Finite groups of Lie type and their representations – Lecture II

Gerhard Hiss

Lehrstuhl D für Mathematik
RWTH Aachen University

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CONTENTS

REPRESENTATIONS: DEFINITIONS

Let G be a group and k a field.

A k-representation of G is a homomorphism $\mathfrak{X} : G \to \mathrm{GL}(V)$, where V is a k-vector space. ($\mathfrak{X}$ is also called a representation of G on V.)

If $d := \dim_k(V)$ is finite, d is called the degree of $\mathfrak{X}$.

$\mathfrak{X}$ reducible, if there exists a G-invariant subspace $0 \neq W \neq V$ (i.e. $\mathfrak{X}(g)(w) \in W$ for all $w \in W$ and $g \in G$).

In this case we obtain a sub-representation of G on W and a quotient representation of G on V/W.

Otherwise, $\mathfrak{X}$ is called irreducible.

There is a natural notion of equivalence of k-representations.

COMPOSITION SERIES

Let $\mathfrak{X}$ be a k-representation of G on V with $\dim V < \infty$.

Consider a chain $\{0\} < V_1 < \cdots < V_l = V$ of G-invariant subspaces, such that the representation $\mathfrak{X}_i$ of G on V_i/V_{i-1} is irreducible for all $1 \leq i \leq l$.

Choosing a basis of V through the V_i, we obtain a matrix representation $\tilde{\mathfrak{X}}$ of G, equivalent to $\mathfrak{X}$, s.t.:

$$\tilde{\mathfrak{X}}(g) = \begin{bmatrix} \mathfrak{X}_1(g) & \star & \cdots & \star \\ 0 & \mathfrak{X}_2(g) & \cdots & \star \\ 0 & 0 & \ddots & \star \\ 0 & 0 & \cdots & \mathfrak{X}_l(g) \end{bmatrix} \quad \text{for all } g \in G.$$

The $\mathfrak{X}_i$ (or the V_i/V_{i-1}) are called the irreducible constituents (or composition factors) of $\mathfrak{X}$ (or of V).

They are unique up to equivalence and ordering.

MODULES AND THE GROUP ALGEBRA

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be a k-representation of G on V.

For $v \in V$ and $g \in G$, write $g.v := \mathfrak{X}(g)(v)$.
This makes V into a left kG-module.

Here, kG denotes the group algebra of G over k:

$$kG := \left\{ \sum_{g \in G} a_g g \mid a_g \in k,\ a_g = 0 \text{ for almost all } g \right\},$$

with multiplication inherited from G.

- $\mathfrak{X}$ is irreducible if and only if V is a simple kG-module.
- $\mathfrak{X}$ and $\mathfrak{Y} : G \to \mathrm{GL}(W)$ are equivalent, if and only if V and W are isomorphic as kG-modules.

CLASSIFICATION OF REPRESENTATIONS

Let G now be **finite.**

1. There are only finitely many irreducible k-representations of G up to equivalence.
2. "Classify" all irreducible representations of all finite simple groups.
3. "Most" finite simple groups are groups of Lie type. Find labels for their irreducible representations, find the degrees of these, etc.

THREE CASES

In the following, let $G = \mathbf{G}^F$ be a finite reductive group.

Recall that $\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_p$ and that F is a Frobenius morphism of $\mathbf{G}$.

Let k be algebraically closed with $\mathrm{char}(k) = \ell \geq 0$.

It is natural to distinguish three cases:

1. $\ell = p$ (usually $k = \bar{\mathbb{F}}_p$); defining characteristic
2. $\ell = 0$; ordinary representations
3. $\ell > 0$, $\ell \neq p$; non-defining characteristic

In this lecture, I will talk about Case 1, and the remaining two lectures are devoted to Cases 2 and 3.

A ROUGH SURVEY

Let $k = \bar{\mathbb{F}}_p$ and let $(\mathbf{G}, F)$ be a finite reductive group over k.

