COMPUTATIONAL REPRESENTATION THEORY – LECTURE IV

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

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Contents

Notation

Throughout this lecture, G denotes a finite group and F a field.

Also, $\mathfrak{A}$ denotes a finite-dimensional F-algebra,
$J(\mathfrak{A})$ the Jacobson radical of $\mathfrak{A}$
(i.e., the intersection of the maximal right ideals of $\mathfrak{A}$).

mod-$\mathfrak{A}$: category of finite-dimensional right $\mathfrak{A}$-modules

Condensation: Motivation

The MeatAxe can reduce representations of degree up to 200 000 over $\mathbb{F}_2$.

Over larger fields, only smaller degrees are feasible.

To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).

Condensation: Theory [J. A. Green 1980]

Let $e \in \mathfrak{A}$ an idempotent, i.e., $0 \neq e = e^2$ (a projection).

Get exact functor: mod-$\mathfrak{A} \to$ mod-$e\mathfrak{A}e$, $V \mapsto Ve$.

If $S \in$ mod-$\mathfrak{A}$ is simple, then $Se = 0$ or simple.

Let $S_1, \ldots, S_n$ be the simple $\mathfrak{A}$-modules (up to isomorphism).

Suppose that $S_1 e \neq 0, \ldots, S_m e \neq 0$, $S_{m+1} e = \cdots = S_n e = 0$.

Then $S_1 e, \ldots, S_m e$ are exactly the simple $e\mathfrak{A}e$-modules (up to isomorphism).

As the Condensation functor is exact, it sends a composition series of $V \in$ mod-$\mathfrak{A}$ to a composition series of $Ve \in$ mod-$e\mathfrak{A}e$.

Condensation: Morita Equivalence

If $Se \neq 0$ for all simple $S \in \text{mod-}\mathfrak{A}$, then Condensation is an equivalence of categories, i.e. $\mathfrak{A}$ and $e\mathfrak{A}e$ are Morita equivalent.

An indecomposable direct summand of $\mathfrak{A}_{\mathfrak{A}}$ is called a PIM.

(A PIM in the sense of Lecture 2 is the Brauer character of a PIM of FG, extended by 0 from $G_{p'}$ to G.)

A projective $\mathfrak{A}$-module is a direct sum of PIMs.

A finite-dimensional F-algebra $\mathfrak{B}$ is Morita equivalent to $\mathfrak{A}$, if $\mathfrak{B} \cong \text{End}_{\mathfrak{A}}(Q)$ for a projective module Q of $\mathfrak{A}$ containing every PIM of $\mathfrak{A}$ (up to isomorphism) as a direct summand.

Morita equivalent algebras have “the same” representations.

CONDENSATION: IDEMPOTENTS

Let $H \leq G$ with $\mathrm{char}(F) \nmid |H|$. Then

$$e := e_H := \frac{1}{|H|} \sum_{x \in H} x \in FG$$

is a suitable idempotent.

Other idempotents can be used, e.g.,

$$e = \frac{1}{|H|} \sum_{x \in H} \lambda(x^{-1}) x \in FG,$$

where $\lambda : H \to F^*$ is a homomorphism (Noeske, 2005).

CONDENSATION: PERMUTATION MODULES, I

Let $e := e_H = 1/|H| \sum_{x \in H}$ be as above.

Let V be the permutation FG-module w.r.t. an action of G on the finite set Ω. Then Ve is the set of H-fixed points in V.

Task: Given $g \in G$, determine the action of ege on Ve,

without the explicit computation of the action of g on V.

THEOREM (THACKRAY AND PARKER, 1981)

This can be done!

CONDENSATION: PERMUTATION MODULES, II

Let $\Omega_1, \ldots, \Omega_m$ be the H-orbits on Ω.

The orbits sums $\widehat{\Omega}_j := \sum_{\omega \in \Omega_j} \omega \in V$ form a basis of Ve.

W.r.t. this basis, the (i, j)-entry a_{ij} of the matrix of ege on Ve equals

$$a_{ij} = \frac{1}{|\Omega_j|} |\Omega_i g \cap \Omega_j|.$$

ENUMERATION OF LONG ORBITS

To perform these computations, we need to be able to

1. compute Ω_j and $|\Omega_j|$, $1 \leq j \leq m$,
2. decide $\omega \in \Omega_j$? for given $\omega \in \Omega$ and $1 \leq j \leq m$.

In actual applications, $|\Omega| \approx 10^{15}$, so the elements of Ω can not be stored in memory.

Parker and Wilson suggested Direct Condensation methods; these were later extended and implemented by Cooperman, Lübeck, Müller and Neunhöffer.

Principal idea: Enumerate the H-orbits Ω_j by suborbits of subgroups $U \leq H$. Iterate this idea.

Details depend on the realisation of the action on Ω.

