ON MAXIMAL EMBEDDINGS OF FINITE QUASISIMPLE GROUPS

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Dedicated to the Memory of my Friend and Colleague Kay Magaard

ABSTRACT. If a finite quasisimple group G with simple quotient S is embedded into a suitable classical group X through the smallest degree of a projective representation of S, then $N_X(G)$ is a maximal subgroup of X, up to two series of exceptions where S is a Ree group, and four exceptions where S is sporadic.

1. INTRODUCTION

With his seminal paper [1], Michael Aschbacher initiated the programme of classifying the maximal subgroups of the finite classical groups. He introduced eight classes of geometric subgroups of such a classical group and showed that a maximal subgroup lies in one of these classes or is what is now called a subgroup of type $\mathcal{S}$. After the monumental works of Kleidman and Liebeck [9] and of Bray, Holt and Roney-Dougal [2], it remains to determine the maximal subgroups of type $\mathcal{S}$ for classical groups of degree at least 13.

If X is a quasisimple classical group, a maximal subgroup of type $\mathcal{S}$ of X is of the form $N_X(G)$, where G is a quasisimple group, acting absolutely irreducibly on the vector space underlying X (and satisfies further conditions). One may now ask conversely: Which absolutely irreducible faithful representations of a quasisimple group G into a classical group X give rise to maximal subgroups of X of type $\mathcal{S}$? By [5, Theorem 7.5(b), Example 7.6], most such representations (in a precise quantitative sense) are imprimitive and thus, in general, do not yield maximal subgroups of X, as their image is contained in the stabilizer of the imprimitivity decomposition of the underlying vector space. In [11, Section 3], Kay Magaard has answered the above question for the

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simple Mathieu group $G = M_{11}$. Surprisingly enough, only the two 5-dimensional representations of M_{11} over the field with three elements give maximal subgroups of X. While this somewhat unexpected behaviour may be due to the small size of the group, another feature of this example is not. The degree 5 representations of M_{11} are its representations of smallest possible degree greater than 1. In this note we show that this is a general phenomenon. Let G be a quasisimple group and let n denote the smallest degree of a non-trivial projective representation of the simple group $G/Z(G)$. Then, up to two series of exceptions and four single cases the following holds: If X is a quasisimple classical group of degree n properly containing G and minimal with this property, then $N_X(G)$ is maximal in X.

2. Minimal projective degrees

Unless otherwise stated, the groups in our paper will be finite. For a prime power q we write $\mathbb{F}_q$ for the finite field with q elements and $\overline{\mathbb{F}}_q$ for its algebraic closure. If H and G are groups, we write $H \preceq G$, if H is isomorphic to a subgroup of G, and we write G' for the derived subgroup and G^∞ for the last term in the derived series of G. We will make use of the following well-known result about quasisimple groups, which follows from the Three Subgroups Lemma.

Lemma 2.1. *Let G be a perfect group, $Z \leq Z(G)$ such that G/Z is quasisimple. Then G is quasisimple.*

The following definitions are taken from [9, (5.3.1),(5.3.2)].

Definition 2.2 (Kleidman-Liebeck, [9])**.** Let G be a finite group.

(a) If p is a prime, set

$$R_p(G) := \min\{0 \neq n \in \mathbb{N} \mid G \preceq \mathrm{PGL}_n(\overline{\mathbb{F}}_p)\}.$$

Notice that

$R_p(G) = \min\{0 \neq n \in \mathbb{N} \mid G \preceq \mathrm{PGL}_n(F), F$ a field of characteristic $p\}$ (see [9, (5.3.2)]).

(b) We also set

$$R(G) := \min\{R_p(G) \mid p \text{ a prime}\}.$$

We collect a few properties of these invariants needed later on.

Remark 2.3. Let S be a non-abelian simple group with universal covering group $\hat{S}$ and let p be a prime.