By a k-representation of $\mathbf{G}$ we understand a rational hom.

1. An irreducible k-representation of $\mathbf{G}$ has finite degree.
2. The irreducible k-representations of $\mathbf{G}$ are classified by dominant weights, i.e. we have labels for these irreducible k-representations.
3. "Every" irreducible k-representation of $G = \mathbf{G}^F$ is the restriction of an irreducible k-representation of $\mathbf{G}$ to G.

What are dominant weights?

Which irreducible representations of $\mathbf{G}$ restrict to irreducible representations of G?

CHARACTER GROUP AND COCHARACTER GROUP

For the remainder of this lecture, let **G** be a connected reductive algebraic group over $k = \bar{\mathbb{F}}_p$ and let **T** be a maximal torus of **G**. (All of these are conjugate.)

Recall $\mathbf{T} \cong k^* \times k^* \times \cdots \times k^*$. The number r of factors is an invariant of **G**, the rank of **G**.

Put $X := X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, k^*)$. Then $X \cong \bigoplus_1^r \mathrm{Hom}(k^*, k^*)$.

Now $\mathrm{Hom}(k^*, k^*) \cong \mathbb{Z}$, so X is a free abelian group of rank r ($\chi \in \mathrm{Hom}(k^*, k^*)$ is of the form $\chi(t) = t^z$ for some $z \in \mathbb{Z}$).

Similarly, $Y := Y(\mathbf{T}) := \mathrm{Hom}(k^*, \mathbf{T})$ is free abelian of rank r.

X and Y are the character group and cocharacter group, resp.

There is a natural duality $X \times Y \to \mathbb{Z}$, $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle$, defined by $\chi \circ \gamma \in \mathrm{Hom}(k^*, k^*) \cong \mathbb{Z}$.

An example: $\mathrm{GL}_n(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$. Take

$$\mathbf{T} := \{\mathrm{diag}[t_1, t_2, \ldots, t_n] \mid t_1, \ldots, t_n \in k^*\},$$

the maximal torus of diagonal matrices.
(Thus $\mathrm{GL}_n(k)$ has rank n.)

X has basis $\varepsilon_1, \ldots, \varepsilon_n$ with

$$\varepsilon_i(\mathrm{diag}[t_1, t_2, \ldots, t_n]) = t_i.$$

Y has basis $\varepsilon'_1, \ldots \varepsilon'_n$ with

$$\varepsilon'_i(t) = \mathrm{diag}[1, \ldots 1, t, 1, \cdots, 1],$$

where the t is on position i.

Clearly, $\{\varepsilon_i\}$ and $\{\varepsilon'_i\}$ are dual with respect to the pairing $\langle -, - \rangle$.

ROOTS AND COROOTS

Let **B** be a Borel subgroup of **G** containing **T**.

Then $\mathbf{B} = \mathbf{UT}$ with $\mathbf{U} \triangleleft \mathbf{B}$ and $\mathbf{U} \cap \mathbf{T} = \{1\}$. (Recall that **G** has a split *BN*-pair of characteristic *p*.)

The minimal subgroups of **U** normalised by **T** are called root subgroups.

A root subgroup is isomorphic to $\mathbf{G}_a := (k, +)$. The action of **T** on a root subgroup gives rise to a homomorphism $\mathbf{T} \to \mathrm{Aut}(\mathbf{G}_a)$.

Since $\mathrm{Aut}(\mathbf{G}_a) \cong k^*$, we obtain an element of X. The characters obtained this way are the positive roots of **G** w.r.t. **T** and **B**.

The set of positive roots is denoted by Φ^+, and the set $\Phi := \Phi^+ \cup (-\Phi^+) \subset X$ is the root system of **G**.

One can also define a set $\Phi^\vee \subset Y$ of coroots of **G** w.r.t. **T** and **B**.

