Lattices and Codes, analogies and interactions.

or

Codes, Invariants and Modular Forms, a conclusion.

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Codes.

R finite ring, A finite left R-module $C \leq A^N$ is called a **code**,
N its **length** and **codewords** $c = (c_1, \ldots, c_N)$ are rows.
The **codepolynomial** of C is

$$p_C := \sum_{c \in C} \prod_{i=1}^{N} x_{c_i} \in \mathbb{C}[x_a \mid a \in A]_N = p_C^{(1)}$$

The **genus m codepolynomial** of C is

$$p_C^{(m)} := \sum_{(c^{(1)},\ldots,c^{(m)}) \in C^m} \prod_{v \in A^m} x_v^{a_v(c^{(1)},\ldots,c^{(m)})} \in \mathbb{C}[x_v : v \in A^m]\,.$$

where

$$a_v(c^{(1)}, \ldots, c^{(m)}) := |\{j \in \{1, \ldots, N\} \mid c_j^{(i)} = v_i \text{ for all } i \in \{1, \ldots, m\}\}|$$

for $v := (v_1, \ldots, v_m) \in A^m$.

For $C \leq A^N$ and $m \in \mathbb{N}$ let

$$C(m) := R^{m\times 1} \otimes C = \{(c^{(1)}, \ldots, c^{(m)})^{\mathsf{Tr}} \mid c^{(1)}, \ldots, c^{(m)} \in C\} \leq (A^m)^N$$

Then

$$p_C^{(m)} = p_{C(m)}.$$

A typical element of $C(m)$ is a matrix in $A^{m\times N}$, where the rows are codewords in C.

$$\begin{array}{cccccc}
c_1^{(1)} & c_2^{(1)} & \ldots & c_j^{(1)} & \ldots & c_N^{(1)} \\
c_1^{(2)} & c_2^{(2)} & \ldots & c_j^{(2)} & \ldots & c_N^{(2)} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
c_1^{(m)} & c_2^{(m)} & \ldots & c_j^{(m)} & \ldots & c_N^{(m)} \\
 & & & \uparrow & & \\
 & & & v \in A^m & &
\end{array}$$

The finite Siegel Φ-operator. (B. Runge, 1995)

$$\Phi_m : p_C^{(m)} \mapsto p_C^{(m-1)}$$

is given by the variable substitution:

$$x_{(v_1,\ldots,v_m)} \mapsto \begin{cases} x_{(v_1,\ldots,v_{m-1})} & \text{if } v_m = 0 \\ 0 & \text{else} \end{cases}$$

$p_C^{(m-1)}$ is obtained from $p_C^{(m)}$ by counting only those matrices

$$\begin{array}{ccccccc}
c_1^{(1)} & c_2^{(1)} & \ldots & c_j^{(1)} & \ldots & c_N^{(1)} \\
c_1^{(2)} & c_2^{(2)} & \ldots & c_j^{(2)} & \ldots & c_N^{(2)} \\
\vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\
c_1^{(m-1)} & c_2^{(m-1)} & \ldots & c_j^{(m-1)} & \ldots & c_N^{(m-1)} \\
0 & 0 & \ldots & 0 & \ldots & 0 \\
 & & & \uparrow & & \\
 & & & v \in A^m & &
\end{array}$$

in which the last row is zero.

Lattices and Theta Series.

$L \leq (\mathbb{R}^N, (,))$ a lattice in Euclidean N-space.

The **theta series** of L is

$$\vartheta_L(z) = \sum_{\ell \in L} q^{(\ell,\ell)}$$

where $q = \exp(\pi i z)$.

The **genus m Siegel theta series** of L is

$$\vartheta_L^{(m)}(Z) = \sum_{\underline{\ell} \in L^m} \exp(\pi i \operatorname{Tr}(Z(\underline{\ell}, \underline{\ell}))).$$

The **Siegel Φ-operator** maps $\vartheta_L^{(m)}$ to $\vartheta_L^{(m-1)}$.

Codes and Lattices: Construction A.

