dividing l are given in the following table:*

		$l = 11$					$l = 14$			$l = 15$		
i	$v(i)$	ev_2	e_2	e_3	e_5	e_7	ev_2	e_3	e_5	ev_2	e_2	e_7
1	-1	9	6	8	12	16	8	8	12	9	6	16
2	0	4	-4	-2	2	-4	2	-4	0	1	-2	0
3	1	-6	1	3	7	6	-4	2	6	-3	2	8

3.7 The genus of the 23-modular lattices in dimension 2.

In the smallest possible dimension 2 the genus $\mathcal{G}(\Gamma_{23})$ consists of only 2 isometry classes and $\mathcal{H}_{\mathbb{Q}} = \langle K_2 \rangle \cong \mathbb{Q}^2 = \hat{\mathcal{H}}_{\mathbb{Q}}$ for the same argument that all lattices in the genus are modular.

Theorem 3.9 *For $l = 23$ one has $\dim(\mathcal{D}_v) = 1$ for $v = -1, 0$. One has $v(1) = -1$, $v(2) = 0$, $d_1 K_2 = 2d_1$ and $d_2 K_2 = -d_2$. For the $T(p)$ for primes $p < 23$ we find $T(2) = T(3) = T(13) = K_2$ and $T(5) = T(7) = T(11) = T(17) = T(19) = 0$.*

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