REPRESENTATION THEORY FOR GROUPS OF LIE TYPE

LECTURE III: DELIGNE-LUSZTIG THEORY

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Contents

RECOLLECTION

Objective: Classify all irreducible representations of all finite simple groups and related finite groups,

find labels for their irreducible representations, find the degrees of these, etc.

In the following: $G = \mathbf{G}^F$ a finite reductive group over $\mathbf{F}$, $\mathrm{char}(\mathbf{F}) = p$,

k an algebraically closed field.

Recall that we distinguish three cases:

1. $\mathrm{char}(k) = p$; defining characteristic
2. $\mathrm{char}(k) = 0$; ordinary representations
3. $0 < \mathrm{char}(k) \neq p$; non-defining characteristic

Today I mainly talk about Case 2, so assume that $\mathrm{char}(k) = 0$ for the time being.

A SIMPLIFICATION: CHARACTERS

Let V, V' be kG-modules.

The character afforded by V is the map

$$\chi_V : G \to k, \quad g \mapsto \mathrm{Trace}(g|V).$$

Characters are class functions.

V and V' are isomorphic, if and only if $\chi_V = \chi_{V'}$.

$\mathrm{Irr}(G) := \{\chi_V \mid V \text{ simple } kG\text{-module}\}$: irreducible characters

$\mathcal{C}$: set of representatives of the conjugacy classes of G

The **square** matrix

$$[\chi(g)]_{\chi \in \mathrm{Irr}(G), g \in \mathcal{C}}$$

is the ordinary character table of G.

AN EXAMPLE: THE ALTERNATING GROUP A_5

EXAMPLE (THE CHARACTER TABLE OF $A_5 \cong \mathrm{SL}_2(4)$)

	$1a$	$2a$	$3a$	$5a$	$5b$
χ_1	1	1	1	1	1
χ_2	3	-1	0	A	$*A$
χ_3	3	-1	0	$*A$	A
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$$A = (1 - \sqrt{5})/2, \qquad *A = (1 + \sqrt{5})/2$$

$$1 \in 1a, \quad (1,2)(3,4) \in 2a, \quad (1,2,3) \in 3a,$$
$$(1,2,3,4,5) \in 5a, \quad (1,3,5,2,4) \in 5b$$

GOALS AND RESULTS

Objective: Describe all ordinary character tables of all finite simple groups and related finite groups.

Almost done:

1. For alternating groups: Frobenius, Schur
2. For groups of Lie type: Green, Deligne, **Lusztig**, Shoji, ... (only "a few" character values missing)
3. For sporadic groups and other "small" groups:

   Atlas of Finite Groups, Conway, Curtis, Norton, Parker, Wilson, 1986

THE GENERIC CHARACTER TABLE FOR $\mathrm{SL}_2(q)$, q EVEN

	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(m)$	$q+1$	1	$\zeta^{am}+\zeta^{-am}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi^{bn}-\xi^{-bn}$

$a, m = 1, \ldots, (q-2)/2, \qquad b, n = 1, \ldots, q/2,$

$\zeta := \exp(\frac{2\pi\sqrt{-1}}{q-1}), \qquad \xi := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

$\begin{bmatrix} \mu^a & 0 \\ 0 & \mu^{-a} \end{bmatrix} \in C_3(a)$ ($\mu \in \mathbb{F}_q$ a primitive $(q-1)$th root of 1)

$\begin{bmatrix} \nu^b & 0 \\ 0 & \nu^{-b} \end{bmatrix} \underset{\sim}{\in} C_4(b)$ ($\nu \in \mathbb{F}_{q^2}$ a primitive $(q+1)$th root of 1)

Specialising q to 4, gives the character table of $\mathrm{SL}_2(4) \cong A_5$.

DRINFELD'S EXAMPLE

The cuspidal simple $k\mathrm{SL}_2(q)$-modules have dimensions $q-1$ and $(q-1)/2$ (the latter only occur if p is odd).

How to construct these?

Consider the affine curve

$$C = \{(x, y) \in \mathbf{F}^2 \mid xy^q - x^q y = 1\}.$$

$G = \mathrm{SL}_2(q)$ acts on C by linear change of coordinates.

Hence G also acts on the étale cohomology group

$$H_c^1(C, \bar{\mathbb{Q}}_\ell),$$

where ℓ is a prime different from p.

It turns out that the simple $\bar{\mathbb{Q}}_\ell G$-submodules of $H_c^1(C, \bar{\mathbb{Q}}_\ell)$ are the cuspidal ones (here $k = \bar{\mathbb{Q}}_\ell$).