Let p be a prime and $(b_1, \ldots, b_N)$ be a basis of $\mathbb{R}^N$ such that

$$(b_i, b_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1/p & \text{if } i = j \end{cases}$$

Let $C \leq \mathbb{F}_p^N = \mathbb{Z}^N/p\mathbb{Z}^N$ be a code. Then the **codelattice** L_C is

$$L_C := \{\sum_{i=1}^{N} a_i b_i \mid (a_1 \pmod{p}, \ldots, a_N \pmod{p}) \in C\}$$

Remark.

(a) $L_C^* = L_{C^\perp}$, so L_C is unimodular, if C is self-dual.

(b) L_C is even unimodular, if $p = 2$ and C is a Type II code.

(c) $\vartheta_{L_C} = p_C(\vartheta_0, \ldots, \vartheta_{p-1})$ where

$$\vartheta_a = \vartheta_{(a+p\mathbb{Z})b_1} = \sum_{n=-\infty}^{\infty} q^{(a+pn)^2/p}$$

similarly for higher genus theta series and codepolynomials.

Theta series are Modular Forms.

If $L = L^*$ and $(\ell,\ell) \in 2\mathbb{Z}$ for all $\ell \in L$, **even unimodular lattice**, then

$$\vartheta_L^{(m)}(Z) = \sum_{\underline{\ell} \in L^m} \exp(\pi i \,\mathrm{Tr}(Z(\underline{\ell},\underline{\ell}))) \in \mathcal{M}_{N/2}(\mathrm{Sp}_{2m}(\mathbb{Z}))$$

where

$$\mathrm{Sp}_{2m}(\mathbb{Z}) = \langle \begin{pmatrix} A & 0 \\ 0 & A^{-tr} \end{pmatrix}, \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix}, \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

$$\mid A \in \mathrm{GL}_m(\mathbb{Z}), B = B^{tr} \in \mathbb{Z}^{m \times m} \rangle.$$

Codepolynomials are Invariants.

R finite ring, A finite left R-module, $\beta : A \times A \to \mathbb{Q}/\mathbb{Z}$ regular.
For $C \leq A^N$ the dual code is

$$C^{\perp} := \{v \in A^N \mid \sum_{i=1}^{N} \beta(v_i, c_i) = 0 \text{ for all } c \in C\}.$$

Let $M := \{\beta^r : (x, y) \mapsto \beta(x, ry) \mid r \in R\}$ and assume that $M \cong R_R$ and is closed under interchanging arguments.
Additional quadratic conditions are given by a subgroup $Q \leq (\mathbb{Q}/\mathbb{Z})^A$, such that:

- For all $\varphi \in Q$, $\lambda(\varphi) : (x, y) \mapsto \varphi(x + y) - \varphi(x) - \varphi(y) \in M$.
- For all $r \in R$, $\varphi \in Q$, $\varphi[r] : x \mapsto \varphi(rx) \in Q$.
- For all $r \in R$, $\{\!\{ \beta^r \}\!\} : x \mapsto \beta(x, rx) \in Q$.

Then (R, A, β, Q) is called a **Type**.

C is called a **Type** T **code**, if
a) $C \leq A^N$ is an R-module.
b) $\sum_{i=1}^{N} \varphi(c_i) = 0$ for all $\varphi \in Q$, $c \in C$ (**isotropic**).
c) $C = C^{\perp}$ (**self-dual**)

Examples.

Type I codes (2_{I})

$$R = \mathbb{F}_2 = A,\ \beta(x,y) = \frac{1}{2}xy,\ Q = \{\varphi : x \mapsto \frac{1}{2}x^2 = \beta(x,x), 0\}$$

Type II code (2_{II}).

$$R = \mathbb{F}_2 = A,\ \beta(x,y) = \frac{1}{2}xy,\ Q = \{\phi : x \mapsto \frac{1}{4}x^2, 2\phi = \varphi, 3\phi, 0\}$$

Type IV codes (4^H).

$$R = \mathbb{F}_4 = A,\ \beta(x,y) = \frac{1}{2}\mathrm{trace}(x\overline{y}),\ Q = \{\varphi : x \mapsto \frac{1}{2}x\overline{x}, 0\}$$

where $\overline{x} = x^2$.