THE ROOTS AND THE COROOTS OF $\mathrm{GL}_n(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$ and $\mathbf{T}$ be as above. We choose $\mathbf{B}$ as group of upper triangular matrices. Then $\mathbf{U}$ is the subgroup of upper triangular unipotent matrices.

The root subgroups are the groups $\mathbf{U}_{ij} := \{I_n + aI_{ij} \mid a \in k\}$, $1 \leq i < j \leq n$, where I_{ij} denotes the elementary matrix with 1 on position (i, j) and 0 elsewhere.

The positive root α_{ij} determined by $\mathbf{U}_{ij}$ equals $\varepsilon_i - \varepsilon_j$.

Indeed, if $\mathbf{t} = \mathrm{diag}[t_1, \ldots, t_n]$, then $\mathbf{t}(I_n + aI_{ij})\mathbf{t}^{-1} = I_n + t_i t_j^{-1} aI_{ij}$. On the other hand, $(\varepsilon_i - \varepsilon_j)(\mathbf{t}) = t_i t_j^{-1}$.

We have $\Phi = \{\alpha_{ij} \mid \alpha_{ij} = \varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq n\}$ and $\Phi^\vee = \{\alpha_{ij}^\vee \mid \alpha_{ij}^\vee = \varepsilon_i' - \varepsilon_j', 1 \leq i \neq j \leq n\}$.

Note that $\mathbb{Z}\Phi$ and $\mathbb{Z}\Phi^\vee$ have rank $n - 1$.

The root datum

The quadruple $(X, \Phi, Y, \Phi^\vee)$ satisfies:

1. X and Y are free abelian groups of the same rank and there is a duality $X \times Y \to \mathbb{Z}$, $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle$.
2. Φ and $\Phi^\vee$ are finite subsets of X and of Y, respectively, and there is a bijection $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$.
3. For $\alpha \in \Phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$. Denote by s_α the "reflection" of X defined by

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha,$$

   and let $s_\alpha^\vee$ be its adjoint ($s_\alpha^\vee(\gamma) = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee$). Then $s_\alpha(\Phi) = \Phi$ and $s_\alpha^\vee(\Phi^\vee) = \Phi^\vee$.

A quadruple $(X, \Phi, Y, \Phi^\vee)$ as above is called a root datum.

G is determined by its root datum up to isomorphism.

THE WEYL GROUP

The Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ acts on X and we have

$$W \cong \langle s_\alpha \mid \alpha \in \Phi \rangle \leq \mathrm{Aut}(X).$$

Suppose that $\mathbf{G}$ is semisimple. Then rank $X = \operatorname{rank} \mathbb{Z}\Phi$.

In this case Φ is a root system in $V := X \otimes_{\mathbb{Z}} \mathbb{R}$ and W is its Weyl group (where V is equipped with an inner product $(-,-)$ satisfying $\langle \beta, \alpha^\vee \rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ for all $\alpha, \beta \in \Phi$).

W is a Coxeter group with Coxeter generators $\{s_\alpha \mid \alpha \in \Pi\}$, where $\Pi \subset \Phi^+$ is a base of Φ (which is uniquely determined by this property).

WEIGHT SPACES

Let M be a finite-dimensional $k\mathbf{G}$-module.

For $\lambda \in X = X(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, k^*)$ put

$$M_\lambda := \{v \in M \mid tv = \lambda(t)v \text{ for all } t \in \mathbf{T}\}.$$

If $M_\lambda \neq \{0\}$, then λ is called a weight of M and M_λ is the corresponding weight space. (Thus M_λ is a simultaneous eigenspace for all $t \in \mathbf{T}$.)

Crucial Fact:

$$M = \bigoplus_{\lambda \in X} M_\lambda,$$

i.e. M is a direct sum of its weight spaces.

This follows from the fact that the elements of $\mathbf{T}$ act as commuting semisimple linear operators on M.

