Basic sets

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Representations of Reductive Groups,
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Modular representation theory of finite groups

G a finite group, p a prime number,
B a union of p-blocks of G

$\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$: class functions on G, respectively $G_{p'}$

$\mathcal{R}(B) := \mathbb{Z}[\mathrm{Irr}(B)]$, $\mathcal{R}_p(B) := \mathbb{Z}[\mathrm{IBr}(B)]$

$d : \mathcal{R}(B) \to \mathcal{R}_p(B), \chi \mapsto \chi^\circ$, decomposition map

Note: d is surjective

Definition (Brauer, from 1961)

Any $\mathbb{Z}$-basis of $\mathcal{R}_p(B)$ is a *basic set* for B.

Motivation: Finiteness results
Let $\mathcal{B} = \{\theta_j\}$ be a basic set for B.

Let $D_{\mathcal{B}} = (d_{ij})$ the *decomposition matrix* of B w.r.t. $\mathcal{B}$, and $C_{\mathcal{B}} := D_{\mathcal{B}}^t D_{\mathcal{B}}$, the *Cartan matrix* of B w.r.t. $\mathcal{B}$.

$D := D_{\mathrm{IBr}(B)}$ is *the decomposition matrix* of B
$C := C_{\mathrm{IBr}(B)} = D^t D$ is the *the Cartan matrix* of B

If $\mathcal{B}'$ is another basic set, $\mathcal{B} = U \cdot \mathcal{B}'$ (with $U \in \mathrm{GL}(\ell, \mathbb{Z})$), then $D_{\mathcal{B}'} = D_{\mathcal{B}} U$ and $C_{\mathcal{B}'} = U^t C_{\mathcal{B}} U$.

The integral quadratic form, represented by $C = D^t D$, is independent of the chosen basic set (up to equivalence).

Basic sets according to Richard Parker

Recall

$\mathcal{R}_p(B) := \mathbb{Z}[\mathrm{IBr}(B)]$

$\mathcal{R}_p^+(B) := \mathbb{N}[\mathrm{IBr}(B)]$:
set of proper Brauer characters

Definition (Parker, from 1984)

A *basic set* for B is a $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathcal{R}_p(B)$ with $\mathcal{B} \subseteq \mathcal{R}_p^+(B)$.

Motivation: Computation of $\mathrm{IBr}(B)$

Define $U_1 \in \mathrm{GL}(\ell, \mathbb{Z})$ by $\mathcal{B} = U_1 \cdot \mathrm{IBr}(B)$. Then

- U_1 has non-negative entries
- $D = D_{\mathcal{B}} U_1$

Note: Knowing $\mathrm{Irr}(B)$ and $\mathcal{B}$, it suffices to compute U_1

How does one check if $\mathcal{B} \subseteq \mathcal{R}_p^+(B)$ is a basic set?

Basic sets of projective characters

$\text{IPr}(B)$: the characters of the PIMs of B

Note: $\text{IPr}(B) = D^t \cdot \text{Irr}(B)$ (Brauer reciprocity)

$\mathcal{K}(B) := \mathbb{Z}[\text{IPr}(B)]$, $\mathcal{K}^+(B) := \mathbb{N}[\text{IPr}(B)]$

Definition (Parker, from 1984)

A *basic set $\mathcal{P}$ of projective characters* for B is a $\mathbb{Z}$-basis of $\mathcal{K}(B)$ with $\mathcal{P} \subseteq \mathcal{K}^+(B)$.

Given basic sets $\mathcal{B} = \{\theta_i\}$ and $\mathcal{P} = \{\Psi_j\}$, put $U := (\langle \theta_i, \Psi_j \rangle)_{i,j}$. Define $U_2 \in \text{GL}(\ell, \mathbb{Z})$ by $\mathcal{P} = U_2^t \cdot \text{IPr}(B)$. Then

- U_2 has non-negative entries
- $U = U_1 U_2$

How does one get a basic set $\mathcal{P}$ of projective characters?

Special basic sets

How does one get a basic set $\mathcal{B}$ of Brauer characters?

Definition (Parker, from 1984)

A special basic set for B is a $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathcal{R}_p(B)$ with $\mathcal{B} \subseteq \{\chi^\circ \mid \chi \in \mathrm{Irr}(B)\}$.

Question

Do special basic sets always exist?