DELIGNE-LUSZTIG VARIETIES

Put $k := \bar{\mathbb{Q}}_\ell$.

Deligne and Lusztig (1976) construct for each pair $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, and $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, a generalised character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ of G.
(A generalised character of G is an element of $\mathbb{Z}[\mathrm{Irr}(G)]$.)

Let $(\mathbf{T}, \theta)$ be a pair as above.

Choose a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{T}$.
(In general $\mathbf{B}$ is **not** F-stable.)

Consider the Deligne-Lusztig variety associated to $\mathbf{U}$,

$$Y_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}.$$

This is an algebraic variety over $\mathbf{F}$.

DELIGNE-LUSZTIG GENERALISED CHARACTERS

The finite groups $G = \mathbf{G}^F$ and $T = \mathbf{T}^F$ act on $Y_{\mathbf{U}}$, and these actions commute.

Thus the étale cohomology group $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)$ is a $\bar{\mathbb{Q}}_\ell G$-module-$\bar{\mathbb{Q}}_\ell T$,

and so its θ-isotypic component $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta$ is a $\bar{\mathbb{Q}}_\ell G$-module,

whose character is denoted by ch $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta$.

Only finitely many of the vector spaces $H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)$ are $\neq 0$.

Now put

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \sum_i (-1)^i \mathrm{ch}\, H_c^i(Y_{\mathbf{U}}, \bar{\mathbb{Q}}_\ell)_\theta.$$

This is a Deligne-Lusztig generalised character.

Properties of Deligne-Lusztig characters

The above construction and the following facts are due to Deligne and Lusztig (1976).

Facts

Let $(\mathbf{T}, \theta)$ *be a pair as above. Then*

1. $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *is independent of the choice of* $\mathbf{B}$ *containing* $\mathbf{T}$*.*
2. *If* θ *is in general position, i.e.* $N_G(\mathbf{T}, \theta)/T = \{1\}$*, then* $\pm R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *is an irreducible character.*

Facts (continued)

3. *For* $\chi \in \mathrm{Irr}(G)$*, there is a pair* $(\mathbf{T}, \theta)$ *such that* χ *occurs in* $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$*.*

A GENERALISATION

Instead of a torus **T** one can consider any F-stable Levi subgroup **L** of **G**.

Warning: **L** does in general not give rise to a Levi subgroup of G in the sense of my first lecture.

Consider a parabolic subgroup **P** of **G** with Levi complement **L** and unipotent radical **U**, not necessarily F-stable.

The corresponding Deligne-Lusztig variety $Y_{\mathbf{U}}$ is defined as before: $Y_{\mathbf{U}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in \mathbf{U}\}$.

One gets a Lusztig-induction map $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} : \mathbb{Z}[\mathrm{Irr}(L)] \to \mathbb{Z}[\mathrm{Irr}(G)], \mu \to R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu)$.

PROPERTIES OF LUSZTIG INDUCTION

The above construction and the following facts are due to Lusztig (1976).

Let **L** be an F-stable Levi subgroup of **G** contained in the parabolic subgroup **P**, and let $\mu \in \mathbb{Z}[\mathrm{Irr}(L)]$.

FACTS

1. $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu)(1) = \pm[G:L]_{p'} \cdot \mu(1)$.
2. *If* **P** *is* F*-stable, then* $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(\mu) = R^G_L(\mu)$ *is the Harish-Chandra induced character.*

It is now (almost) known, that $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ is independent of **P**.

UNIPOTENT CHARACTERS

DEFINITION (LUSZTIG)

A character χ of G is called unipotent, if χ is irreducible, and if χ occurs in $R_{\mathbf{T}}^{\mathbf{G}}(\mathbf{1})$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, where $\mathbf{1}$ denotes the trivial character of $T = \mathbf{T}^F$.

We write $\mathrm{Irr}^u(G)$ for the set of unipotent characters of G.

The above definition of unipotent characters uses étale cohomology groups.

So far, no elementary description known, except for $\mathrm{GL}_n(q)$; see below.

Lusztig classified $\mathrm{Irr}^u(G)$ in all cases, **independently** of q.

Harish-Chandra induction preserves unipotent characters (i.e. $\mathrm{Irr}^u(G)$ is a union of Harish-Chandra series), so it suffices to construct the **cuspidal** unipotent characters.

THE UNIPOTENT CHARACTERS OF $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$ and T the torus of diagonal matrices.

Then $\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ occurs in } R_T^G(\mathbf{1})\}$.

