A NOTE ON INVARIANT TRANSVERSALS FOR NORMAL SUBGROUPS

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Abstract. The existence of invariant transversals for a normal subgroup H in a group G is investigated. This yields counterexamples to a conjecture in case H is abelian and G is finite.

Dedicated to the memory of my Doktorvater Otto Kegel

1. Introduction

Let G be a group and let $H, L \leq G$ be subgroups of G. Let $H\backslash G := \{Hg \mid g \in G\}$ denote the set of right H-cosets in G. A transversal for $H\backslash G$ is a set of representatives. A subset S of G is called L-invariant, if S is invariant under conjugation by L or, in other words, if S is a union of L-conjugacy classes of G. If there exists an L-invariant transversal for $H\backslash G$, we say that H admits an L-invariant transversal. A G-invariant transversal for $H\backslash G$ is, in particular, a loop transversal, i.e., a transversal for the cosets of $g^{-1}Hg\backslash G$ for all $g \in G$. Loop transversals carry the structure of a loop.

In [6], the authors put forward the following conjecture, where G' denotes the commutator subgroup of G.

Conjecture 1.1. *Let G be finite and let $H \leq G$ be abelian admitting a G-invariant transversal. Then $H \cap G' = \{1\}$.*

This conjecture, for which the author of this note takes full responsibility, already appears in Ortjohann's Master thesis [11]. What was the motivation for this conjecture? Firstly, this was the observation that if $H \cap G' = \{1\}$ (in which case H is necessarily abelian), then any transversal S of $HG'\backslash G$ yields the G-invariant transversal $T = \{gs \mid g \in G', s \in S\}$ for $H\backslash G$. Thus the truth of Conjecture 1.1 would have simplified the classification of the instances (G, H, T) with H abelian

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and T a G-invariant transversal for $H\backslash G$. Secondly, Artic in her PhD-thesis [1] determined the pairs (G, H) with G finite, $[G\colon H] \leq 30$ and core-free H, admitting a G-invariant transversal. All of Artic's examples satisfy Conjecture 1.1. In her Master thesis [11], Ortjohann verified the conjecture for all groups G of order at most 40. Finally, [6, Theorem 3.1] reproduces a positive result of Kochendörffer and Zappa, in case H is an abelian Hall subgroup of G.

The main purpose of this note is to exhibit counterexamples to Conjecture 1.1. We came across these in our attempt to prove the conjecture for abelian normal subgroups H. There is a second reason for looking at normal subgroups. Suppose that T is a G-invariant transversal for $H\backslash G$ and that L is a subgroup of G with $H \leq L$. Then $T \cap L$ is an L-invariant transversal for $H\backslash L$. This reasoning, which does not assume that H is abelian, applies in particular when L is the normalizer of H in G.

In Section 2, not assuming that the groups are finite, we give a straightforward criterion for the existence of invariant transversals for normal subgroups. In Section 3 we discuss finite groups, where the conjecture leads to a question on central extensions and corresponding projective representations. Fortunately, this question had been answered a long time ago. The answer yields counterexamples to Conjecture 1.1. Gunter Malle called the attention of the author to the results in [9] and, via his survey article [10], to earlier results [2] by Blau. The latter leads to two non-solvable counterexamples.

2. Characterizations

In this section G is a group and $H \trianglelefteq G$ is a normal subgroup of G. Our notation is standard. In particular, $Z(G)$ and G' denote, respectively, the center of G and the commutator subgroup of G. For $g, x \in G$, we put $g^x := x^{-1}gx$ and $[g, x] := g^{-1}x^{-1}gx$. Also, $C_G(x)$ and $C_G(L)$ denote the centralizer in G of $x \in G$, respectively the subgroup $L \leq G$. Since $H \trianglelefteq G$, we have $H\backslash G = G/H$, and we write $\pi\colon G \to G/H$ for the canonical epimorphism.

Lemma 2.1. *Let $H \trianglelefteq G$ and let T be an H-invariant transversal for G/H. Then $\langle T\rangle \leq C_G(H)$ and $G = H\langle T\rangle = HC_G(H)$.*

Proof. Since $H \trianglelefteq G$, we have $Hg = Hg^h$ for all $g \in G$ and $h \in H$. Since T is H-invariant, this yields $t = t^h$ for all $t \in T$ and all $h \in H$, and thus our first assertion.

