Computational Representation Theory – Lecture II

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Contents

Notation

Throughout this lecture, G denotes a finite group and F a field.

Brauer Characters: Motivation

Assume from now on that F is algebraically closed and has prime characteristic p.

Let $\mathfrak{X}$ be an F-representation of G of degree d.

The character $\chi_{\mathfrak{X}}$ of $\mathfrak{X}$ as defined in Lecture 1 does not convey all the desired information, e.g.,

$\chi_{\mathfrak{X}}(1)$ only gives the degree d of $\mathfrak{X}$ modulo p.

Instead one considers the Brauer character $\varphi_{\mathfrak{X}}$ of $\mathfrak{X}$.

This is obtained by consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{p'}$ to $\mathbb{C}$.

Here, $G_{p'}$ is the set of p-regular elements of G ($g \in G$ is p-regular, if $p \nmid |g|$).

More precisely: Write $|G| = p^a m$ with $p \nmid m$, and put $\zeta := \exp(2\pi i/m) \in \mathbb{C}$.

Let $R := \mathbb{Z}[\zeta]$ denote the ring of algebraic integers in $\mathbb{Q}(\zeta)$.

Choose a ring homomorphism $\alpha : R \to F$ sending ζ to a primitive m-th root of unity $\bar{\zeta} \in F$.

Notice that the restriction of α to $\langle \zeta \rangle$ is injective.

The Brauer character of $\mathfrak{X}$ (with respect to α) is the map

$$\varphi_{\mathfrak{X}} : G_{p'} \to R \subseteq \mathbb{C}$$

defined as follows:

If $g \in G_{p'}$, the eigenvalues of $\mathfrak{X}(g)$ are of the form $\bar{\zeta}^i$, since $g^m = 1$.

Let $g \in G_{p'}$ and let $\bar{\zeta}^{i_1}, \ldots, \bar{\zeta}^{i_d}$ denote the eigenvalues of $\mathfrak{X}(g)$, counting multiplicities. Then $\varphi_{\mathfrak{X}}(g) := \sum_{j=1}^{d} \zeta^{i_j}$.

In particular, $\alpha(\varphi_{\mathfrak{X}}(g)) = \chi_{\mathfrak{X}}(g)$ for all $g \in G_{p'}$.

Fact

Two irreducible F-representations of G are equivalent if and only if their Brauer characters are equal.

Put $\mathrm{IBr}_p(G) :=$ set of irreducible Brauer characters of G (all with respect to the same α), $\mathrm{IBr}_p(G) = \{\varphi_1, \ldots, \varphi_l\}$.

If $p \nmid |G|$, then $\mathrm{IBr}_p(G) = \mathrm{Irr}(G)$.

Let $g_1, \ldots, g_l$ be representatives of the conjugacy classes contained in $G_{p'}$ (same l as above!).

The square matrix

$$\left[\varphi_i(g_j)\right]_{1 \leq i,j \leq l}$$

is called the Brauer character table or p-modular character table of G.

Example (The 3-Modular Character Table of M_{11}, (James, '73))

	$1a$	$2a$	$4a$	$5a$	$8a$	$8b$	$11a$	$11b$
φ_1	1	1	1	1	1	1	1	1
φ_2	5	1	-1	.	α	$\bar{\alpha}$	γ	$\bar{\gamma}$
φ_3	5	1	-1	.	$\bar{\alpha}$	α	$\bar{\gamma}$	γ
φ_4	10	2	2	.	.	.	-1	-1
φ_5	10	-2	.	.	β	$-\beta$	-1	-1
φ_6	10	-2	.	.	$-\beta$	β	-1	-1
φ_7	24	.	.	-1	2	2	2	2
φ_8	45	-3	1	.	-1	-1	1	1

$(\alpha = -1 + \sqrt{-2}, \beta = \sqrt{-2}, \gamma = (-1 + \sqrt{-11})/2)$

For $\chi \in \mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$, write $\hat{\chi}$ for the restriction of χ to $G_{p'}$.

Then there are integers $d_{ij} \geq 0$, $1 \leq i \leq k$, $1 \leq j \leq l$ such that

$$\hat{\chi}_i = \sum_{j=1}^{l} d_{ij}\,\varphi_j.$$

These integers are called the decomposition numbers of G modulo p.

The matrix $D = [d_{ij}]$ is the decomposition matrix of G.

REMARKS

$\mathrm{IBr}_p(G)$ is linearly independent (in $\mathrm{Maps}(G_{p'}, \mathbb{C})$) and so the decomposition numbers are uniquely determined.

The elementary divisors of D are all 1 (i.e., the decomposition map defined by $\chi \mapsto \hat{\chi}$ is surjective). Thus:

Knowing $\mathrm{Irr}(G)$ and D is equivalent to knowing $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$.