DOMINANT WEIGHTS AND SIMPLE MODULES

The elements of the set

$$X^+ := \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle \text{ for all } \alpha \in \Phi^+\} \subset X$$

are called dominant weights of $\mathbf{T}$ (w.r.t. Φ^+).
Order X^+ by $\mu \leq \lambda$ if and only if $\lambda - \mu \in \mathbb{N}\Phi^+$.

THEOREM (CHEVALLEY, LATE 1950S)

1. *For each $\lambda \in X^+$ there is a simple $k\mathbf{G}$-module $L(\lambda)$.*
2. $\dim L(\lambda)_\lambda = 1$. *If μ is a weight of $L(\lambda)$, then $\mu \leq \lambda$. (Thus λ is called the highest weight of $L(\lambda)$.)*
3. *If M is a simple $k\mathbf{G}$-module, then $M \cong L(\lambda)$ for some $\lambda \in X^+$.*

$\dim L(\lambda)$ is not known in general.

NATURAL AND ADJOINT REPRESENTATIONS OF $\mathrm{GL}_n(k)$

Let $\mathbf{G} = \mathrm{GL}_n(k)$.

EXAMPLE

1. *Let $M := k^n$ be the natural module of $k\mathbf{G}$.*
   The weights of M are the ε_i, $1 \leq i \leq n$.
   The highest of these is ε_1 (recall that $\varepsilon_i - \varepsilon_j \in \Phi^+$ for $i < j$).
   Thus $M = L(\varepsilon_1)$.
2. *Let $M := \{x \in k^{n\times n} \mid tr(x) = 0\}$. Then M is a simple $k\mathbf{G}$-module by conjugation (the adjoint module).*
   The weights of M are the roots α_{ij} and 0.
   The highest one of these is $\alpha_{1n} = \varepsilon_1 - \varepsilon_n$.
   Thus $M = L(\varepsilon_1 - \varepsilon_n)$.

Steinberg's tensor product theorem

For $q = p^m$, put

$$X_q^+ := \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < q \text{ for all } \alpha \in \Pi\} \subset X^+.$$

Let F denote the standard Frobenius morphism $(a_{ij}) \mapsto (a_{ij}^p)$.
If M is a $k\mathbf{G}$-module, we put $M^{[i]} := M$, with twisted action $g.v := F^i(g).v$, $g \in G$, $v \in M$.

Theorem (Steinberg's tensor product theorem, 1963)

*For $\lambda \in X_q^+$ write $\lambda = \sum_{i=0}^{m-1} p^i \lambda_i$ with $\lambda_i \in X_p^+$.
Then $L(\lambda) = L(\lambda_0) \otimes_k L(\lambda_1)^{[1]} \otimes_k \cdots \otimes_k L(\lambda_{m-1})^{[m-1]}$.*

Theorem (Steinberg, 1963)

*If $\lambda \in X_q^+$, then the restriction of $L(\lambda)$ to $G = \mathbf{G}^{F^m}$ is simple.
If $\mathbf{G}$ is simply connected, i.e. $Y = \mathbb{Z}\Phi^\vee$, then every simple kG-module arises this way.*

The irreducible representations of $\mathrm{SL}_2(k)$ (Brauer-Nesbitt, 1941)

Let $\mathbf{G} = \mathrm{SL}_2(k)$.

Then G acts as group of k-algebra automorphisms on the polynomial ring $k[x_1, x_2]$ in two variables, the action being defined by:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}.$$

For $d = 0, 1, \ldots$ let M_d denote the set of homogeneous polynomials in $k[x_1, x_2]$ of degree d. Then M_d is $\mathbf{G}$-invariant, hence a $k\mathbf{G}$-module, and $\dim M_d = d + 1$.

Moreover, M_d is a simple $k\mathbf{G}$-module, in fact $M_d = L(d\varepsilon_1)$.

Thus $\mathrm{SL}_2(p)$ has exactly the simple modules $M_0, \ldots, M_{p-1}$ of dimensions $1, \ldots, p$.

WEYL MODULES

From now on assume that **G** is simply connected, i.e. $Y = \mathbb{Z}\Phi^\vee$.