(a) Let F be a field of characteristic p and ρ an F-representation of $\hat{S}$ of degree $R_p(S)$ with non-trivial image. Then ρ is absolutely irreducible.

(b) If $U \lneq S$ is a proper subgroup, then $R(S) < [S\!:\!U]$.

(c) Let G and K be quasisimple with $G \leq K$. Then $R_p(K/Z(K)) \geq R_p(G/Z(G))$. In particular, $R(K/Z(K)) \geq R(G/Z(G))$.

If S is a non-abelian simple group, then $R_p(S)$ is the smallest degree of a non-trivial projective representation of S over a field of characteristic p (see [9, Proposition 5.3.1(ii)]). Using this, part (a) of Remark 2.3 is easily verified. Using in addition that the universal covering group of $K/Z(K)$ contains a covering group of $G/Z(G)$ by Lemma 2.1, we obtain part (c). To prove (b), consider the permutation representation of S on the cosets of U.

If S is a sporadic simple group, $R(S)$ can be determined from [7]. If $S = A_m$ is an alternating group of degree $m \geq 5$, then $R(A_m) < 5$ for $5 \leq m \leq 8$, and $R(A_m) = m - 2$ for $m \geq 9$ (see [9, Proposition 5.3.7]). The value $R(S)$ for a simple group S of Lie type can be determined from [9, Propositions 5.4.13, 5.4.15, Corollary 5.4.14] and [8].

3. The main result

Keep the notation of the previous section. We shall consider the following hypothesis.

Hypothesis 3.1. Let S be a finite non-abelian simple group and put $n := R(S)$. Assume that $n \geq 5$. Let p be a prime such that $R(S) = R_p(S)$. Let G be a covering group of S such that G has a faithful representation ρ of degree n over $\overline{\mathbb{F}}_p$. Let q be a power of p minimal with the property that ρ is realizable over $\mathbb{F}_q$.

Let V be an $\mathbb{F}_qG$-module affording ρ and identify G with $\rho(G) \leq \mathrm{SL}(V)$. Finally, let $X \leq \mathrm{SL}(V)$ denote the smallest quasisimple classical group containing G.

Let us clarify what we mean by the smallest quasisimple classical group containing G. If G stabilizes a non-degenerate quadratic form on V, then X is the commutator subgroup of the full isometry group of this form in $\mathrm{SL}(V)$. Suppose that G does not stabilize any non-degenerate quadratic form on V. If G stabilizes a non-degenerate symplectic or hermitian form on V, then X is the full isometry group of this form in $\mathrm{SL}(V)$. Otherwise $X = \mathrm{SL}(V)$. Notice that X is uniquely determined by G (see [2, Lemma 1.8.8]). Moreover, X is quasisimple, as $n \geq 5$. Finally, X is isomorphic to one of $\Omega_n(q)$ (where n is odd),

$\Omega_n^{\pm}(q)$ (where n is even), $\mathrm{Sp}_n(q)$ (where n is even), $\mathrm{SU}_n(q_0)$ (where $q = q_0^2$) or $\mathrm{SL}_n(q)$.

Lemma 3.2. *Assume* Hypothesis 3.1. *Let K be quasisimple such that $G \leq K \leq X$, and put $T := K/Z(K)$. Then $R_p(T) = R_p(S)$ and $R(T) = R(S)$. If T is a group of Lie type of characteristic ℓ, then $\ell = p$ unless $T = \Omega_8^+(2)$ or $T = {}^2F_4(2)'$.*

If T is a classical group, then $K = X$, unless $T = \Omega_8^+(2)$ and p is odd. In the latter case, $G = K$.

If S and T are exceptional groups of Lie type, or if $T = {}^2F_4(2)'$, then $G = K$, unless $(G, K) = ({}^2G_2(q), G_2(q))$, $q = 3^{2m+1}$ with $m \geq 1$, or $(G, K) = ({}^2F_4(q)', F_4(q))$, $q = 2^{2m+1}$ with $m \geq 0$.