Additive codes over $\mathbb{F}_4$. (4^{H+})

$$R = \mathbb{F}_2,\ A = \mathbb{F}_4,\ \beta(x,y) = \frac{1}{2}\mathrm{trace}(x\overline{y}),\ Q = \{\varphi : x \mapsto \frac{1}{2}x\overline{x}, 0\}$$

Clifford-Weil groups.

Let $T := (R, A, \beta, Q)$ be a Type. Then the **associated Clifford-Weil group** $\mathcal{C}(T)$ is a subgroup of $\mathrm{GL}_{|A|}(\mathbb{C})$

$\mathcal{C}(T) = \langle m_r, d_\varphi, h_{e,u_e,v_e} \mid r \in R^*, \varphi \in Q, e = u_e v_e \in R$ symmetric idempotent $\rangle$

Let $(x_a | a \in A)$ denote a basis of $\mathbb{C}^{|A|}$. Then

$$m_r : x_a \mapsto x_{ra}, \quad d_\varphi : x_a \mapsto \exp(2\pi i \varphi(a)) x_a$$

$$h_{e,u_e,v_e} : x_a \mapsto |eA|^{-1/2} \sum_{b \in eA} \exp(2\pi i \beta(b, v_e a)) x_{b+(1-e)a}$$

Similarly the **genus m Clifford-Weil group**

$\mathcal{C}_m(T) = \langle m_r, d_\varphi, h_{e,u_e,v_e} \mid r \in \mathrm{GL}_m(R), \varphi \in Q^{(m)}, e = u_e v_e \in R^{m \times m}$ sym. id. $\rangle$

$$\leq \mathrm{GL}_{|A|^m}(\mathbb{C})$$

$$m_r : x_a \mapsto x_{ra}, \quad d_\varphi : x_a \mapsto \exp(2\pi i \varphi(a)) x_a$$

$$h_{e,u_e,v_e} : x_a \mapsto |eA|^{-1/2} \sum_{b \in eA} \exp(2\pi i \beta(b, v_e a)) x_{b+(1-e)a}$$

Theorem.
Let $C \leq A^N$ be a self-dual isotropic code of Type T. Then $p_C^{(m)}$ is invariant under $\mathcal{C}_m(T)$.
Proof.
Invariance under m_r ($r \in \mathrm{GL}_m(R)$) because C is a code.
Invariance under d_φ ($\varphi \in Q^{(m)}$) because C is isotropic.
Invariance under h_{e,u_e,v_e} because C is self dual.

The main theorem.(N, Rains, Sloane (1999-2006))
If R is a direct product of matrix rings over chain rings, then

$$\mathrm{Inv}(\mathcal{C}_m(T)) = \langle p_C^{(m)} \mid C \text{ of Type } T \rangle.$$

Example: $\mathcal{C}_2(\mathrm{II})$.

$$R = \mathbb{F}_2^{2\times 2}, R^* = \mathsf{GL}_2(\mathbb{F}_2) = \langle a := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ b := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

$A = \mathbb{F}_2^2 = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \}$, symmetric idempotent $e = \mathrm{diag}(1,0)$

$$\mathcal{C}_2(\mathrm{II}) \quad = \langle m_a = \begin{pmatrix} 1000 \\ 0010 \\ 0100 \\ 0001 \end{pmatrix},\ m_b = \begin{pmatrix} 1000 \\ 0001 \\ 0100 \\ 0010 \end{pmatrix},$$

$$h_{e,e,e} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\,1\,0\,0 \\ 1\text{-}1 0\,0 \\ 0\,0\,1\,1 \\ 0\,0\,1\text{-}1 \end{pmatrix},\ d_{\phi e} = \mathrm{diag}(1,i,1,i)\rangle.$$

$\mathcal{C}_2(\mathrm{II})$ has order 92160 and Molien series

$$\frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})}$$

where the generators correspond to the genus 2 codepolynomials of the codes:

$$e_8, g_{24}, d_{24}^+, d_{40}^+, \text{ and } d_{32}^+$$

$\mathcal{C}_2(\mathrm{II})$ has a reflection subgroup of index 2, No. 31 on the Shephard-Todd list.

