Imprimitive irreducible representations of finite quasisimple groups

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Contents

PROLOGUE: A JOINT PROJECT

The following joint project with William J. Husen and Kay Magaard started back in 1999.

PROJECT

Classify the pairs $(G, G \to \mathrm{SL}(V))$ *such that*

1. G *is a finite quasisimple group,*
2. V *a finite dimensional vector space over a field* $k = \bar{k}$*,*
3. $G \to \mathrm{SL}(V)$ *is absolutely irreducible and imprimitive.*

EXPLANATIONS

1. G *is quasisimple, if* $G = G'$ *and* $G/Z(G)$ *is simple.*
2. $G \to \mathrm{SL}(V)$ *is imprimitive, if* $V = V_1 \oplus \cdots \oplus V_t$, $t > 1$, *the action of* G *permuting the* V_i*.*
   Equivalently, $V \cong \mathrm{Ind}_H^G(V_1) := kG \otimes_{kH} V_1$ *as* kG*-modules, where* $H := \mathrm{Stab}_G(V_1)$*.*

THE FINITE CLASSICAL GROUPS

From now on, k is a finite field, V an n-dimensional k-vector space, and X a finite classical group on V.

To be more specific, $V = \mathbb{F}_q^n$ (i.e. $k = \mathbb{F}_q$), and

1. $X = \mathrm{SL}_n(q) \quad (n \geq 2)$, or
2. $X = \mathrm{Sp}_n(q) \quad (n \geq 4 \text{ even})$, or
3. $X = \Omega_n(q) \quad (n \geq 7 \text{ odd})$, or
4. $X = \Omega_n^{\pm}(q) \quad (n \geq 8 \text{ even})$, or

$V = \mathbb{F}_{q^2}^n$ (i.e. $k = \mathbb{F}_{q^2}$), and

5. $X = \mathrm{SU}_n(q) \quad (n \geq 3)$.

In Cases 2–5, the group X is the stabilizer of a non-degenerate form (symplectic, quadratic or hermitian) on V.

THE ASCHBACHER CLASSIFICATION

Let X be a finite classical group as above.

Overall objective: Determine the maximal subgroups of X.

Approach: Aschbacher's subgroup classification theorem.

There are nine classes of subgroups $\mathcal{C}_1(X), \ldots, \mathcal{C}_8(X)$ and $\mathcal{S}(X)$ of X such that the following holds.

THEOREM (ASCHBACHER, '84)

Let $H \leq X$ be a maximal subgroup of X. Then

$$H \in \cup_{i=1}^{8} \mathcal{C}_i(X) \cup \mathcal{S}(X).$$

But: An element in $\cup_{i=1}^{8} \mathcal{C}_i(X) \cup \mathcal{S}(X)$ is not necessarily a maximal subgroup of X.

Kleidman-Liebeck and Bray-Holt-Roney-Dougal: Determine the maximal subgroups among the members of $\cup_{i=1}^{8} \mathcal{C}_i(X)$ (amot).

THE CLASS $\mathcal{S}(X)$

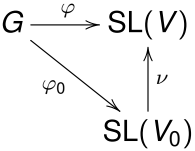
Let $X \leq \mathrm{SL}(V)$ be a finite classical group as above, and $H \lneq X$.

DEFINITION

$H \in \mathcal{S}(X)$, if $H = N_X(G)$, where
$G \leq X$ is quasisimple, such that

1. $\varphi : G \to X \leq \mathrm{SL}(V)$ is absolutely irreducible, and
2. not realizable over a smaller field.

[$\varphi : G \to \mathrm{SL}(V)$ is realizable over a smaller field, if φ factors as



for some proper subfield $k_0 \lneq k$, a k_0-vector space V_0 with $V = k \otimes_{k_0} V_0$, and a representation $\varphi_0 : G \to \mathrm{SL}(V_0)$.]

The structure of $H \in \mathcal{S}(X)$

Let $H = N_X(G) \in \mathcal{S}(X)$.

Put $Z := Z(X)$.

Then

$$C_X(G) = Z = C_X(H) = Z(H),$$

as $\varphi : G \to X$ is absolutely irreducible.

Also, $H/ZG \leq \mathrm{Out}(G)$ is solvable by Schreier's conjecture.

Hence $F^*(H) = ZG$ and thus $G = F^*(H)' = H^\infty$.

