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Exercise sheet for the Tutorial on Computational Representation Theory within the program *Group Theory and Computational Methods*, ICTS-TIFR, Bangalore, 05 – 14 November 2016

In the following exercises, G is a finite group and F a field.

1. Let N be a normal subgroup of G. Show that every (irreducible) F-representation of G/N yields an (irreducible) F-representation of G.

2. Show that two F-representations of G of degree 1 are equivalent, if and only if they are equal.

3. Let G' denote the commutator subgroup of G. Prove that G and G/G' have the same number of irreducible F-representation of degree 1.

4. Compute (by hand) the ordinary character table of the symmetric group S_3.

5. Compute (by hand) the ordinary character table of the symmetric group S_4. Use the fact that S_4 has a factor group isomorphic to S_3. Construct an irreducible $\mathbb{C}$-representation of S_4 of degree 3 by finding an invariant subspace in a 4-dimensional $\mathbb{C}$-vector space V, on which S_4 acts by permuting four basis vectors $v_1, \ldots, v_4$ in the same way as it permutes the letters $1, \ldots 4$.

6. Let χ and λ be two $\mathbb{C}$-characters of G with $\lambda(1) = 1$. Show that $\chi \cdot \lambda : G \to \mathbb{C}, g \mapsto \chi(g)\lambda(g)$ is a $\mathbb{C}$-character of G.

7. Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote two F-representations of G on the vector spaces U and W, respectively. Show that there is an F-representation $\mathfrak{Z}$ of G on $U \otimes_F W$ such that

   $$(u \otimes w)\mathfrak{Z}(g) = u\mathfrak{X}(g) \otimes w\mathfrak{Y}(g)$$

   for all $u \in U$, $w \in W$ and $g \in G$.

   Compute the character $\chi_{\mathfrak{Z}}$ from the characters $\chi_{\mathfrak{X}}$ and $\chi_{\mathfrak{Y}}$.

8. Let χ be a $\mathbb{C}$-character of G and let $g \in G$. Show that $\chi(g^{-1}) = \overline{\chi(g)}$, where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

9. Prove the *Second Orthogonality Relations*: Let $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_k\}$ denote the set of irreducible $\mathbb{C}$-characters of G, and let $g, h \in G$. Show that

$$\sum_{i=1}^{k} \chi_i(g)\chi_i(h^{-1}) = \begin{cases} |C_G(g)|, & \text{if } g \text{ and } h \text{ are conjugate in } G \\ 0, & \text{otherwise} \end{cases}$$

   Hint: Write the Orthogonality Relations as a matrix equation and use the fact that an invertible matrix commutes with its inverse.

10. Use the character table of the sporadic group M_{11} to show that the alternating group A_5 is a subgroup of M_{11}.

    Hint: Use GAP to compute structure constants of M_{11} and a presentation of A_5, e.g. $A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$.

11. Let F be finite of characteristic p and let G be a p-group. Show that G has a unique irreducible F-representation.

12. Use GAP to compute the p-modular character tables of the alternating group A_5 for $p = 3, 5$. What about $p = 2$? (The p-modular character table of G is the Brauer character table of G with respect to an algebraically closed field F of characteristic p.)

    Hint: Produce projective characters using suitable products of characters or induced characters. Use the fact that $|G|_p$, the p-part of $|G|$, divides $\Phi(1)$ for every PIM Φ.

References

[1] I. M. Isaacs, Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, RI, 2006.

[2] K. Lux and H. Pahlings, Representations of Groups. A computational approach. Cambridge University Press, 2010.