Applications of Representation Theory to Compact Flat Riemannian Manifolds

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Joint work with

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(and work of many others)

Flat Manifolds (Geometry)

A Riemannian manifold X is a real, connected, differentiable n-manifold,

equipped with a Riemannian metric, i.e., a scalar product on each tangent space $T_x X$, depending smoothly on x.

X is flat, if its sectional curvature is 0.

For short: A flat manifold is a compact, flat, Riemannian manifold.

In the following, X denotes a flat manifold of dimension n.

From Geometry to Algebra, I

– The universal covering space of X equals $\mathbb{R}^n$.

– $\Gamma := \pi_1(X, x_0)$, the fundamental group of X, acts as group of deck transformations on $\mathbb{R}^n$.

– $X \cong \mathbb{R}^n/\Gamma$ (isometric).

– $\Gamma \leq E(n)$, the group of rigid motions of $\mathbb{R}^n$.

– Γ is discrete and torsion free.

The Affine Group

$A(n) := \mathsf{GL}(n) \ltimes \mathbb{R}^n$, the affine group.

$$A(n) \cong \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in \mathsf{GL}(n), v \in \mathbb{R}^n \right\} \leq \mathsf{GL}(n+1).$$

$E(n) := \mathrm{O}(n) \ltimes \mathbb{R}^n \leq A(n)$.

From Geometry to Algebra, II

Let Γ be the fundamental group of X. Then we have:

Bieberbach 1:
(1) $L := \Gamma \cap \mathbb{R}^n$ is a free abelian group of rank n, and a maximal abelian subgroup of Γ,
(2) $G := \Gamma / L$ is finite, the holonomy group of X.

Conversely, if $\Gamma \leq E(n)$ is torsion free and satisfies (1) and (2), then $\mathbb{R}^n/\Gamma$ is a flat manifold.

From Geometry to Algebra, III

Let X_i be flat n-manifolds with fundamental groups $\Gamma_i \leq E(n)$, $i = 1, 2$.

Bieberbach 2: The following are equivalent:
(1) X_1 and X_2 are affine equivalent.
(2) Γ_1 and Γ_2 are isomorphic.
(3) Γ_1 and Γ_2 are conjugate in $A(n)$.

Bieberbach 3: Up to affine equivalence, there are only finitely many flat n-manifolds.

Classification, I

$n = 1$: One flat manifold: $S^1 = \mathbb{R}/\mathbb{Z}$.

$n = 2$: Two flat manifolds:
(1) The torus $\mathbb{R}^2/\mathbb{Z}^2$,
(2) The Klein bottle $\mathbb{R}^2/\Gamma$, with

$$\Gamma := \left\langle \frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\rangle .$$

$n = 3$: Ten flat manifolds (Hantzsche and Wendt, '35), called platycosms by Conway and Rossetti.

Classification, II

$n = 4$: Classification by Brown, Bülow, Neubüser, Wondratschek and Zassenhaus, '78, (74 flat manifolds).

$n = 5$: Classification by Sczcepański and Cid and Schulz, '01, (1060 flat manifolds).

$n = 6$: Classification by Cid and Schulz, '01, (38746 flat manifolds).

Classification for $n = 5, 6$ completed with CARAT (http://wwwb.math.rwth-aachen.de/carat/).

Bieberbach Groups, I

A Bieberbach group of rank n is a torsion free group Γ given by a s.e.s.

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,$$

where G is finite and L is a free abelian group of rank n and a maximal abelian subgroup of Γ.

Thus G acts on L by conjugation, and we get a faithful holonomy representation $\rho : G \to \mathrm{O}(n)$.

Moreover, Γ can be embedded into $E(n)$, so that $\mathbb{R}^n/\Gamma$ is a flat manifold.

Bieberbach Groups, II

To define a Bieberbach group we need:

(1) $G \leq \mathrm{GL}(L)$ finite, where L is a free abelian group (equivalently: $G \leq \mathrm{GL}_n(\mathbb{Z})$ finite),

(2) A special element $\alpha \in H^2(G, L)$.