Moreover, H/Z is almost simple, i.e. there is a nonabelian simple group S such that H/Z fits into a short exact sequence

$$1 \to S \to H/Z \to \mathrm{Aut}(S) \to 1$$

ON THE MAXIMALITY OF THE ELEMENTS OF $\mathcal{S}(X)$

Let $H = N_X(G) \in \mathcal{S}(X)$.

QUESTION

Is H a maximal subgroup of X?

If not, there is a maximal subgroup K of X with

$$H \lneq K \lneq X.$$

By the definiton of the classes $\mathcal{C}_i(X)$ and $\mathcal{S}(X)$, we have

$$K \in \mathcal{C}_2(X) \cup \mathcal{C}_4(X) \cup \mathcal{C}_7(X) \cup \mathcal{S}(X).$$

In this talk we investigate the possibility $K \in \mathcal{C}_2(X)$, which we call a *$\mathcal{C}_2$-obstruction* to the maximality of H.

THE SUBGROUP CLASS $\mathcal{C}_2(X)'$

Let $H \leq X$. We say that $H \in \mathcal{C}_2(X)'$, if

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_t \tag{1}$$

such that
(a) H permutes the set $\{V_1, \ldots, V_t\}$;
(b) $t \geq 2$;
(c) if $X \neq \mathrm{SL}_n(q)$, then either

- the V_i are non-degenerate and pairwise orthogonal, or
- $t = 2$ and V_1, V_2 are totally singular.

In particular, if $H \in \mathcal{C}_2(X)'$, then $\varphi : H \to X$ is imprimitive.

The group H belongs to $\mathcal{C}_2(X)$, if H is the **full** stabilizer of a decomposition (1) satisfying (a)–(c).

THE $\mathcal{C}_2$-OBSTRUCTION

Let $H = N_X(G) \in \mathcal{S}(X)$. (Recall that $G = H^\infty$.)

PROPOSITION

There exists $H \lneq K \lneq X$ with $K \in \mathcal{C}_2(X)$ if and only if $H \in \mathcal{C}_2(X)'$.

Proof. The *only if* direction is trivial.

Suppose that $H \in \mathcal{C}_2(X)'$, stabilizing a decomposition (1) satisfying (a) – (c).

Let K denote the stabilizer in X of this decomposition.

Then K^∞ does not act absolutely irreducibly on V (by the explicit description of K in all cases for X).

In particular, $H \lneq K$. □

If $H \in \mathcal{C}_2(X)'$, then $G \in \mathcal{C}_2(X)'$, and $\varphi : G \to X$ is imprimitive.

AN EXAMPLE: THE MATHIEU GROUP M_{11}

Let X be a finite classical group.

Let $\varphi : M_{11} \to X$ be absolutely irreducible, faithful, and not realizable over a smaller field. (All such (φ, X) are known.)

Put $G := \varphi(M_{11})$. Then $N_X(G) = Z(X) \times G$.

Is $Z(X) \times G$ maximal in X?

NO, except for $\varphi : M_{11} \to \mathrm{SL}_5(3)$.

EXAMPLES

(1) $M_{11} \to A_{11} \to \Omega^+_{10}(3)$ *($\mathcal{S}$-obstruction).*

(2) $M_{11} \to \mathrm{SO}_{55}(\ell)$ *is imprimitive,* $\ell \geq 5$ *($\mathcal{C}_2$-obstruction).*

(3) Also: $M_{11} \to M_{12} \to A_{12} \to \mathrm{SO}_{11}(\ell) \to \mathrm{SO}_{55}(\ell)$, $\ell \geq 5$.

(4) $M_{11} \to 2.M_{12} \to \mathrm{SL}_{10}(3)$ *($\mathcal{S}$-obstruction).*

(5) $M_{11} \to \mathrm{SL}_5(3) \to \Omega^-_{24}(3)$ *($\mathcal{S}$-obstruction).*

What about $\varphi : M \to \Omega^-_{196882}(2)$?   ($M$: Monster)

THE MAIN OBJECTIVE

To determine the triples (G, k, V) allowing a $\mathcal{C}_2$-obstruction, we plan to perform the following steps.

OBJECTIVE (THE H.-HUSEN-MAGAARD PROJECT)

Determine all triples $(G, \mathbf{k}, \mathbf{V})$ *such that*

- G *is a finite quasisimple group,*
- $\mathbf{k}$ *is an algebraically closed field,*
- $\mathbf{V}$ *is an irreducible, imprimitive* $\mathbf{k}G$*-module,*
- *faithfully representing* G*.*

The case $\text{char}(\mathbf{k}) = 0$ is included as a model for the desired classification; it has provided most of the ideas for an approach to the general case.