Higher genus Clifford-Weil groups for the classical Types of codes over finite fields.

$$\mathcal{C}_m(T) = S.(\mathsf{ker}(\lambda) \times \mathsf{ker}(\lambda)).\mathcal{G}_m(T)$$

$\lambda(\varphi) : (x, y) \mapsto \varphi(x + y) - \varphi(x) - \varphi(y)$

R	J	ϵ	$\mathcal{G}_m(T)$
$\mathbb{F}_q \oplus \mathbb{F}_q$	$(r,s)^J = (s,r)$	1	$\mathsf{GL}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_{q^2}$	$r^J = r^q$	1	$U_{2m}(\mathbb{F}_{q^2})$
$\mathbb{F}_q$, q odd	$r^J = r$	1	$\mathsf{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q$, q odd	$r^J = r$	-1	$O^+_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q$, q even	doubly even		$\mathsf{Sp}_{2m}(\mathbb{F}_q)$
$\mathbb{F}_q$, q even	singly even		$O^+_{2m}(\mathbb{F}_q)$

Hecke operators for codes.

Motivation.

Determine linear relations between $p_C^{(m)}$ for $C \in M_N(T) = \{C = C^{\perp} \leq A^N \mid C$ isotropic $\}$.

$M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$ and these two codes have the same genus 1 and 2 codepolynomials, but $p^{(3)}(e_8 \perp e_8)$ and $p^{(3)}(d_{16}^+)$ are linearly independent.

$h(M_{24}(\mathrm{II})) = 9$ and only the genus 6 codepolynomials are linearly independent, there is one relation for the genus 5 codepolynomials.

$h(M_{32}(\mathrm{II})) = 85$ and here the genus 10 codepolynomials are linearly independent, whereas there is a unique relation for the genus 9 codepolynomials.

Three different approaches:

1) Determine all the codes and their codepolynomials.
If $\dim(C) = n = N/2$ there are $\prod_{i=0}^{d-1}(2^n - 2^i)/(2^d - 2^i)$ subspaces of dimension d in C.
$N = 32, d = 10$ yields more than 10^{18} subspaces.

2) Use Molien's theorem:
$\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})) = \langle p_C^{(m)} \mid C \in M_N(\mathrm{II})\rangle$
and if $a_N := \dim(\mathrm{Inv}_N(\mathcal{C}_m(\mathrm{II})))$ then

$$\sum_{N=0}^{\infty} a_N t^N = \frac{1}{|\mathcal{C}_m(\mathrm{II})|} \sum_{g\in\mathcal{C}_m(\mathrm{II})} (\det(1 - tg))^{-1}$$

Problem: $\mathcal{C}_{10}(\mathrm{II}) \leq \mathrm{GL}_{1024}(\mathbb{C})$ has order $> 10^{69}$.

3) Use Hecke operators.

Fix a Type $T = (\mathbb{F}_q, \mathbb{F}_q, \beta, Q)$ of self-dual codes over a finite **field** with q elements.

$$M_N(T) = \{C = C^\perp \leq \mathbb{F}_q^N \mid C \text{ isotropic } \} = [C_1] \dot\cup \ldots \dot\cup [C_h]$$