Proof. The embedding of K into X is a faithful representation of K of degree n, and thus $R_p(T) \leq n$ by the comment following Remark 2.3. As $G \leq K$, we conclude from Remark 2.3(c) that $n \geq R_p(T) \geq R_p(S) = R(S) = n$. It follows that $R_p(T) = R_p(S)$ and also that $n = R_p(T) \geq R(T) \geq R(S) = n$, i.e. $R(T) = R(S)$.

Suppose now that T is a group of Lie type of defining characteristic ℓ. As $R(T) \geq 5$, we conclude from [9, Propositions 5.4.13, 5.4.15, Corollary 5.4.14] and [8], that $R_\ell(T) = R(T)$. As $R(T) = R_p(T) = n$ by the first paragraph of this proof, the same references also give $\ell = p$, unless $T = \Omega_8^+(2)$ or $T = {}^2F_4(2)'$, and if T is a classical group of natural dimension d, then $d = R(T) = n$. In the latter case we get $K = X$, unless $T = \Omega_8^+(2)$ and p is odd. Indeed, K and X are then quasisimple classical groups of the same characteristic and degree, they stabilize the same form by [2, Lemma 1.8.8], $K \leq X$ and K is not realizable over a smaller field than $\mathbb{F}_q$. Suppose now that $T = \Omega_8^+(2)$ and p is odd. Then $n = R(T) = 8$ by [8]. Suppose that $S \lneq T$ and consider the list of maximal subgroups of T (see [4, p. 85]). This implies that $S \preceq S_1$, where S_1 is one of the simple groups $\mathrm{Sp}_6(2)$, A_8, A_9, $\mathrm{SU}_4(2)$ or A_5. But then $R(S_1) < 8$, and hence $R(S) < 8$ by Remark 2.3(c), a contradiction. Thus $S = T$ and hence $G = K$.

Suppose now that S and T are exceptional groups of Lie type, but that $T \neq {}^2F_4(2)'$. By [9, Table 5.4.C] we find that $R(T)$ determines the Dynkin type of T. Thus S and T are of the same Dynkin type, and either both are untwisted, both are twisted, or one is twisted and the other one is untwisted. In the first two alternatives, the exceptional groups of Lie type $G \leq K$ are of the same twisted type and the same characteristic. In this case, we get $G = K$, as the minimal field of definition of a representation of K of degree $R(K)$ determines K (see [9, Propositions 5.4.6(i), 5.4.17, 5.4.18, Remark 5.4.7]). Now suppose that we are in the third alternative. We begin by considering the Dynkin type E_6, so that $\{S, T\} = \{{}^2E_6(q_1), E_6(q_2)\}$ where q_1 and q_2 are powers of p. By [9, Proposition 5.4.17], we have $q = q_1^2$ and $X = \mathrm{SU}_{27}(q_1)$ if $S = {}^2E_6(q_1)$, and we have $q = q_2$ and $X = \mathrm{SL}_{27}(q)$ if $S = E_6(q_2)$. Both alternatives first lead to $q_2 = q_1^2$ and then to a contradiction. The cases of Dynkin type F_4 and G_2 yield the pairs (G, K) with $G \lneq K$ listed in the last statement of the lemma. (The case $G = {}^2G_2(3)'$ is excluded, as $R({}^2G_2(3)') = 2$ and $R(G_2(3)) = 7$.)

Suppose finally that $T = {}^2F_4(2)'$. Then $K = T$ and $n = 26$. If $G \lneq K = T$, then S is a proper section of T. But this would imply $R(S) < 26$ by the list of maximal subgroups of T (see [4, p. 74]), a contradiction. Thus $G = K$ and we are done. $\diamond$

We can now state our main theorem.