The finite quasisimple groups

Theorem

A finite quasisimple group is one of

1. *a covering group of a sporadic simple group;*
2. *a covering group of an alternating group A_n, $n \geq 5$;*
3. *an exceptional covering group of a simple finite reductive group or the Tits simple group;*
4. *a quotient of a quasisimple finite reductive group.*

FINITE REDUCTIVE GROUPS

Let **G** denote a reductive algebraic group over $\bar{\mathbb{F}}$, the algebraic closure of the prime field $\mathbb{F}_p$.

Let F denote a Steinberg morphism of **G**.

Then $G := \mathbf{G}^F$ is a finite reductive group of characteristic p.

An F-stable Levi subgroup **L** of **G** is split, if **L** is a Levi complement in an F-stable parabolic subgroup **P** of **G**.

Such a pair $(\mathbf{L}, \mathbf{P})$ gives rise to a parabolic subgroup $P = \mathbf{P}^F$ of G with Levi complement $L = \mathbf{L}^F$.

THE CHARACTERISTIC 0 CASE

THEOREM

The triples $(G, \mathbf{k}, \mathbf{V})$ *of the main objective, i.e.*

- *G is a finite quasisimple group,*
- *$\mathbf{k}$ is an algebraically closed field,*
- *$\mathbf{V}$ is an irreducible, imprimitive $\mathbf{k}G$-module,*

are known if $char(\mathbf{k}) = 0$.

Here are some references:

1. sporadic groups [H.-Husen-Magaard, ’15]
2. alternating [Djoković-Malzan, ’76; Nett-Noeske, ’11]
3. exceptional covering and Tits [H.-Husen-Magaard, ’15]
4. finite reductive groups [H.-Husen-Magaard, 15’; H.-Magaard, ’16+]

Reductive groups in defining characteristics

The following result of Seitz contains the classification for finite reductive groups in defining characteristic.

Theorem (Gary Seitz, '88)

Let G be a quasisimple finite reductive group over $\mathbb{F}$.

Suppose that $\mathbf{V}$ is an irreducible, imprimitive $\mathbb{F}G$-module.

Then G is one of

$$\mathrm{SL}_2(5), \mathrm{SL}_2(7), \mathrm{SL}_3(2), \mathrm{Sp}_4(3),$$

and $\mathbf{V}$ is the Steinberg module.

Thus it remains to study finite reductive groups in non-defining characteristics.

THE POSITIVE CHARACTERISTIC CASE

THEOREM (H.-HUSEN-MAGAARD, '15)

The triples $(G, \mathbf{k}, \mathbf{V})$ of the main objective, i.e.

- *G is a finite quasisimple group,*
- *$\mathbf{k}$ is an algebraically closed field,*
- *$\mathbf{V}$ is an irreducible, imprimitive $\mathbf{k}G$-module,*

are known if

- *G is sporadic;*
- *G is an exceptional covering group of a finite reductive group or the Tits simple group;*
- *G is a Suzuki or Ree group, $G = G_2(q)$, or G is a Steinberg triality group.*

It remains to consider alternating groups or finite reductive groups in case $p \neq \mathrm{char}(\mathbf{k}) > 0$.

THE MAIN REDUCTION THEOREM

Let G be a quasisimple finite reductive group of characteristic p.

Suppose that G

1. does not have an exceptional Schur multiplier,
2. is not isomorphic to a finite reductive group of a different characteristic.

Let $\mathbf{k}$ be an algebraically closed field with $\mathrm{char}(\mathbf{k}) \neq p$.

THEOREM (H.-HUSEN-MAGAARD, '15)

Let G and $\mathbf{k}$ be as above. Let $H \leq G$ be a maximal subgroup. Suppose that $\mathrm{Ind}_H^G(V_1)$ is irreducible for some $\mathbf{k}H$-module V_1.

Then $H = P$ is a parabolic subgroup of G.

PARABOLIC INDUCTION

Let G be a quasisimple finite reductive group of characteristic p, and let $\mathbf{k}$ be an algebraically closed field with $\mathrm{char}(\mathbf{k}) \neq p$.

According to our main reduction theorem, we may restrict our investigation to parabolic subgroups.