[$\alpha \in H^2(G, L)$ is special, if $\mathrm{res}^G_U(\alpha) \neq 0$ for every $U \leq G$ of prime order.]

Finite groups as holonomy groups, I

Let G be a finite group.

Is G the holonomy group of a flat manifold?

Need:

(1) $\mathbb{Z}G$-lattice L, with G acting faithfully,
(2) $\alpha \in H^2(G, L)$ special.

Auslander-Kuranishi, '57: Take $L := \oplus\mathbb{Z}[G/U]$, where U runs over all subgroups of prime order, up to conjugacy.

Finite groups as holonomy groups, II

For $\mathbb{Z}G$-lattice L put $\mathbb{Q}L := \mathbb{Q} \otimes_{\mathbb{Z}} L$, a $\mathbb{Q}G$-mod.

Conjecture (Szczepański): G is the holonomy group of a flat manifold with translation lattice L, such that $\mathbb{Q}L$ is multiplicity free.

$m(G)$: smallest n s.t. G is the holonomy group of a flat n-manifold.

Hiller: $m(C_{p^r}) = p^{r-1}(p-1)$.

Plesken, '89: determines $m(\text{PSL}_2(p))$, e.g., $m(A_5) = 15$, $m(\text{PSL}_2(7)) = 23$.

Reducibility of holonomy represent'n

Let Γ be a Bieberbach group:

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Theorem (Szczepański-H., '91): $\mathbb{Q}L$ is reducible.

Proved before for soluble G by Gerald Cliff.

Our proof uses the classification of finite simple groups.

Ingredients in Proof, I

Suppose $\mathbb{Q}L$ is irreducible.

Let $\{\chi_i\}$ be the set of irreducible characters of $\mathbb{C}L$. Let p be a prime with $p \mid |G|$. Then:

(1) χ_i is in the principal p-block of G.

(2) (Plesken, '89): Suppose G has a cyclic Sylow p-subgroup. Let the Brauer tree of the principal block be $1 — \mu — \theta — \cdots$

Then $\theta \in \{\chi_i\}$.

Ingredients in Proof, II

Usually θ as in (2) is not in principal q-block for some prime $q \neq p$.

Generalize to non-abelian simple subnormal subgroups, use classification.

If G has normal p-subgroup N, then:

Case 1: Some prime $q \neq p$ divides $|G|$. Then $N \subseteq \ker(\mathbb{C}L)$ (since all χ_i are in principal q-block).

Case 2: G is a p-group. Easy.

In the following, Γ denotes the fundamental group of X, the extension

$$0 \longrightarrow L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

being described by $\alpha \in H^2(G, L)$.

Affine Selfequivalences, I

Aff(X): group of affine self equivalences of X,
Aff$_0(X)$ connected component of 1.

Aff$_0(X)$ is a torus of dimension $b_1(X)$.
($b_1(X) =$ rk $H^0(G, L)$.)

Aff$(X)/$Aff$_0(X) \cong$ Out(Γ).

How small or large can Aff(X) be?

Affine Selfequivalences, II

Theorem (Hiller-Sah, '86): Suppose for some prime p dividing $|G|$, a Sylow p-subgroup of G is cyclic and has a normal complement.
Then $b_1(X) \neq 0$. In particular $\text{Aff}_0(X) \neq 1$.

Theorem (Szczepański-H., '97): Suppose for some prime p dividing $|G|$, a Sylow p-subgroup of G has a normal complement (G is p-nilpotent).
Then $\text{Aff}(X) \neq 1$.

The outer automorphism group of Γ

Theorem (Charlap-Vasquez, '73): There is a short exact sequence

$$0 \longrightarrow H^1(G,L) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow N_\alpha/\rho(G) \longrightarrow 1,$$

where $N_\alpha = N_{\mathrm{O}(n)}(\rho(G),\alpha)$.

Sketch of proof of "p-nilpotency theorem":

– May assume $H^0(G,L) = 0$. Then:

– $p \mid |H^1(G,L)|$ iff L/pL has a trivial submodule.