From this and $G = \langle H, T\rangle$, we obtain $G = H\langle T\rangle = HC_G(H)$. □

Corollary 2.2. *Suppose that H is abelian and that $H \trianglelefteq G$. Then there exists an H-invariant transversal for G/H, if and only if $H \leq Z(G)$.*

Proof. Suppose first that there exists an H-invariant transversal for G/H. Since H is abelian, $H \leq C_G(H)$ and so $C_G(H) = HC_G(H) = G$ by Lemma 2.1. Thus $H \leq Z(G)$.

Conversely, if $H \leq Z(G)$, every transversal for G/H is H-invariant. This concludes the proof. □

Let $T \subseteq G$. By definition, T is a transversal for G/H, if and only if π restricts to a bijection $\pi|_T \colon T \to G/H$. If T is a G-invariant transversal for G/H, then $\pi|_T$ induces a bijection between the conjugacy classes of G contained in T and the conjugacy classes of G/H. The existence of G-invariant transversals for normal subgroups is characterized by the following result.

Proposition 2.3. *Let $H \trianglelefteq G$. There is a G-invariant transversal for G/H, if and only if*

$$G = HC_G(H) \tag{1}$$

and

$$C_{G/H}(Hg) = HC_G(g)/H \qquad \textit{for all } g \in G. \tag{2}$$

Proof. Suppose first that T is a G-invariant transversal for G/H. Then $G = HC_G(H)$ by Lemma 2.1, so (1) holds.

To prove (2), let $g \in G$. It suffices to show that $C_{G/H}(Hg) \leq HC_G(g)/H$. For this purpose, let $x \in G$ with $[x, g] \in H$. Write $g = ht$ and $x = h't'$ with $h, h' \in H$ and $t, t' \in T$. Since $t, t' \in C_G(H)$, once more by Lemma 2.1, we obtain $[x, g] = [t', t][h', h]$, and so $[t', t] \in H$. It follows that $Ht = tH = t^{t'}H = Ht^{t'}$. Since T is G-invariant, we conclude that t' centralizes t, and thus g. Hence $x \in HC_G(g)$, as claimed.

Now suppose that (1) and (2) are satisfied. For $g \in G$, let C_g denote the G-conjugacy class of g. Choose a transversal S for G/H with $S \subseteq C_G(H)$. For $s \in S$ we have $H \leq C_G(s)$, and thus $\pi(C_G(s)) = C_{G/H}(\pi(s))$ from (2). It follows that π restricts to a bijection between C_s and the G/H-conjugacy class of $\pi(s)$.

Let $\overline{\mathcal{C}}$ denote the set of conjugacy classes of G/H. For every $\bar{C} \in \overline{\mathcal{C}}$, choose $s(\bar{C}) \in S$ with $\pi(s(\bar{C})) \in \bar{C}$. Then

$$T := \bigcup_{\bar{C} \in \overline{\mathcal{C}}} C_{s(\bar{C})}$$

is G-invariant. Moreover, T is a transversal for G/H, since π maps T bijectively onto G/H. □

The next results prepare for a reduction.

Lemma 2.4. *Let $H \trianglelefteq G$. Suppose that $G = HC_G(H)$ and let $T \subseteq G$. Then T is a G-invariant transversal for G/H if and only if $T \subseteq C_G(H)$ and T is a $C_G(H)$-invariant transversal for $C_G(H)/Z(H)$.*

Proof. Suppose that T is a G-invariant transversal for G/H. Then $T \subseteq C_G(H)$ by Lemma 2.1. Thus T is a $C_G(H)$-invariant transversal for $C_G(H)/(H \cap C_G(H))$.