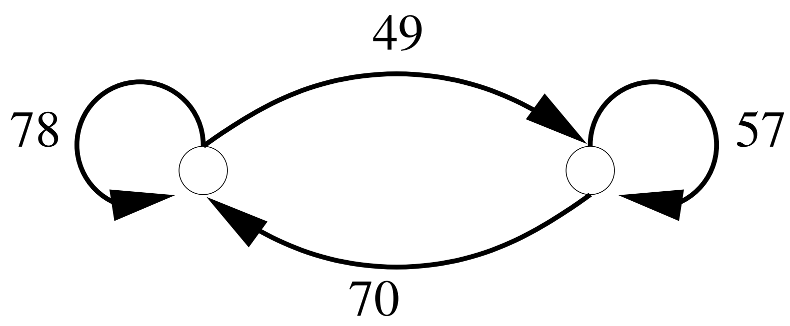
where $[C]$ denotes the **permutation equivalence** class of the code C. Then $n := \frac{N}{2} = \dim(C)$ for all $C \in M_N(T)$.
$C, D \in M_N(T)$ are called **neighbours**, if $\dim(C) - \dim(C \cap D) = 1$, $C \sim D$.

$$\mathcal{V} = \mathbb{C}[C_1] \oplus \ldots \oplus \mathbb{C}[C_h] \cong \mathbb{C}^h$$

$$K_N(T) \in \mathrm{End}(\mathcal{V}),\ K_N(T) : [C] \mapsto \sum_{D \in M_N(T), D \sim C} [D].$$

Kneser-Hecke operator.
(adjacency matrix of neighbouring graph)

Example. $M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^{+}]$



$$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

$\mathcal{V}$ has a Hermitian positive definite inner product defined by

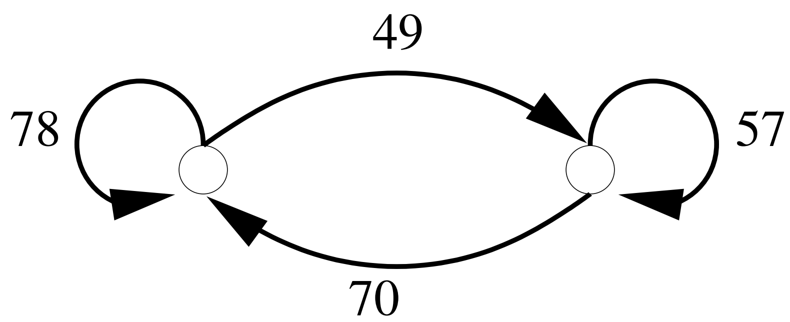
$$\langle [C_i], [C_j] \rangle := |\operatorname{Aut}(C_i)| \delta_{ij}.$$

Theorem. (N. 2006)
The Kneser-Hecke operator K is a self-adjoint linear operator.

$$\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.$$

Example. $\frac{7}{10} = \frac{|\operatorname{Aut}(e_8 \perp e_8)|}{|\operatorname{Aut}(d_{16}^+)|} = \frac{49}{70}$ hence

$$\operatorname{diag}(7, 10) K_{16}(\mathrm{II})^{\mathrm{Tr}} = K_{16}(\mathrm{II}) \operatorname{diag}(7, 10).$$



$$p^{(m)} : \mathcal{V} \to \mathbb{C}[X], \sum_{i=1}^{h} a_i[C_i] \mapsto \sum_{i=1}^{h} a_i p_{C_i}^{(m)}$$

is a linear mapping with kernel

$$\mathcal{V}_m := \mathrm{ker}(p^{(m)}).$$

Then

$$\mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_n = \{0\}.$$

is a filtration of $\mathcal{V}$ yielding the orthogonal decomposition

$$\mathcal{V} = \bigoplus_{m=0}^{n} \mathcal{Y}_m \text{ where } \mathcal{Y}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^{\perp}.$$

$$\mathcal{V}_0 = \{\sum_{i=1}^{h} a_i[C_i] \mid \sum_{i=1}^{h} a_i = 0\}$$

and

$$\mathcal{V}_0^{\perp} = \mathcal{Y}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\mathrm{Aut}(C_i)|}[C_i] \rangle.$$

Theorem. (N. 2006)
The space $\mathcal{Y}_m = \mathcal{Y}_m(N)$ is the $K_N(T)$-eigenspace to the eigenvalue $\nu_N^{(m)}(T)$ with $\nu_N^{(m)}(T) > \nu_N^{(m+1)}(T)$ for all m.