Moreover, there is bijection

$$\mathcal{P}_n \leftrightarrow \mathrm{Irr}^u(G), \quad \lambda \leftrightarrow \chi_\lambda,$$

where $\mathcal{P}_n$ denotes the set of partitions of n.

This bijection arises from $\mathrm{End}_{kG}(R_T^G(\mathbf{1})) \cong \mathcal{H}_{k,q}(S_n) \cong kS_n$.

The degrees of the unipotent characters are "polynomials in q":

$$\chi_\lambda(1) = q^{d(\lambda)} \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{\prod_{h(\lambda)}(q^h-1)},$$

with $d(\lambda) \in \mathbb{N}$, and $h(\lambda)$ runs through the hook lengths of λ.

Degrees of the unipotent characters of $GL_5(q)$

λ	$\chi_\lambda(1)$
(5)	1
$(4,1)$	$q(q+1)(q^2+1)$
$(3,2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3,1^2)$	$q^3(q^2+1)(q^2+q+1)$
$(2^2,1)$	$q^4(q^4+q^3+q^2+q+1)$
$(2,1^3)$	$q^6(q+1)(q^2+1)$
(1^5)	q^{10}

JORDAN DECOMPOSITION OF CONJUGACY CLASSES

This is a model classification for $\mathrm{Irr}(G)$.

For $g \in G$ with Jordan decomposition $g = us = su$, we write $C^G_{u,s}$ for the G-conjugacy class containing g.

This gives a labelling

$$\{\text{conjugacy classes of } G\}$$
$$\updownarrow$$
$$\{C^G_{s,u} \mid s \text{ semisimple}, u \in C_G(s) \text{ unipotent}\}.$$

(In the above, the labels s and u have to be taken modulo conjugacy in G and $C_G(s)$, respectively.)

Moreover, $|C^G_{s,u}| = |G : C_G(s)||C^{C_G(s)}_{1,u}|$.

This is the Jordan decomposition of conjugacy classes.

EXAMPLE: THE GENERAL LINEAR GROUP ONCE MORE

$G = \mathrm{GL}_n(q)$, $s \in G$ semisimple. Then

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$$

with $\sum_{i=1}^m n_i d_i = n$. (This gives finitely many class types.)

Thus it suffices to classify the set of unipotent conjugacy classes $\mathcal{U}$ of G.

By Linear Algebra we have

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$C^G_{1,u} \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of q.

In general, i.e. for other groups, it depends slightly on q.

JORDAN DECOMPOSITION OF CHARACTERS

Let $\mathbf{G}^*$ denote the reductive group dual to $\mathbf{G}$.

(Every reductive group has a dual, also reductive.)

EXAMPLES

(1) If $\mathbf{G} = \mathrm{GL}_n(\mathbf{F})$, then $\mathbf{G}^* = \mathbf{G}$.
(2) If $\mathbf{G} = \mathrm{SO}_{2m+1}(\mathbf{F})$, then $\mathbf{G}^* = \mathrm{Sp}_{2m}(\mathbf{F})$.

F gives rise to a Frobenius map on $\mathbf{G}^*$, also denoted by F.

MAIN THEOREM (LUSZTIG; JORDAN DEC. OF CHAR'S, 1984)

Suppose that $Z(\mathbf{G})$ is connected. Then there is a bijection

$$\mathrm{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \textit{ semisimple }, \lambda \in \mathrm{Irr}^u(C_{G^*}(s))\}$$

(where the $s \in G^*$ are taken modulo conjugacy in G^*).
Moreover, $\chi_{s,\lambda}(1) = |G^* \colon C_{G^*}(s)|_{p'}\, \lambda(1)$.

THE IRREDUCIBLE CHARACTERS OF $\mathrm{GL}_n(q)$

Let $G = \mathrm{GL}_n(q)$. Then

$$\mathrm{Irr}(G) = \{\chi_{s,\lambda} \mid s \in G \text{ semisimple}, \lambda \in \mathrm{Irr}^u(C_G(s))\}.$$

We have $C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_m}(q^{d_m})$ with $\sum_{i=1}^m n_i d_i = n$.

Thus $\lambda = \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_m$ with $\lambda_i \in \mathrm{Irr}^u(\mathrm{GL}_{n_i}(q^{d_i})) \longleftrightarrow \mathcal{P}_{n_i}$.