Theorem 3.3. *Assume* Hypothesis 3.1. *Then one of the following holds:*

(a) *We have $G = X$.*

(b) *The normalizer $N_X(G)$ is a maximal subgroup in X.*

(c) *We have $G = {}^2G_2(q)$ with $q = 3^{2m+1}$, $m \geq 1$ and $n = 7$. In this case, $X = \Omega_7(q)$ and $G \lneq G_2(q) \lneq X$ for all q.*

(d) *We have $G = {}^2F_4(q)'$ with $q = 2^{2m+1}$, $m \geq 0$ and $n = 26$. In this case, $X = \Omega_{26}^+(q)$ and $G \lneq F_4(q) \lneq X$ for all q.*

(e) *We have $G = J_2$, $n = 6$ and $q = 4$. In this case $X = \mathrm{Sp}_6(4)$ and $G \lneq G_2(4) \lneq X$.*

(f) *We have $G = M_{23}$, $n = 11$ and $q = 2$. In this case, $X = \mathrm{SL}_{11}(2)$ and $G \lneq M_{24} \lneq X$.*

(g) *We have $G = 3.\mathrm{Fi}_{22}$, $n = 27$ and $q = 4$. In this case, $X = \mathrm{SU}_{27}(2)$, and $G \lneq 3.{}^2E_6(2) \lneq X$.*

(h) *We have $G = \mathrm{Th}$, $n = 248$ and $q = 3$. In this case, $X = \Omega_{248}^+(3)$, and $G \times 2 \lneq E_8(3) \times 2 \lneq X$.*

Proof. If S is a classical group, we get $G = X$ and hence conclusion (a) by Lemma 3.2 (applied with $K = G$), unless $S = \Omega_8^+(2)$, $n = 8$ and p is odd. Suppose in the following that conclusion (a) does not hold. Then $N_X(G)$ is a proper subgroup of X, as G and X are quasisimple. Choose a maximal subgroup L of X with $N_X(G) \leq L$. We consider the possible Aschbacher classes containing L. For the definition of Aschbacher classes we follow [2, Subsections 2.2.1–2.2.8, Definition 2.1.3].

By construction, V is absolutely irreducible. Clearly, V is tensor indecomposable, as $\dim(V) = R_p(G)$. Also, V is primitive by Remark 2.3(b). By definition, G and hence L do not lie in Aschbacher class $\mathcal{C}_5$. It follows that L lies in one of the Aschbacher classes $\mathcal{C}_3, \mathcal{C}_6, \mathcal{C}_8$ or $\mathcal{S}$. Put $K := L^\infty$. Then $G = N_X(G)^\infty \leq K$. In particular, K acts absolutely irreducibly on V. But then L cannot lie in Aschbacher class $\mathcal{C}_3$, as in that case L^∞ does not act absolutely irreducibly (see [2, Lemma 2.2.7] and the remark following [2, Definition 2.1.4]). Suppose that L lies in Aschbacher class $\mathcal{C}_6$. Then $n = r^m$ for some prime r and some positive integer m. As $n \geq 5$ by our assumption, we have $m \geq 3$ if $r = 2$ and $m \geq 2$ if $r = 3$. Moreover, $K = U.C$, where U is a normal r-subgroup of K and C is a quasisimple classical group of degree $2m$ (see [9, (7.6.1)] or [2, Table 2.9]). It follows that $G/(U \cap Z(G))$ embeds into C, and hence G has a nontrivial irreducible representation over a field of characteristic r of degree at most $2m$. Thus $r^m = n = R(S) \leq 2m$, contradicting $n \geq 5$. Suppose that L lies in Aschbacher class $\mathcal{C}_8$. Then K is a quasisimple classical group of characteristic p, and Lemma 3.2 yields the contradiction $L \geq K = X \gneq L$.