Type	$\nu_N^{(m)}(T)$
q_{I}^E	$(q^{n-m} - q - q^m + 1)/(q-1)$
q_{II}^E	$(q^{n-m-1} - q^m)/(q-1)$
q^E	$(q^{n-m} - q^m)/(q-1)$
q_1^E	$(q^{n-m-1} - q^m)/(q-1)$
q^H	$(q^{n-m+1/2} - q^m - q^{1/2} + 1)/(q-1)$
q_1^H	$(q^{n-m-1/2} - q^m - q^{1/2} + 1)/(q-1)$

Corollary. The neighbouring graph is connected.

Proof. The maximal eigenvalue ν_0 of the adjacency matrix is simple with eigenspace $\mathcal{Y}_0$.

Example: $M_{16}(\mathrm{II}) = [e_8 \perp e_8] \cup [d_{16}^+]$
$(2^{8-m-1} - 2^m : m = 0,1,2,3) = (127, 62, 28, 8)$

$$K_{16}(\mathrm{II}) = \begin{pmatrix} 78 & 49 \\ 70 & 57 \end{pmatrix}$$

has eigenvalues 127 and 8 with eigenvectors $(7, 10)$ and $(1, -1)$.
Hence

$$\mathcal{Y}_0 = \langle 7[e_8 \perp e_8] + 10[d_{16}^+] \rangle$$

$$\mathcal{Y}_1 = \mathcal{Y}_2 = 0$$

$$\mathcal{Y}_3 = \langle [e_8 \perp e_8] - [d_{16}^+] \rangle.$$

Even unimodular lattices.

$$\mathcal{L}_N = \{L = L^* \leq \mathbb{R}^N \mid L \text{ even } \} = [L_1] \,\dot\cup\, \ldots \,\dot\cup\, [L_h]$$

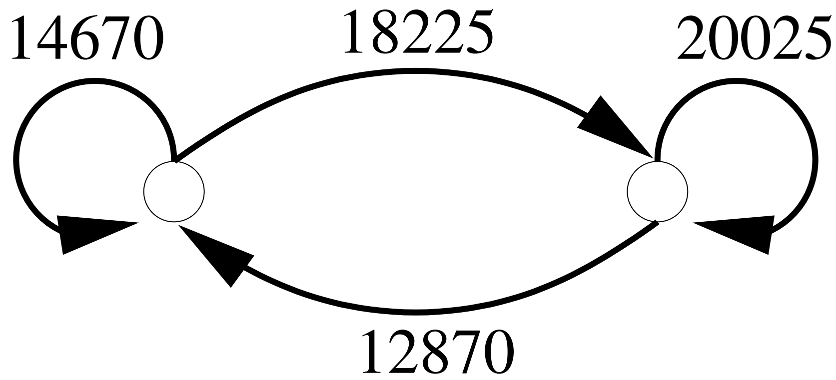
where $[L]$ denotes the **isometry** class of the lattice L.
$L, M \in \mathcal{L}_N$ are called **p-neighbours**, if $[L : L \cap M] = p$,
notation: $L \sim M$.

$$\mathcal{V} = \mathbb{C}[L_1] \oplus \ldots \oplus \mathbb{C}[L_h] \cong \mathbb{C}^h$$

$$K_{N/2}(p) \in \mathrm{End}(\mathcal{V}),\ K_{N/2}(p) : [L] \mapsto \sum_{M \in \mathcal{L}_N, M \sim L} [M].$$

Kneser-Hecke operator.
(adjacency matrix of neighbouring graph)