Moreover,

$$\chi_{s,\lambda}(1) = \frac{(q^n-1)\cdots(q-1)}{\prod_{i=1}^m \left[(q^{d_i n_i}-1)\cdots(q^{d_i}-1)\right]} \prod_{i=1}^m \lambda_i(1).$$

DEGREES OF THE IRREDUCIBLE CHARACTERS OF $\mathrm{GL}_3(q)$

$C_G(s)$	λ	$\chi_{s,\lambda}(1)$
$\mathrm{GL}_1(q^3)$	(1)	$(q-1)^2(q+1)$
$\mathrm{GL}_1(q^2)\times\mathrm{GL}_1(q)$	$(1)\boxtimes(1)$	$(q-1)(q^2+q+1)$
$\mathrm{GL}_1(q)^3$	$(1)\boxtimes(1)\boxtimes(1)$	$(q+1)(q^2+q+1)$
$\mathrm{GL}_2(q)\times\mathrm{GL}_1(q)$	$(2)\boxtimes(1)$	q^2+q+1
	$(1,1)\boxtimes(1)$	$q(q^2+q+1)$
$\mathrm{GL}_3(q)$	(3)	1
	$(2,1)$	$q(q+1)$
	$(1,1,1)$	q^3

(This example was already known to Steinberg.)

LUSZTIG SERIES

Lusztig (1988) also obtained a Jordan decomposition for $\mathrm{Irr}(G)$ in case $Z(\mathbf{G})$ is not connected, e.g. if $\mathbf{G} = \mathrm{SL}_n(\mathbf{F})$ or $\mathbf{G} = \mathrm{Sp}_{2m}(\mathbf{F})$ with p odd.

For such groups, $C_{\mathbf{G}^*}(s)$ is not always connected, and the problem is to define $\mathrm{Irr}^u(C_{G^*}(s))$, the unipotent characters.

The Jordan decomposition yields a partition

$$\mathrm{Irr}(G) = \bigcup_{[s] \subset G^*} \mathcal{E}(G, s),$$

where $[s]$ runs through the semisimple G^*-conjugacy classes of G^* and $s \in [s]$.

By definition, $\mathcal{E}(G, s) = \{\chi_{s,\lambda} \mid \lambda \in \mathrm{Irr}^u(C_{G^*}(s)\}$.

For example $\mathcal{E}(G, 1) = \mathrm{Irr}^u(G)$.

The sets $\mathcal{E}(G, s)$ are called rational Lusztig series.

JORDAN DECOMPOSITION IN POSITIVE CHARACTERISTIC?

Now assume that $0 < \mathrm{char}(k) = \ell \neq p$. Write $\mathrm{Irr}_\ell(G) := \mathrm{Irr}_k(G)$. Here, we also have a notion of unipotent characters, $\mathrm{Irr}_\ell^u(G)$. Investigations are guided by the following main conjecture.

CONJECTURE

Suppose that $Z(\mathbf{G})$ is connected. Then there is a labelling

$$\mathrm{Irr}_\ell(G) \leftrightarrow \{\varphi_{s,\mu} \mid s \in G^* \text{ semisimple }, \ell \nmid |s|, \mu \in \mathrm{Irr}_\ell^u(C_{G^*}(s))\},$$

such that $\varphi_{s,\mu}(1) = |G^* \colon C_{G^*}(s)|_{p'}\, \mu(1)$.

Moreover, $\varphi_{s,\mu}$ can be computed from μ.

Known to be true for $\mathrm{GL}_n(q)$ (Dipper-James, 1980s) and if $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ (Bonnafé-Rouquier, 2003). The truth of this conjecture would reduce the computation of $\mathrm{Irr}_\ell(G)$ to unipotent characters.

GENERICITY

Recall that $|G| = q^N \prod_{i=1}^m \Phi_i(q)^{a_i}$, where Φ_i is the i th cyclotomic polynomial and $a_i \in \mathbb{N}$.

If $\ell \mid |G|$ but $\ell \nmid q$, there is a smallest $e \leq m$ with $a_e > 0$ such that $\ell \mid \Phi_e(q)$.

CONJECTURE (GECK)

If ℓ is not too small, there is a natural bijection $\mathrm{Irr}^u(G) \leftrightarrow \mathrm{Irr}^u_\ell(G)$.

In particular, $\mathrm{Irr}^u_\ell(G)$ can be classified independently of q.

CONJECTURE (GENERICITY CONJECTURE)

If ℓ is not too small, the values of the element of $\mathrm{Irr}^u_\ell(G)$ only depend on e, not on ℓ.

This would reduce the computation of $\mathrm{Irr}^u_\ell(G)$ to finitely many cases: finitely many $e \leq m$, finitely many small primes ℓ.

End of Lecture III.

Thank you for your listening!