Tensor Products and Induced Modules

Let V and W be two FG-modules.

Task: Given $g \in G$, determine the action of ege on $(V \otimes W)e$, without the explicit computation of the action of g on $V \otimes W$.

Theorem (Lux and Wiegelmann, 1997)

This can be done!

Let M be a subgroup of G and let W be an FM-module.

The induced module is the FG-module $W \otimes_{FM} FG$.

Task: Given $g \in G$, determine action of ege on $(W \otimes_{FM} FG)e$, without the explicit computation of the action of g on $W \otimes_{FM} FG$.

Theorem (Müller and Rosenboom, 1997)

This can be done!

HOMOMORPHISM SPACES

Let M be a subgroup of G, W an FM-module and V an FG-FM-bimodule.
Then $\mathrm{Hom}_{FM}(V, W)$ is a right FG-module:

$$v(\varphi g) := (gv)\varphi, \quad v \in V, \varphi \in \mathrm{Hom}_{FM}(V, W), g \in G.$$

EXAMPLES

1. $\mathrm{Hom}_{FM}(FG, F) \cong$ *permutation module corresponding to permutation action of G on* $\Omega := M \backslash G$.
2. $\mathrm{Hom}_F(V^*, W) \cong V \otimes W$ *for* $V, W \in$ mod-FG. ($V^* = \mathrm{Hom}_F(V, F)$.)
3. $\mathrm{Hom}_{FM}(FG, W) \cong W \otimes_{FM} FG$.

Lux, Neunhöffer, Noeske develop general Condensation programs for such homomorphism spaces.

Condensation: Some Applications

Benson, Conway, Parker, Thackray, Thompson, 1980:
Existence of J_4.

Thackray, 1981:
2-modular character table of McL.
Answer to a question of Brauer.

Cooperman, H., Lux, Müller, 1997:
Brauer tree of Th modulo 19.
$\dim(V) = 976\,841\,775$, $\dim(Ve) = 1403$.

Müller, Neunhöffer, Röhr, Wilson, 2002:
Brauer trees of Ly modulo 37 and 67.
$\dim(V) = 1\,113\,229\,656$.

More applications later.

The Basic Algebra

Definition

A finite-dimensional F-algebra $\mathfrak{B}$ is called basic, if

$$\mathfrak{B}_{\mathfrak{B}} = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$$

with PIMs Q_i such that $Q_i \not\cong Q_j$ for $1 \leq i \neq j \leq n$.

Alternatively, if $\mathfrak{B}/J(\mathfrak{B})$ is a direct sum of division algebras.

Facts

Let $P_1, \ldots, P_n$ be the PIMs of $\mathfrak{A}$ (up to isomorphism). Then $\mathfrak{B} := \mathrm{End}_{\mathfrak{A}}(P_1 \oplus \cdots \oplus P_n)$ is a basic algebra Morita equivalent to $\mathfrak{A}$, the basic algebra of $\mathfrak{A}$.

This is the smallest algebra Morita equivalent to $\mathfrak{A}$.

MORITA EQUIVALENT ALGEBRAS

If $\dim(\mathfrak{A})$ is large, it may be too difficult to construct the basic algebra of $\mathfrak{A}$ explicitly.

Klaus Lux uses Condensation to construct algebras of feasible dimensions, Morita equivalent to (blocks of) group algebras FG.

Need idempotent $e \in FG$ with $Se \neq 0$ for all simple FG-modules S (or all simple modules in a block).

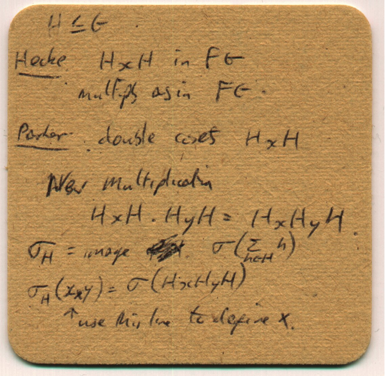
This can be checked with the modular character table of G, if $e = e_H$ for some $H \leq G$ with $\mathrm{char}(F) \nmid |H|$.

Example: Principal block $\mathfrak{B}_0$ of HS modulo 5, $|H| = 192$. $\dim(\mathfrak{B}_0) = 15\,364\,500$, $\dim(e_H \mathfrak{B}_0 e_H) = 767$.

See KLAUS LUX, *Faithful Condensation for Sporadic Groups*, (http://math.arizona.edu/˜klux/habil.html).

Applications: Cartan matrices for group algebras, cohomology computations.

Condensation: History



THE GENERATION PROBLEM

We investigate Ve through the MeatAxe, using matrices of generators of $eFGe$.

QUESTION (THE GENERATION PROBLEM)

How can eFGe be generated with "a few" elements?