For each $\lambda \in X^+$, there is a distinguished finite-dimensional k**G**-module $V(\lambda)$. The $V(\lambda)$s are called Weyl modules.

Construction of $V(\lambda)$ through reduction modulo p.

Recall that **G** is obtained as group of automorphisms of $\mathfrak{g}_k$, where $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{C}$.

For $\lambda \in X^+$, let $V(\lambda)_\mathbb{C}$ be a simple $\mathfrak{g}$-module. This has a suitable $\mathbb{Z}$-form $V(\lambda)_\mathbb{Z}$. Then $V(\lambda) := k \otimes_\mathbb{Z} V(\lambda)_\mathbb{C}$ can be equipped with the structure of a k**G**-module.

FORMAL CHARACTERS

Let M be a finite-dimensional $k\mathbf{G}$-module. Recall that

$$M = \bigoplus_{\lambda \in X} M_\lambda.$$

Clearly, dim M can be recovered by the vector $(\dim M_\lambda)_{\lambda \in X}$.

It is convenient to view this as an element of $\mathbb{Z}X$.

Introduce a $\mathbb{Z}$-basis e^λ, $\lambda \in X$, of $\mathbb{Z}X$ with $e^\lambda e^\mu = e^{\lambda+\mu}$.

DEFINITION

The formal character of M is the element

$$\operatorname{ch} M := \textstyle\sum_{\lambda \in X} \dim M_\lambda \, e^\lambda$$

of $\mathbb{Z}X$.

Characters of Weyl modules

The characters of the Weyl modules $V(\lambda)$ can be computed from Weyl's character formula. In particular, $\dim V(\lambda)$ is known.

Put $a_{\lambda,\mu} := [V(\lambda) \colon L(\mu)] :=$ multiplicity of $L(\mu)$ as a composition factor of $V(\lambda)$.

Fact

$a_{\lambda,\lambda} = 1$, and if $a_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$.

We obviously have

$$\operatorname{ch} V(\lambda) = \operatorname{ch} L(\lambda) + \sum_{\mu<\lambda} a_{\lambda,\mu} \operatorname{ch} L(\mu).$$

Once the $a_{\lambda,\mu}$ are known, $\operatorname{ch} L(\lambda)$ and thus $\dim L(\lambda)$ can be computed recursively from $\operatorname{ch} V(\mu)$ with $\mu \leq \lambda$ (there are only finitely many such μ).

COXETER GROUPS

Let $M = (m_{ij})_{1 \le i,j \le r}$ be a symmetric matrix with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ii} = 1$ and $m_{ij} > 1$ for $i \neq j$.

The group

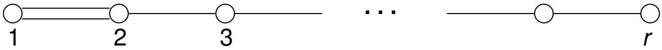
$$W := W(M) := \left\langle s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} = 1 (i \neq j), s_i^2 = 1 \right\rangle_{\text{group}},$$

is called the Coxeter group of M, the elements $s_1, \ldots, s_r$ are the Coxeter generators of W.

The relations $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$ are called the braid relations.

In view of $s_i^2 = 1$, they can be written as $s_i s_j s_i \cdots = s_j s_i s_j \cdots$

The matrix M is usually encoded in a Coxeter diagram, e.g.

B_r:


with number of edges between nodes $i \neq j$ equal to $m_{ij} - 2$.

THE IWAHORI-HECKE ALGEBRA

Let W be a Coxeter group w.r.t. the matrix $M = (m_{ij})$.

Let A be a commutative ring and $v \in A$. The algebra

$$\mathcal{H}_{A,v}(W) := \left\langle T_{s_1}, \ldots, T_{s_r} \mid T_{s_i}^2 = v1 + (v-1)T_{s_i},\ \text{braid rel's} \right\rangle_{A\text{-alg.}}$$

is called the Iwahori-Hecke algebra of W over A with parameter v.