If G is p-soluble, D has shape

$$D = \left[\begin{array}{c} I_l \\ \hline D' \end{array}\right],$$

where I_l is the $(l \times l)$ identity matrix (Fong-Swan theorem).

Example: Decomposition Numbers of M_{11}

		φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
		1	5	5	10	10	10	24	45
χ_1	1	1	.	.	.	.	.	.	.
χ_2	10	.	.	.	1	.	.	.	.
χ_3	10	.	.	.	.	1	.	.	.
χ_4	10	.	.	.	.	.	1	.	.
χ_5	11	1	1	1	.	.	.	.	.
χ_6	16	1	1	.	.	.	1	.	.
χ_7	16	1	.	1	.	1	.	.	.
χ_8	44	.	1	1	1	.	.	1	.
χ_9	45	.	.	.	.	.	.	.	1
χ_{10}	55	1	1	1	.	1	1	1	.

AIM

Describe all Brauer character tables of all finite simple groups and related finite groups.

In contrast to the case of ordinary character tables (cf. Lecture 1), this is wide open:

1. For alternating groups: complete up to A_{17}
2. For groups of Lie type: only partial results
3. For sporadic groups up to McL and other "small" groups (of order $\leq 10^9$): *An Atlas of Brauer Characters*, Jansen, Lux, Parker, Wilson, 1995

The program to compute the modular (= Brauer) character tables of the ATLAS groups.

ATLAS := *Atlas of Finite Groups*, Conway et al., 1986

Start: PhD-thesis of Gordon James on Mathieu groups (1973)

The tables of Jansen et al. and more are collected on the web site of the Modular Atlas Project (http://www.math.rwth-aachen.de/~MOC/).

Methods: GAP, MOC, Meat-Axe, Condensation

The Players

Wilson
Waki
Thackray
Ryba
Parker
Noeske
Neunhöffer
Müller
Lux
Lübeck
Jansen
James
H.
and many others

The Brauer character tables are completely known for the following groups:

M_{11}, M_{12}, J_1, M_{22}, J_2, M_{23}, HS, J_3, M_{24}, McL (10 groups)

An Atlas of Brauer Characters, Jansen, Lux, Parker, Wilson, 1995

He, Ru, Suz, $O'N$, Co_3, Co_2, Fi_{22}, HN, Fi_{23} (9 groups)

various authors (1988 – 2016)

State of the Art, cont.

Grp	Characteristic	
	Known	Not Completely Known
Ly	7, 11, 31, 37, 67	2, 3*, 5*
Th	19	2–7, $13^\dagger$, $31^\dagger$
Co_1	7–13, 23	2, 3, 5
J_4	5, 7, 37	2, 3, 11, $23^\dagger$, $29^\dagger$, $31^\dagger$, $43^\dagger$
Fi_{24}'	11, 23	2–7, $13^\dagger$, $17^\dagger$, $29^\dagger$
B	11, 23	2–7, 13°, $17^\dagger$, 19°, 31°, 47°
M	17, 19, 23, 31	2–13, 29°, 41°, 47°, 59°, 71°

*: Known "up to condensation" (mod 3: Thackray, mod 5: Lux & Ryba)

†: Cyclic defect, degrees known

$^\circ$: Cyclic defect, degrees unknown

Remaining Problems for *Th* (neglecting $p = 13, 31$)

p	No. irr. char's	No. known char's	missing
2	21	8	13
3	16	14	2
5	41	33	8
7	44	30	14

Partial character tables for *HN* (mod 2), *Th* (mod 3, 5, 7), and Co_1 (mod 5) are now on the Modular Atlas Homepage.

These contain bounds for the degrees of the missing irreducibles.

Recently Solved Problems, Work in Progress

1. 2-modular table of Fi_{23}, H., Neunhöffer, Noeske, 2006
2. 5-modular table of HN, Lux, Noeske, Ryba, 2008
3. 2-modular and 3-modular table of HN, H., Müller, Noeske, Thackray, 2012
4. 3-modular table of Fi_{23}, Görgen, Lux, 2016

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5. 2-modular tables of Ly, Th, Co_1, and J_4, Thackray
6. 3-modular table of Ly, Thackray
7. 5-modular table of Th, Carlson, H., Lux, Noeske
8. 5-modular table of Ly, Lux, Ryba

What is MOC?

MOC is a collection of stand-alone programs (in C and FORTRAN), linked via shell scripts, for computing with Brauer characters and projective characters.

Purpose: Assist the computation of decomposition numbers (equivalently: Brauer character tables).

Authors: Richard Parker, Christoph Jansen, Klaus Lux, H.

Developed: 1984 – 1987

Projective Characters: Definition

Let $\text{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$, $\text{IBr}_p(G) = \{\varphi_1, \ldots, \varphi_l\}$ and $D = [d_{ij}]$ the decomposition matrix.