If $\mathcal{E} \subseteq FG$ with $F\langle\mathcal{E}\rangle = FG$, then in general $F\langle e\mathcal{E}e\rangle \lneq eFGe$.

- Let $\mathfrak{C} := F\langle e\mathcal{E}e\rangle \leq eFGe$.
  Instead of Ve we consider the $\mathfrak{C}$-module $Ve|_{\mathfrak{C}}$.
- We can draw conclusions on V from Ve, but not from $Ve|_{\mathfrak{C}}$.

Generation and Matching

Theorem (F. Noeske, 2005)

Let $H \trianglelefteq N \leq G$. If $\mathcal{T}$ is a set of double coset representatives of $N \backslash G / N$ and $\mathcal{N}$ a set of generators of N, then we have for $e = e_H$:

$$eFGe = F\langle e\mathcal{N}e, e\mathcal{T}e\rangle$$

as F-algebras.

More sophisticated results by Noeske on generation are available, but have not found applications yet.

Matching Problem: Let $e, e' \in FG$ be idempotents. Suppose $S, S' \in \text{mod-}FG$ are simple, and we know Se and $S'e'$. Can we decide if $S \cong S'$? Yes! (Noeske, 2008)

Condensing Projective Modules

Not a new idea, but now feasible through

- improved Condensation techniques
- programs by Jon Carlson for matrix algebras (see next lecture)

If $P = eFG$ is projective, then $\mathrm{End}_{FG}(P) = eFGe = Pe$, and "generation" can be checked. E.g. $\dim(\mathrm{End}_{FG}(P))$ known.

Example ($G = Th$, $p = 5$)

1. *(Done in 2007 with Jon Carlson):*
   $P = e_H FG$, *for* $H = 3xG_2(3)$, $\dim(P) = 7\,124\,544\,000$,
   $\dim_F(\mathrm{End}_{FG}(P)) = 788 \rightsquigarrow$ *some progress*
2. *(Envisaged):*
   $\dim(Q) = 43\,957\,879\,875$, $\dim_F(\mathrm{End}_{FG}(Q)) = 21\,530$
   $\rightsquigarrow$ *almost finish Th modulo* 5

THE FISCHER GROUP Fi_{23}

Let G denote the Fischer group Fi_{23}.

This is a sporadic simple group of order

$$4\,089\,470\,473\,293\,004\,800.$$

G has a maximal subgroup M of index 31 671, isomorphic to $2.Fi_{22}$, the double cover of the Fischer group Fi_{22}.

In joint work with Max Neunhöffer and F. Noeske we have computed the 2-modular character table of G.

SOME REPRESENTATIONS OF Fi_{23}

In the following, let $F = \mathbb{F}_2$, the field with 2 elements.

Let $\Omega := M\backslash G$ and let V denote the corresponding permutation module over F (thus $\dim_F(V) = 31\,671$).

Using the MeatAxe we found: V contains composition factors $1, 782, 1\,494, 3\,588, 19\,940$ (denoted by their degrees).
(This took about 4 days of CPU time in 8 GB main memory.)

Using Condensation we analysed the ten tensor products:

$$782 \otimes 782, 782 \otimes 1\,494, \ldots, 19\,940 \otimes 19\,940.$$

Note: $\dim_F(19\,940 \otimes 19\,940) = 367\,603\,600$.
One such matrix over $\mathbb{F}_2$ would need $\approx 18\,403\,938$ GB.

The Condensation for Fi_{23}

1. We took $H \leq G$, $|H| = 3^9 = 19\,683$.

2. We found that $eFGe$ and FG are Morita equivalent (a posteriori).

3. $\dim_F (19\,940 \otimes 19\,940)e = 25\,542$.

   One such matrix over $\mathbb{F}_2$ needs ≈ 77.8 MB.

   About 1 week of CPU time to compute the action of one element ege on $(19\,940 \otimes 19\,940)e$.

4. Every irreducible FG-module (of the principal 2-block) occurs in $19\,940 \otimes 19\,940$.

THE IRREDUCIBLE BRAUER CHARACTERS OF Fi_{23}

The results of the Condensation and further computations with Brauer characters using GAP and MOC gave all the irreducible 2-modular characters of G.

Degrees of the irreducible 2-modular characters of Fi_{23}:

1,	782,	1 494,	3 588,
19 940,	57 408,	79 442,	94 588,
94 588,	583 440,	724 776,	979 132,
1 951 872,	1 997 872,	1 997 872,	5 812 860,
7 821 240,	8 280 208,	17 276 520,	34 744 192,
73 531 392,	97 976 320,	166 559 744,	504 627 200,
504 627 200.			

Using similar methods, Görgen and Lux have recently computed the irreducible characters of Fi_{23} over $\mathbb{F}_3$.
(Largest condensed module:184 644,
largest module found: 34 753 159.)

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Thank you for your attention!