PROPOSITION (H.-HUSEN-MAGAARD, '15)

Let P be a parabolic subgroup of G with unipotent radical U.

Let V_1 be a $\mathbf{k}P$-module such that $\mathrm{Ind}_P^G(V_1)$ is irreducible. Then U is in the kernel of V_1.

In other words, $\mathrm{Ind}_P^G(V_1)$ is Harish-Chandra induced.

This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups or Iwahori-Hecke algebras.

LUSZTIG SERIES

Let $G = \mathbf{G}^F$ be a finite reductive group.
Let $G^* = \mathbf{G}^{*F}$ denote a dual reductive group.
We have

$$\mathrm{Irr}(G) = \bigcup_{[s]} \mathcal{E}(G, [s]),$$

a disjoint union into rational Lusztig series ($[s]$ runs through the G^*-conjugacy classes of semisimple elements of G^*).

THEOREM (LUSZTIG, H.-HUSEN-MAGAARD, '15)

(a) If $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$, where $\mathbf{L}^* \leq \mathbf{G}^*$ is a split Levi subgroup, then every $\chi \in \mathcal{E}(G, [s])$ is Harish-Chandra induced from L.

*(b) Suppose that $C_{\mathbf{G}^*}(s)$ is connected and **not** contained in a proper split Levi subgroup of $\mathbf{G}^*$.*

Then every element of $\mathcal{E}(G, [s])$ is Harish-Chandra primitive.

In particular, the elements of $\mathcal{E}(G, [1])$ are HC-primitive.

NON-CONNECTED CENTRALIZERS

Write $C^\circ_{\mathbf{G}^*}(s)$ for the connected component of $C_{\mathbf{G}^*}(s)$.
Lusztig's generalized Jordan decomposition: There is an equivalence relation $\sim$ on $\mathcal{E}(G,[s])$ and a bijection

$$\mathcal{E}(G,[s])/_\sim \to \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])/_\approx, \quad [\chi] \mapsto [\lambda],$$

where $\approx$ denotes $C_{\mathbf{G}^*}(s)^F$-orbits on $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])$.

THEOREM (H.-MAGAARD, '16+)

Let $\chi \in \mathcal{E}(G,[s])$ and $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])$ with $[\chi] \mapsto [\lambda]$.
(a) *If*

$$C_{\mathbf{G}^*}(s)^F_\lambda\ C^\circ_{\mathbf{G}^*}(s) \leq \mathbf{L}^*, \tag{2}$$

($\mathbf{L}^* \leq \mathbf{G}^*$ split Levi), then χ is Harish-Chandra induced from L.
(b) *Suppose that $\mathbf{G}$ is simple and simply connected.*
If χ is Harish-Chandra imprimitive,
there is a proper split F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ such that Condition (2) is satisfied.

THE CASE OF $\mathrm{SL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then $G^* = G$. Let $s \in G$ be semisimple.

We may write $s = s_1 \oplus s_2 \oplus \cdots \oplus s_e$ with $\mathrm{EV}(s_i) \cap \mathrm{EV}(s_j) = \emptyset$ for $i \neq j$ (where $\mathrm{EV}(s_i)$ = multiset of eigenvalues of s_i). Then

$$C_G(s) = \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_e}(q^{d_e}).$$

We have $\mathcal{E}(G, [s]) = \{\chi_{s,\lambda} \mid \lambda \in \mathcal{E}(C_G(s), [1])\}$, and $\mathcal{E}(C_G(s), [1]) \leftrightarrow \{(\pi_1, \ldots, \pi_e) \mid \pi_i \vdash n_i, 1 \leq i \leq e\}$.

THEOREM (H.-MAGAARD '16+)

Let $\chi := \chi_{s,\lambda} \in \mathcal{E}(G, [s])$ with $\lambda \leftrightarrow (\pi_1, \ldots, \pi_e)$.
Let χ' be any constituent of $\mathrm{Res}^G_{\mathrm{SL}_n(q)}(\chi)$.
Then χ' is Harish-Chandra primitive, if and only if $n_1 = n_2 = \cdots = n_e$, $d_1 = d_2 = \cdots = d_e$, $\pi_1 = \pi_2 = \cdots = \pi_e$, and $\mathrm{EV}(s_i) = \alpha^i \mathrm{EV}(s_1)$ for some $\alpha \in \mathbb{F}_q$.

Thank you for listening!