Example. $\mathcal{L}_{16} = [E_8 \perp E_8] \cup [D_{16}^+]$



$$K_8(2) = \begin{pmatrix} 14670 & 18225 \\ 12870 & 20025 \end{pmatrix}$$

$\mathcal{V}$ has a Hermitian positive definite inner product defined by

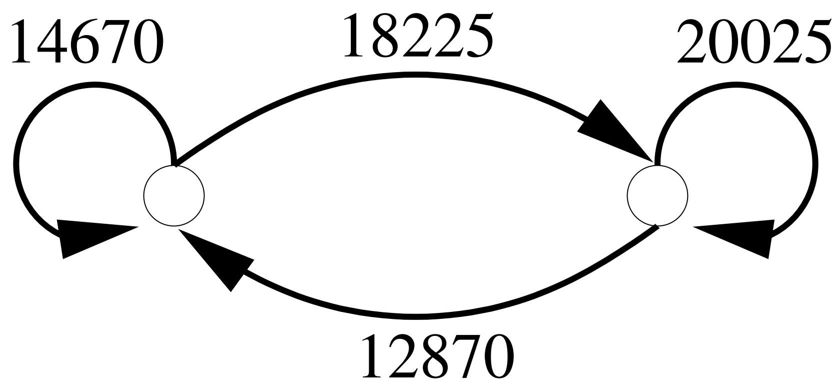
$$\langle [L_i], [L_j] \rangle := |\operatorname{Aut}(L_i)| \delta_{ij}.$$

Theorem. (Venkov, N. 2001)
The Kneser-Hecke operator K is a self-adjoint linear operator.

$$\langle v, Kw \rangle = \langle Kv, w \rangle \text{ for all } v, w \in \mathcal{V}.$$

Example. $\frac{405}{286} = \frac{|\operatorname{Aut}(E_8 \perp E_8)|}{|\operatorname{Aut}(D_{16}^+)|} = \frac{18225}{12870}$ hence

$$\operatorname{diag}(405, 286) K_8(2)^{\mathrm{Tr}} = K_8(2) \operatorname{diag}(405, 286).$$



$$\vartheta^{(m)} : \mathcal{V} \to \mathcal{M}_{N/2}(\mathsf{Sp}_{2m}(\mathbb{Z})), \sum_{i=1}^{h} a_i[L_i] \mapsto \sum_{i=1}^{h} a_i \vartheta_{L_i}^{(m)}$$

is a linear mapping with kernel

$$\mathcal{V}_m := \ker(\vartheta^{(m)}).$$

Then

$$\mathcal{V} =: \mathcal{V}_{-1} \geq \mathcal{V}_0 \geq \mathcal{V}_1 \geq \ldots \geq \mathcal{V}_N = \{0\}.$$

is a filtration of $\mathcal{V}$ yielding the orthogonal decomposition

$$\star_L \quad \mathcal{V} = \bigoplus_{m=0}^{N} \mathcal{Y}_m \text{ where } \mathcal{Y}_m = \mathcal{V}_{m-1} \cap \mathcal{V}_m^{\perp}.$$

$$\mathcal{V}_0 = \{\sum_{i=1}^{h} a_i[L_i] \mid \sum a_i = 0\} \text{ and } \mathcal{V}_0^{\perp} = \mathcal{Y}_0 = \langle \sum_{i=1}^{h} \frac{1}{|\operatorname{Aut}(L_i)|}[L_i] \rangle.$$

Theorem. $\star_L$ is invariant under $K_{N/2}(p)$
(but the eigenspace decomposition is usually much finer and I do not know how to predict eigenvalues).

Example: $\mathcal{L}_{16} = [E_8 \perp E_8] \cup [D_{16}^{+}]$

$$K_8(2) = \begin{pmatrix} 14670 & 18225 \\ 12870 & 20025 \end{pmatrix}$$

has eigenvalues 32895 and 1800 with eigenvectors $(286, 405)$ and $(1, -1)$.
Here $\vartheta^{(m)} = p^{(m)}(\vartheta_a : a \in \mathbb{F}_2^m)$ and all lattices come from codes.

$$\mathcal{Y}_0 = \langle 286[E_8 \perp E_8] + 405[D_{16}^{+}] \rangle$$

$$\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3 = 0$$

$$\mathcal{Y}_4 = \langle [E_8 \perp E_8] - [D_{16}^{+}] \rangle.$$

Dimension 24: The 24 Niemeier lattices. (N, Venkov)

Here $h = 24$ and only 9 of the lattices are codelattices. With B. Venkov we calculated $K_{12}(2)$ and its eigenspace decomposition.

j	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim(\mathcal{Y}_j)$	1	1	1	1	2	2	3	3*	4*	2*	2*	1	1

$*$ means that the dimension is only conjectured.