Definition

The (ordinary) character

$$\Phi_i := \sum_{j=1}^{k} d_{ji}\chi_j$$

is called the projective indecomposable character (PIM) associated to φ_i ($1 \leq i \leq l$). Put $\text{IPr}_p(G) := \{\Phi_1, \ldots, \Phi_l\}$.
A projective character is a sum of PIMs.

Expanding a projective character in $\text{Irr}(G)$ yields a sum of columns of the decomposition matrix.

Put $\mathbb{Z}[\mathrm{IBr}_p(G)] := \{\sum_{i=1}^{l} z_i \varphi_i \mid z_i \in \mathbb{Z}, 1 \leq i \leq l\}$ (generalised Brauer characters) and $\mathbb{Z}[\mathrm{IPr}_p(G)] := \{\sum_{i=1}^{l} z_i \Phi_i \mid z_i \in \mathbb{Z}, 1 \leq i \leq l\}$ (generalised projective characters).

These are free abelian groups with bases $\mathrm{IBr}_p(G)$ and $\mathrm{IPr}_p(G)$, respectively.

Define

$$\langle -, - \rangle' : \mathbb{Z}[\mathrm{IBr}_p(G)] \times \mathbb{Z}[\mathrm{IPr}_p(G)] \to \mathbb{Z}$$

$$\langle \chi, \psi \rangle' := \frac{1}{|G|} \sum_{g \in G_{p'}} \chi(g)\, \psi(g^{-1})$$

THEOREM (ORTHOGONALITY RELATIONS)

$\langle \varphi_i, \Phi_j \rangle' = \delta_{ij}$.

Basic Sets

Definition

(1) A basic set of Brauer characters is a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathrm{IBr}_p(G)]$ consisting of Brauer characters.
(2) A basic set of projective characters is a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathrm{IPr}_p(G)]$ consisting of projective characters.

Lemma

Let $B_{\mathcal{B}}$ and $B_{\mathcal{P}}$ be sets of Brauer characters, respectively projective characters.
Then $B_{\mathcal{B}}$ and $B_{\mathcal{P}}$ are basic sets if and only if $U := \langle B_{\mathcal{B}}, B_{\mathcal{P}} \rangle'$ is square and invertible over $\mathbb{Z}$.

Proof. Let $X_1, X_2 \in \mathbb{N}^{l \times l}$ be the matrices expressing $B_{\mathcal{B}}$ in $\mathrm{IBr}_p(G)$ and $B_{\mathcal{P}}$ in $\mathrm{IPr}_p(G)$, respectively. Then, by the orthogonality relations, $U = X_1 X_2^{tr}$.

Constructions of Characters

Facts (Constructions of Brauer Characters)

1. χ *ordinary character, then* $\hat{\chi}$ *Brauer character*
2. ψ *Brauer character of* $H \leq G$*, then* ψ^G *Brauer character*
3. *products of Brauer characters are Brauer characters*

Facts (Constructions of Projective Characters)

1. $\Phi \in \mathrm{Irr}(G)$ *is a projective character, if and only if* $p \nmid |G|/\Phi(1)$
2. *if* $p \nmid |G|$*, then every ordinary character is projective*
3. ψ *projective character of* $H \leq G$*, then* ψ^G *is projective*
4. φ *Brauer character,* Φ *projective character, then* $\varphi \cdot \Phi$ *(extended by* 0 *on* $G \setminus G_{p'}$*) is projective*

1. Compute sets $\mathcal{B}$ of Brauer characters and $\mathcal{P}$ of projective characters using the constructions above.
2. Select maximal linearly independent subsets $B_{\mathcal{B}} \subseteq \mathcal{B}$ and $B_{\mathcal{P}} \subseteq \mathcal{P}$.
3. Compute $U := \langle B_{\mathcal{B}}, B_{\mathcal{P}} \rangle'$.
4. If $\det U \neq \pm 1$, go to Step 1.
5. Otherwise, $\mathrm{IBr}_p(G) = X_1^{-1} \cdot B_{\mathcal{B}}$, $\mathrm{IPr}_p(G) = X_2^{-1} \cdot B_{\mathcal{P}}$, with (unknown) $X_1, X_2 \in \mathbb{N}^{l \times l}$ such that
   - $U = X_1 X_2^{tr}$,
   - $\langle \mathcal{B}, B_{\mathcal{P}} \rangle' \cdot (X_2^{tr})^{-1} \geq 0$,
   - $X_1^{-1} \cdot \langle B_{\mathcal{B}}, \mathcal{P} \rangle' \geq 0$.

Try to determine X_1, X_2 from these conditions.

Works well, if U is “sparse”. Finds D, if G is p-soluble.

References

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3. C. Jansen, K. Lux, R. A. Parker R. A. Wilson, An Atlas of Brauer Characters, Clarendon Press, 1995.
4. K. Lux and H. Pahlings, Representations of Groups. A computational approach. Cambridge University Press, 2010.

Thank you for your attention!