We conclude that L lies in Aschbacher class $\mathcal{S}$. In particular, K is quasisimple and $L = N_X(K)$. If $G = K$, we get $N_X(G) = N_X(K) = L$, hence $N_X(G)$ is a maximal subgroup of X and conclusion (b) holds. So let us assume that $G \lneq K$ in the following and put $T := K/Z(K)$. As $K \leq L \lneq X$, Lemma 3.2 implies that T is not a classical group. If both of S or T are exceptional groups of Lie type, then (G, K) is as in (c) or (d), again by Lemma 3.2. The structure of X in these cases follows from [9, Table 5.4.C] and the fact that the Tits group ${}^2F_4(2)'$ embeds into $\Omega_{26}^+(2)$; see [8]. We may hence assume that not both of S or T are exceptional groups of Lie type, and $T \neq {}^2F_4(2)'$.

Suppose that T is an exceptional group of Lie type. Then p is the defining characteristic of T by Lemma 3.2. By the considerations in the previous paragraph and at the beginning of our proof, we may then assume that S is a sporadic group, an alternating group, or $S = \Omega_8^+(2)$, $n = 8$ and p is odd. Since $R(T) \leq 248$ by [9, Table 5.4.C], and G is a quasisimple group with an absolutely irreducible representation of degree $R(T)$, we can use [6] to find candidates for G and thus S. If $R(T) = 248$, then $S = \mathrm{Th}$ is the only candidate, and this gives rise to case (f) (see [4, p. 177] and [10]). Next, suppose that $R(T) = 56$. Then $R(S) < R(T)$ for all possible G by [6] and [7]. If $R(T) = 27$, then $R(S) < R(T)$ unless $S = \mathrm{Fi}_{22}$, $p = 2$ and $T = {}^2E_2(2)$. This gives rise to case (e) by [9, Proposition 5.4.17] and [4, p. 162]. If $R(T) = 26$ or $R(T) = 25$, then $R(S) < R(T)$ for all possible G. If $R(T) = 8$, then $T = {}^3D_4(p^a)$ for some positive integer a. Here, $R(S) < R(T)$, unless $S = A_{10}$ and $p \in \{2, 5\}$, or $S = \Omega_8^+(2)$ and p is odd. Moreover, $X = \mathrm{Sp}_8(2)$ if $p = 2$, and $X = \Omega_8^+(p)$, if p is odd. Neither of these groups X contains ${}^3D_4(p^a)$ as a subgroup. Next, suppose that $R(T) = 7$. Then $T = {}^2G_2(3^a)$, where a is an odd positive integer greater than 1. In particular, $p = 3$. This excludes the possibility $S = J_1$ by [6] and [7]. If $G = A_9$, then $X = \Omega_7(3)$, and $N_X(G)$ is maximal in X (see [4, p. 109]). Finally, suppose that $R(T) = 6$. Then $T = G_2(2^a)$, where a is a positive integer greater than 1. In particular, $p = 2$. The tables in [6] leave the options $S = J_2$ and $S = M_{22}$ (with $G = 3.M_{22}$). The first case gives rise to case (c) by [4, p. 97], [10] and [8, p. 273]. The second case yields $X = \mathrm{SU}_6(2)$ by [8, p. 93] and the remark in [8, p. xi], and $3.M_{22}$ is maximal in X by [4, p. 115].

Let $\bar{G}$ denote the image of G in T. Then $\bar{G}$ is quasisimple and $\bar{G} \lneq T$. Suppose now that T is an alternating group of degree m. As $R(T) \geq 5$, we find $n = R(T) = m - 2$ (see [9, Proposition 5.3.7]), i.e., $T = A_{n+2}$, and $p \mid n + 2$. The embedding $\bar{G} \to T$ yields a permutation representation of degree $n+2$ of $\bar{G}$. Let k be the size of a nontrivial orbit of $\bar{G}$. Then $\bar{G}$ has a subgroup of index k, and thus some quasisimple quotient of $\bar{G}$ embeds into A_k. By Remark 2.3(c) we find $R(A_k) \geq R(S) = n \geq 5$, and hence $R(A_k) = k - 2$. As $k \leq n + 2$, we conclude $k = n + 2$. Also, the action of $\bar{G}$ is two-fold transitive. Otherwise, the corresponding ordinary permutation character of $\bar{G}$ would have a non-trivial constituent of degree smaller than n, which would imply $R(S) < n$, a contradiction. All possibilities for S and n are listed in [3, Table 7.4]. In each case $R(S) < n$, as can be checked for sporadic groups S from [7].