Now suppose that $T \subseteq C_G(H)$ is a $C_G(H)$-invariant transversal for $C_G(H)/Z(H)$. Then T is a transversal for G/H. Moreover, T is G-invariant, since H centralizes $C_G(H)$. □

Let T be a G-invariant transversal for G/H, and put $L := C_G(H)$. If $Z(H) = \{1\}$ we obtain the uninteresting case $G = H \times L$. In general, $G = H \circ_Z L$ is a central product, where $Z = Z(H)$ is amalgamated with a central subgroup of L, and $T \subseteq L$ is an L-invariant transversal for L/Z. Thus the classification of the G-invariant transversals for G/H can be reduced to the case when H is abelian.

Corollary 2.5. *Suppose that H is abelian and that $H \trianglelefteq G$. Then there is a G-invariant transversal for G/H, if and only if*

$$H \leq Z(G) \tag{3}$$

and

$$C_{G/H}(Hg) = C_G(g)/H \qquad \text{for all } g \in G. \tag{4}$$

Proof. If H is abelian, $H \leq C_G(H)$, and so Condition (1) is equivalent to (3). Moreover, if $H \leq Z(G)$, then $H \leq C_G(g)$ for all $g \in G$, so that Condition (2) is equivalent to (4). The result follows from Proposition 2.3. □

Notice that if $H \leq Z(G)$, then Condition (4) is equivalent to the statement that no non-trivial commutator of G lies in H. The latter condition holds for $G = \mathrm{SL}_2(\mathbb{R})$ with $H = Z(G)$. Since $\mathrm{SL}_2(\mathbb{R})$ is perfect, this yields an infinite "counterexample" to Conjecture 1.1. In contrast, by a theorem of Ree [12], every element in a semisimple algebraic group over an algebraically closed field is a commutator. With a finite number of exceptions, the same is true for every finite quasisimple group by the work of Liebeck, O'Brien, Shalev and Tiep [9, Theorem 1.1]. (A finite group is quasisimple, if it is perfect and simple modulo its center.) As will be discussed at the end this, two of these exceptions, already contained in the work of Blau [2], give rise to non-solvable counterexamples to Conjecture 1.1.

3. Finite groups and G-invariant transversals

In this section G is a finite group and $H \trianglelefteq G$. For a finite group L, let $k(L)$ denote the number of conjugacy classes of L. In [3], Gallagher has shown that Condition (2) is equivalent to

$$k(G) = k(G/H) \cdot k(H). \tag{5}$$

The article of Gallagher also contains a character theoretic interpretation of this relation. A search with GAP [4] using (5), discloses a group G of order 64 with a subgroup H of index 2 such that (G, H) satisfies (2) but not (1). If G is the dihedral group of order 8 and $H = Z(G)$, then (G, H) satisfies (1) but not (2). Thus neither of these conditions implies the other one.

From now on, we write $\bar{G} := G/H$ and use the bar-convention to denote the elements of $\bar{G}$, i.e., $\bar{g} \in \bar{G}$ is the image of $g \in G$ under the canonical epimorphism $G \to \bar{G}$. The following reasoning, based on [13], leads the way to counterexamples to Conjecture 1.1.

Suppose that $H \leq Z(G)$. Thus

$$1 \longrightarrow H \longrightarrow G \longrightarrow \bar{G} \longrightarrow 1$$

is a central extension of $\bar{G}$ with kernel H. Let $T = \{t_{\bar{x}} \mid \bar{x} \in \bar{G}\}$ be a transversal for G/H. From T we obtain a 2-cocycle $\gamma \in Z^2(\bar{G}, H)$ by the rule

$$t_{\bar{x}} t_{\bar{y}} = \gamma(\bar{x}, \bar{y}) t_{\bar{x}\bar{y}}$$

for $\bar{x}, \bar{y} \in \bar{G}$. Let $\lambda \in \mathrm{Hom}(H, \mathbb{C}^\times)$. Then $\alpha := \lambda \circ \gamma$ is a 2-cocycle in $Z^2(\bar{G}, \mathbb{C}^\times)$, and we write $[\alpha]$ for its image in $H^2(\bar{G}, \mathbb{C}^\times)$. Then $[\alpha]$ is non-trivial, if and only if $H \cap Z(G') \not\leq \ker(\lambda)$; see [8, Theorem (11.19)]. By definition, $\bar{x} \in \bar{G}$ is called α-regular, if $\alpha(\bar{x}, \bar{y}) = \alpha(\bar{y}, \bar{x})$ for all $\bar{y} \in C_{\bar{G}}(\bar{x})$.