This question is still open today.

Answer is Yes,

- if B is a block of a sporadic group (computer calculations)
- if G is a p-solvable group (see next slide)
- in many more cases to follow.

Triangular shape

Definition

$A \in \mathbb{Z}^{k \times \ell}$ (with $k \geq \ell$) has *triangular shape*, if $A = \begin{bmatrix} U \\ A' \end{bmatrix}$,
where $U \in \mathbb{Z}^{\ell \times \ell}$ is lower uni-triangular.

Suppose that $\mathcal{P}$ is a basic set of projective characters for B.
Define $A \in \mathbb{N}^{k \times \ell}$ by $\mathcal{P} = A^t \cdot \mathrm{Irr}(B)$.

- If A has triangular shape, then D has.
  Indeed, $A = DU_2$ with $U_2 \in \mathbb{N}^{\ell \times \ell}$.
- If D has triangular shape, a special basic set exists.

If G is p-solvable, D has shape $\begin{bmatrix} I_\ell \\ D' \end{bmatrix}$ with identity matrix I_ℓ
(Fong-Swan theorem)

An example (thanks to Olivier B.)

Let $G = \mathrm{SL}(2,9) \cong 2.A_6$, $p = 3$, and B the "faithful" block.

$$D = \begin{bmatrix} 1 & 1 & . & . \\ 1 & 1 & . & . \\ . & 1 & 1 & . \\ 1 & . & . & 1 \\ 1 & 1 & 1 & . \\ 1 & 1 & . & 1 \end{bmatrix}$$

(dots represent zeros)

No triangular shape, yet a special basic set exists:

Indeed, $\begin{bmatrix} 1 & 1 & . & . \\ . & 1 & 1 & . \\ 1 & 1 & 1 & . \\ 1 & 1 & . & 1 \end{bmatrix}$ has determinant 1.

A remedy would be to look at $3.\mathrm{SL}(2,9)$, but this does not work for $G = \mathrm{SL}(2,p^2)$, $p \geq 5$ a prime.

James, Dipper and Geck

Theorem (James)

If $G = S_n$, then D has triangular shape.

Special basic set: $\{\chi_\nu^\circ \mid \nu$ p-regular partition of n$\}$.

Theorem (Dipper and Geck)

Let $G = \mathrm{GL}_n(q)$ (Dipper) or $G = \mathrm{GU}_n(q)$ (Geck) with $p \nmid q$. Let B be the union of the unipotent blocks.
Then D has triangular shape.

Special basic set: $\{\chi_\nu^\circ \mid \nu$ partition of n$\}$.

Produce triangular shape basic set of projective characters by

- Harish-Chandra induction of projective characters (Dipper)
- Generalized Gelfand-Graev characters (Geck)

Some history

Arcata 1986, Josie Shamash: Brauer trees for $G_2(q)$
Left open some cases.

Back to Aachen experimented with Klaus Lux:
Used Maple to compute tensor products of unipotent characters of $G_2(q)$ generically.

Could solve Shamash's problems.
This was the begin of Chevie.

Talked in a seminar in Aachen.
This lead to the topic of Meinolf's diploma thesis (Pahlings):
Compute decomposition numbers for $SU_3(q)$.

Only solved much later by Okuyma and Waki (2002), and, with different methods, by Dudas in 2013.

Theorem (Brunat-Gramain, 2010 (two joint papers))

Every p-block of A_n has a special basic set.

Theorem (Brunat-Gramain, 2020)

If p is odd, every p-block of $2.A_n$ or $2.S_n$ has a special basic set.

Here, $2.A_n$ or $2.S_n$ denote double coverings of A_n and S_n, respectivley.

The case $p = 2$ is contained in the first theorem.

Example (Brunat-Gramain-Jacon, 2023)

D does not have triangular shape for $G = A_{18}, A_{19}$ and $p = 3$.

Let $G := \mathbf{G}^F = G(q)$ be a finite reductive group.
$\mathrm{Irr}(G)$ is organized in Lusztig series.
G^* dual reductive group;
$s \in G^*$ semisimple $\rightsquigarrow$ *Lusztig series* $\mathcal{E}(G, s) \subseteq \mathrm{Irr}(G)$;
$\mathcal{E}(G, s) = \mathcal{E}(G, s')$ if and only if s, s' conjugate in G^*;

$$\mathrm{Irr}(G) = \bigcup \mathcal{E}(G, s)$$

Theorem (Broué-Michel, 1989)

Assume $p \nmid q$. Let $s \in G^*$ be a semisimple p'-element. Then

$$\mathcal{E}_p(G, s) := \bigcup_{t \in C_{G^*}(s)_p} \mathcal{E}(G, st)$$

is a union of p-blocks.