Suppose finally that T is a sporadic group. In this case we use the Atlas [4] and [13, Section 4] to find the candidates for $\bar{G}$ and hence S, and [7] to rule out all possibilities except $S = \bar{G} = M_{23} \leq M_{24} = T$, $n = 11$ and $p = 2$. Using the remark in [8, p. xi] and the character tables in [8, p. 177, p. 267], we find that $X = \mathrm{SL}_{11}(2)$ and that $M_{23} \leq M_{24} \leq X$. $\diamond$

Remark 3.4. If $G = X \lneq \mathrm{SL}(V)$ in Theorem 3.3, then G is in Aschbacher class $\mathcal{C}_8$. In this case, replace X by the smallest classical group Y properly containing G. In matrix notation, $Y = \mathrm{Sp}_n(q)$ if n and q are even and $G = \Omega_n^{\pm}(q)$. In all other cases $Y = \mathrm{SL}_n(q)$. Then $N_Y(G)$ is a maximal subgroup of Y (see [9, Tables 3.5.A, 3.5.C] and [2, Proposition 2.3.32]).

Remark 3.5. Assume Hypothesis 3.1, except for the condition that $n \geq 5$; suppose instead that $n \in \{2, 3, 4\}$. Then one of the conclusions of Theorem 3.3 (a) or (b) holds for G. The statement of Remark 3.4 is also true.

Indeed, let S be a classical simple group of one of the following series: $\mathrm{PSL}_n(q)$, $2 \leq n \leq 4$, $\mathrm{PSU}_n(q)$, $3 \leq n \leq 4$, $\mathrm{PSp}_n(q)$, $n = 4$, where q is a power of the prime p. Then $R(S) = R_p(S) = n$ and $R_\ell(S) > n$ for all primes $\ell \neq p$, unless

$$S \in \{\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5), \mathrm{PSL}_3(2), \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)\}. \tag{1}$$

Moreover, every finite simple group T with $R(T) \leq 4$ is isomorphic to one of the classical groups listed above, to one of A_m, $5 \leq m \leq 8$, or to a Suzuki group. All these assertions follow from [9, Propositions 5.4.13, 5.4.15, Corollary 5.4.14] (if T is a sporadic group use [7]). As the alternating groups A_m, $5 \leq m \leq 8$, as well as the groups listed in (1) are contained in the Modular Atlas [8], it is easy to prove the initial statements using [2, Tables 8.1–8.17].

Remark 3.6. Let $G \lneq K \lneq X$ be as in one of the cases (c)–(h) of Theorem 3.3, where K denotes the unique quasisimple group disproving the maximality of $N_X(G)$. Then $N_X(K)$ is maximal in X. Moreover, $N_X(K) = K$, except in cases (g) and (h), where $N_X(K) = 3.{}^2E_6(2).3$ and $2 \times E_8(3)$, respectively. Indeed, in cases (c) and (d), $\mathrm{Aut}(K)$ does not have an irreducible representation of degree $R(K)$, as follows from inspection of $\mathrm{Aut}(F_4(2))$ and $\mathrm{Aut}(G_2(3))$, respectively. To prove the statement in case (g), we use [14] to obtain a 27-dimensional irreducible representation of $3.{}^2E_6(2).3$ over $\mathbb{F}_4$, which can be shown with the help of the MeatAxe [12] to preserve a hermitian form. In the remaining cases, $\mathrm{Aut}(K) = K$ so that $N_X(K) = C_X(K) = KZ(X)$.

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