Braid rel's: $T_{s_i} T_{s_j} T_{s_i} \cdots = T_{s_j} T_{s_i} T_{s_j} \cdots$ (m_{ij} factors on each side)

FACT

$\mathcal{H}_{A,v}(W)$ is a free A-algebra with A-basis T_w, $w \in W$.

Note that $\mathcal{H}_{A,1}(W) \cong AW$, so that $\mathcal{H}_{A,v}(W)$ is a deformation of AW, the group algebra of W over A.

KAZHDAN-LUSZTIG POLYNOMIALS

Let W be a Coxeter group as above and let $\leq$ denote the Bruhat order on W.

Let v be an indeterminate, put $A := \mathbb{Z}[v, v^{-1}]$ and $u := v^2$.

There is an involution ι on $\mathcal{H}_{A,u}(W)$ determined by $\iota(v) = v^{-1}$ and $\iota(T_w) = (T_{w^{-1}})^{-1}$ for all $w \in W$.

THEOREM (KAZHDAN-LUSZTIG, 1979)

There is a unique basis C'_w, $w \in W$ of $\mathcal{H}_{A,u}(W)$ such that

1. $\iota(C'_w) = C'_w$ *for all* $w \in W$;
2. $C'_w = v^{-\ell(w)} \sum_{y \leq w} P_{y,w} T_w$ *with* $P_{w,w} = 1$, $P_{y,w} \in \mathbb{Z}[u]$, $\deg P_{y,w} \leq (\ell(w) - \ell(y) - 1)/2$ *for all* $y < w \in W$.

The $P_{y,w} \in \mathbb{Z}[u]$, $y \leq w \in W$, are called the Kazhdan-Lusztig polynomials of W.

THE AFFINE WEYL GROUP

Recall that the Weyl group W acts on X as a group of linear transformations.

Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, and define the dot-action of W as follows:

$$w.\lambda := w(\lambda+\rho)-\rho, \qquad \lambda\in X, w\in W.$$

Define

$$W_p = \langle s_{\alpha,z} \mid \alpha\in\Phi^+, z\in\mathbb{Z}\rangle.$$

Here, $s_{\alpha,z}(\lambda) = s_\alpha.\lambda + zp\alpha$ is an affine reflection of X.

W_p is a Coxeter group, called the affine Weyl group.

Each W_p-orbit on X contains a unique element in $\bar{C} := \{\lambda\in X \mid 0 \le \langle\lambda+\rho,\alpha^\vee\rangle \le p \text{ for all } \alpha\in\Phi^+\}$.

LUSZTIG'S CONJECTURE

Let $\lambda_0 \in X$ with $0 < \langle \lambda_0 + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in \Phi^+$ (such a λ_0 only exists if $p \geq h := h(W)$). Fix $w \in W_p$ such that $w.\lambda_0 \in X_p^+$.

THEOREM (ANDERSEN, JANTZEN, EARLY 80S)

$\mathrm{ch}\, L(w.\lambda_0) = \sum_{w'} b_{w,w'} \,\mathrm{ch}\, V(w'.\lambda_0)$, *with* $w' \in W_p$ *such that* $w'.\lambda_0 \leq w.\lambda_0$ *and* $w'.\lambda_0 \in X^+$. *The* $b_{w,w'}$ *are independent of* λ_0.

For $p \geq h$, the computation of $\mathrm{ch}\, L(\lambda)$ for any $\lambda \in X^+$ can be reduced to one of these cases.

CONJECTURE (LUSZTIG'S CONJECTURE, 1980)

$b_{w,w'} = (-1)^{\ell(w)+\ell(w')} P_{w_0 w', w_0 w}(1)$, *in particular, the* $b_{w,w'}$ *are also independent of p. (*w_0*: longest element in* $W \leq W_p$*.)*

THEOREM (ANDERSEN-JANTZEN-SOERGEL, 1994)

Lusztig's conjecture is true provided $p >> 0$.

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Thank you for your listening!