Here is the link to invariant transversals. By the definition of γ, the centralizer condition (4) is satisfied for (G, H), if and only if

$$\gamma(\bar{x}, \bar{y}) = \gamma(\bar{y}, \bar{x}) \quad \text{for all } \bar{x} \in \bar{G} \text{ and all } \bar{y} \in C_{\bar{G}}(\bar{x}). \tag{6}$$

If λ is faithful, (6) is equivalent to the condition that every $\bar{x} \in \bar{G}$ is α-regular. If this holds, the number of irreducible complex characters of G which restrict to H as a multiple of λ equals the number of irreducible complex characters of $\bar{G}$. Can this happen if $[\alpha]$ is non-trivial? This is the question addressed by Reynolds in [13].

Let p be a prime. In [13, §3], Reynolds constructs a metabelian group G of order p^8 with a subgroup $H \leq Z(G) \cap G'$ of order p, such that, for every $\lambda \in \mathrm{Hom}(H, \mathbb{C}^\times)$, every element of $\bar{G}$ is α-regular. It follows that the pair (G, H) is a counterexample to Conjecture 1.1. The final section of Gallagher's article [3] contains further counterexamples, which the author attributes to Dade.

A computation with GAP shows that there are no counterexamples (G, H) to Conjecture 1.1 with $|G| < 128$ and $H \leq Z(G)$. On the other hand, we found 52 isomorphism classes of groups G of order 2^7, containing a subgroup $H \leq Z(G) \cap G'$ of order 2 such that

$$k(G) = k(G/H) \cdot 2. \tag{7}$$

Thus (5), and hence (4) is satisfied for (G, H), so that these groups yield counterexamples as well. Let G be one of these groups. Then $Z(G) \cong G/G'$ is elementary abelian of order 8, $Z(G) \leq G'$, G' is abelian of exponent 4, and $Z(G)$ contains a unique subgroup H of order 2 satisfying (7).

We conclude this note with a going down result.

Lemma 3.1. *Let $H \leq Z(G)$. Suppose that p is a prime dividing $|H \cap G'|$. Let P be a Sylow p-subgroup of G and let $Q \leq H \cap G'$ be of order p. Then $Q \leq P'$.*

If H admits a G-invariant transversal, then Q admits a P-invariant transversal.

Proof. The first assertion follows from a transfer result; see [7, Satz IV.2.2].

If H admits a G-invariant transversal, there is a G-invariant transversal for $G\backslash Q$, since $Q \leq H \leq Z(G)$. Intersecting this with P, we obtain a P-invariant transversal for $Q\backslash P$. □

In the following, C_n denotes a cyclic group of order n. Let G be quasisimple and let $\{1\} \lneq H \leq Z(G)$ be such that no non-trivial element of H is a commutator. Then $G \in \{G_1, G_2\}$, where G_i is a covering group of $\mathrm{PSL}_3(4)$ for $i = 1, 2$, $Z(G_1) \cong C_2 \times C_{12}$ and $Z(G_2) \cong C_2 \times C_4$; moreover, $|H| = 2$ (see [2, Theorem 1] or [9, Theorem 1.1 and Table 1]). Up to isomorphism, the covering groups G_1 and G_2 of $\mathrm{PSL}_3(4)$ are determined by their centers. For G_1, this follows from the fact that there is an automorphism of the universal covering group of $\mathrm{PSL}_3(4)$ permuting the three involutions in its center transitively; see [5, Table 6.3.1]. For the uniqueness of G_2 see [5, Theorem 6.3.2]. Using GAP [4], one can show that $Z(G_i)$ contains a unique subgroup H_i of order 2 with the given property, $i = 1, 2$. Up to isomorphism, we thus obtain two non-solvable counterexamples (G_1, H_1) and (G_2, H_2) to Conjecture 1.1. Let S be a Sylow 2-subgroup of G_1. Then $|S| = 2^9$ and S is isomorphic to a Sylow 2-subgroup of G_2. By Lemma 3.1, the pair (S, H_1) also yields a counterexample.

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