Hecke operators as double cosets. (Lattices.)

$$t_m^{(m-1)}(p^2) := \mathrm{diag}(1, \underbrace{p, \ldots, p}_{m-1}, p^2, \underbrace{p, \ldots, p}_{m-1}) \in \mathrm{GSp}_{2m}(\mathbb{Z})$$

Then the double coset

$$T_m^{(m-1)}(p^2) := \mathrm{Sp}_{2m}(\mathbb{Z}) t_m^{(m-1)}(p^2)\, \mathrm{Sp}_{2m}(\mathbb{Z}) = \dot\cup_{j=1}^{d} \mathrm{Sp}_{2m}(\mathbb{Z})\gamma_j$$

acts on the space of modular forms $\mathcal{M}_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ and also on the subspace spanned by theta series by

$$\delta_k(T_m^{(m-1)}(p^2)) : f \mapsto \sum_{j=1}^{d} f_{|_k\gamma_j}$$

The Kneser-Hecke operator $K_k(p)$ also acts on this space via $\Delta_m(K_k(p))$.

Theorem. (Yoshida 1985) There are explicit constants $c = c(m, k, p), d = d(m, k, p)$ such that

$$\delta_k(T_m^{(m-1)}(p^2)) = c\, \mathrm{id} + d\Delta_m(K_k(p)).$$

Hecke operators as double cosets. (Codes.)

Let (R, A, β, Q) be a Type.

The associated extraspecial group

$$\begin{array}{ll} \mathcal{E}_m := (A^m \times A^m) \rtimes \mathbb{Q}/\mathbb{Z}, & \text{with multiplication} \\ (a,b,q)(a',b',q') = & (a+a', b+b', q+q'+\beta(b',a)) \end{array}$$

acts irreducibly on $\mathbb{C}[A^m] = \langle x_v : v \in A^m \rangle_{\mathbb{C}}$ via

$$(a,b,q)x_v := \exp(2\pi i(q + \beta(v,a)))x_{v+b}$$

Remark. The associated Clifford-Weil group $\mathcal{C}_m \leq \mathrm{GL}(\mathbb{C}[A^m])$ normalizes $\mathcal{E}_m$.

$\mathcal{U}_j := \{(a,0,0) \mid a = (0^{m-j}, a_1, \ldots, a_j) \in A^m\} \leq \mathcal{E}_m$ and $\mathcal{T}_j = \mathcal{C}_m p_{\mathcal{U}_j} \mathcal{C}_m$ where for $U \leq \mathcal{E}_m$ the endomorphism

$$p_U := \frac{1}{|U|} \sum_{u \in U} u$$

denotes the orthogonal projection onto the fixed space of U. Note that $p_U = 0$ if $U \cap Z \neq \{(0,0,0)\}$ where $Z = \{(0,0,q) \mid q \in \mathbb{Q}/\mathbb{Z}\} = Z(\mathcal{E}_m)$.

Theorem. (N. 2006) If $A = R = \mathbb{F}_q$ is a finite field, then

$$\mathcal{H}(\mathcal{C}_m) = \langle \mathcal{T}_j \mid 0 \leq j \leq m \rangle_{\mathbb{C}-algebra} = \mathbb{C}[\mathcal{T}_1]$$

is a commutative subalgebra of End(Inv($\mathcal{C}_m$)) consisting of self-adjoint linear operators acting on the subspace of degree N invariants via, say, δ_N.
Then there are explicit constants c, d
(depending on q, the Type T, the genus m and the length N)
such that

$$\delta_N(\mathcal{T}_1) = c\,\mathrm{id} + d\Delta_m(K_N(T)).$$