– G is p-nilpotent and $p \mid |H^2(G,L)|$, hence L/pL has a trivial submodule.

Finiteness of outer automorphism grp

N_α is finite if and only if $C_{\mathrm{O}(n)}(\rho(G))$ is finite. This gives the following theorem, implicitly contained in Brown et al.

Theorem: Equivalent are:

(1) Out(Γ) is finite.

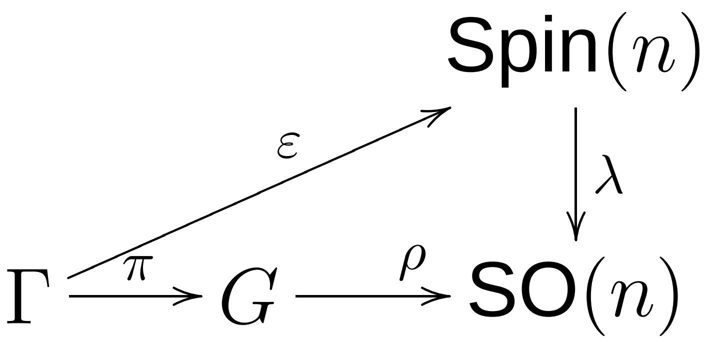
(2) $\mathbb{Q}L$ is multiplicity free, and $\mathbb{R}S$ is simple for every simple constituent S of $\mathbb{Q}L$.

Waldmüller's Example

Theorem (Waldmüller, '02): There is a 141-dimensional flat manifold X with holonomy group M_{11} such that Aff$(X) = 1$.

Spin Structures, I

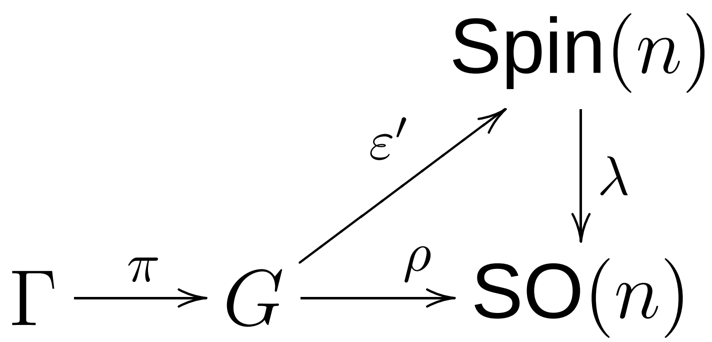
X is oriented, if $\rho(G) \leq \mathrm{SO}(n)$.
From now on assume X oriented.
A spin structure on X allows to define a Dirac operator on X.
The spin structures on X correspond to the lifts $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ with $\rho \circ \pi = \lambda \circ \varepsilon$:



Special Spin Structures, I

A special spin structure is a spin structure ε with $\varepsilon(L) = 1$, i.e., there exists

$\varepsilon' : G \to \mathrm{Spin}(n)$ with $\rho = \lambda \circ \varepsilon'$:



Special Spin Structures, II

There is a nice **negative** criterion by Griess and Gagola and Garrison, '82:

Theorem: Suppose there is an involution $g \in G$, such that

$$\frac{1}{2}(n - \text{trace}(\rho(g))) \equiv 2 (\text{mod } 4).$$

Then no such ε' exists.

Spin Structures, II

Let S_2 be a Sylow 2-subgroup of G.
Put $\Gamma_2 := \pi^{-1}(S_2)$, $X_2 := \mathbb{R}^n/\Gamma_2$.

Proposition (Dekimpe et al., '04): X has spin structure if and only if X_2 has spin structure.

Examples: (1) (Pfäffle, '00): All flat 3-manifolds have spin structures.

(2) (Miatello-Podestá, '04): There is $G \leq \mathrm{GL}_6(\mathbb{Z})$, $G \cong C_2 \times C_2$, and special $\alpha_i \in H^2(G, \mathbb{Z}^6)$, $i = 1, 2$, such that Γ_1 has spin structure and Γ_2 doesn't.

Thank you for your attention!