Henceforth: Fix s, put $B := \mathcal{E}_p(G, s)$

Theorem (Geck-H., 1991)

Suppose $Z(\mathbf{G})$ is connected and p is good for $\mathbf{G}$. Then $\{\chi^\circ \mid \chi \in \mathcal{E}(G,s)\}$ is a basic set for $\mathcal{E}_p(G,s)$.

Theorem (Geck, 1993)

Suppose $p \nmid (Z(\mathbf{G})/Z(\mathbf{G})^\circ)_F$ and p is good for $\mathbf{G}$. Then the same conclusion holds.

Corollary

Under the above hypotheses, $|\mathrm{IBr}(\mathcal{E}_p(G,s))| = |\mathcal{E}(G,s)|$.

Examples

$G = \mathrm{SL}_3(q)$, $p = 3 \mid q-1$: $|\mathcal{E}(G,1)| = 3$, $|\mathrm{IBr}(\mathcal{E}_3(G,1))| = 5$.

$G = G_2(q)$, $p = 2$, q odd: $|\mathcal{E}(G,1)| = 10$, $|\mathrm{IBr}(\mathcal{E}_2(G,1))| = 9$.

Meinolf saw: Under our hypothesis, $\mathbf{L}^* := C_{\mathbf{G}^*}(t)$ is a Levi subgroup of $\mathbf{G}^*$ for all p-elements $1 \neq t$.

Let $\mathbf{L} \leq \mathbf{G}$ be a Levi subgroup dual to $\mathbf{L}^*$.

Since $C_{\mathbf{G}^*}(st) \leq \mathbf{L}^*$, there is a bijection (Lusztig)

$$\mathcal{E}(L, st) \to \mathcal{E}(G, st), \psi \mapsto \varepsilon_{\mathbf{L}}\varepsilon_{\mathbf{G}} R_{\mathbf{L}}^{\mathbf{G}}(\psi)$$

Also, $R_{\mathbf{L}}^{\mathbf{G}}(\psi)^\circ = R_{\mathbf{L}}^{\mathbf{G}}(\psi^\circ)$ and $\psi^\circ \in \mathbb{Z}[\mathcal{E}(L, s)]$, since $t \in Z(L^*)$.

As $R_{\mathbf{L}}^{\mathbf{G}}$ preserves Lusztig series, $\{\chi^\circ \mid \chi \in \mathcal{E}(G, s)\}$ is a generating set for $\mathcal{E}_p(G, s)$.

The rest of the proof is a clever counting argument due to Meinolf.

Suppose $Z(\mathbf{G})$ is connected. Then there exists a bijection

$$\mathcal{E}(G, s) \xrightarrow{\mathcal{L}_s} \mathcal{E}(C_{G^*}(s), 1)$$

(Lusztig's Jordan decomposition of characters)

CONJECTURE (GECK-H., 1991)

Suppose $s \in G^*$ is a semisimple p'-element. Then

$$\mathcal{E}_p(G, s) \text{ and } \mathcal{E}_p(C_{G^*}(s), 1)$$

have the same decomposition matrices (w.r.t. $(\mathcal{L}_{st})_t$).

CONJECTURE (GECK, CA. 1990)

Same hypotheses and p good for $\mathbf{G}$.
Then D has triangular shape, giving rise to the special basic set $\mathcal{B} = \{\chi^\circ \mid \chi \in \mathcal{E}(G, s)\}$.

Let the hypotheses be as on the previous slide.

THEOREM (BONNAFÉ-DAT-ROUQUIER, 2017)

Assume s not quasi-isolated. Then $\mathcal{E}_p(G, s)$ and $\mathcal{E}_p(C_{G^*}(s), 1)$ are Morita equivalent.

In fact, these authors prove a much stronger and more precise result.

THEOREM (BRUNAT-DUDAS-TAYLOR, 2021)

Suppose that p is good for $\mathbf{G}$. Then D has triangular shape.

More on this in a later talk.