Problems in the Representation Theory of Finite Groups of Lie Type

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Bristol University, Algebra and Geometry Seminar
10. December 2008

Notation

Throughout this lecture, G denotes a finite group and F a field of characteristic $p \geq 0$.

For simplicity we assume F algebraically closed.

Finite Groups of Lie Type

Finite groups of Lie type are finite analogues of Lie groups.

(There are some series which do not have continuous analogues.)

Examples

- *Classical Groups:* $GL_n(q)$, $GU_n(q)$, $Sp_{2n}(q)$, ... *(q a prime power)*
- *E.g.,* $Sp_{2n}(q) = \{A \in GL_{2n}(q) \mid A^{tr}\tilde{J}A = \tilde{J}\}$, *where*

$$\tilde{J} = \begin{bmatrix} & J \\ -J & \end{bmatrix} \text{ with } J = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{n \times n}.$$

- *Exceptional Groups:* $G_2(q)$, $F_4(q)$, ..., $E_8(q)$
- *Suzuki groups, Ree groups, Steinberg triality groups*

An F-representation of G of degree d is a homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}(V),$$

where V is a d-dimensional F-vector space. (This is also called a representation of G on V.)

$\mathfrak{X}$ reducible, if there exists a G-invariant subspace $0 \neq W \neq V$, (i.e., $\mathfrak{X}(g)w \in W$ for all $w \in W$ and $g \in G$).

In this case we obtain a sub-representation of G on W.

Otherwise, $\mathfrak{X}$ is called irreducible.

There is a natural notion of equivalence of F-representations.

Representations: Classification

1. There are only finitely many irreducible F-representations of G up to equivalence.
2. "Classify" all irreducible representations of all finite simple groups.
3. "Most" finite simple groups are groups of Lie type. Find labels for their irreducible representations, find the degrees of these, etc.

Instead of irreducible representations we can classify their characters.

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be an F-representation of G.

The character afforded by $\mathfrak{X}$ is the map

$$\chi_{\mathfrak{X}} : G \to F, \quad g \mapsto \mathrm{Trace}(\mathfrak{X}(g)).$$

If $\mathfrak{X}$ is irreducible, $\chi_{\mathfrak{X}}$ is called an irreducible character.

Two **irreducible** F-representations are equivalent, if and only if their characters are equal.

$\mathrm{Irr}_F(G) := \{$irreducible F-characters of $G\}$, $\mathrm{Irr}(G) := \mathrm{Irr}_{\mathbb{C}}(G)$.

Three Cases

In the following, let $G = G(q)$ be finite group of Lie type.

Recall that F is an algebraically closed field, $\mathrm{char}(F) = p \geq 0$.

It is natural to distinguish three cases:

- Case 1: $p = 0$ (usually $F = \mathbb{C}$); ordinary representations
- Case 2: $p \mid q$; defining characteristic
- Case 3: $p > 0$, $p \nmid q$; non-defining characteristic

In my talk I will only address Cases 1 and 3.

There is a distinguished class of subgroups of G, the parabolic subgroups.

(In classical groups, parabolic subgroups are the stabilisers of isotropic subspaces.)

A parabolic subgroup P has a Levi decomposition $P = LU$, where U is the unipotent radical of P, and L its Levi subgroup.

Levi subgroups of G resemble G; in particular, they are again groups of Lie type.

Let $G = \mathrm{GL}_n(q)$, $n_1, \ldots, n_k \in \mathbb{N}$ with sum n. Then

$$L = \left\{ \begin{bmatrix} \mathrm{GL}_{n_1}(q) & & \\ & \ddots & \\ & & \mathrm{GL}_{n_k}(q) \end{bmatrix} \right\} \cong GL_{n_1}(q) \times \cdots \times \mathrm{GL}_{n_k}(q)$$

is a typical Levi subgroup of G. A corresponding parabolic is

$$P = \left\{ \begin{bmatrix} \mathrm{GL}_{n_1}(q) & & \\ \star & \ddots & \\ \star & \star & \mathrm{GL}_{n_k}(q) \end{bmatrix} \right\}.$$

In particular, G, and T, the group of diagonal matrices, are Levi subgroups. Corresponding parabolic subgroups are G, and the group of lower triangular matrices, respectively.

Harish-Chandra Induction

Assume from now on $p \nmid q$ (this includes the case $p = 0$).

Let L be a Levi subgroup of G, and $\mathcal{Y} : L \to \mathrm{GL}(V)$ an F-representation of L on V.

View $\mathcal{Y}$ as an F-representation of P via $P \to L$.

Get an F representation $R_L^G(\mathcal{Y})$ on

$$R_L^G(V) := \left\{ f : G \to V \mid \mathcal{Y}(a)f(b) = f(ab) \text{ for all } a \in P, b \in G \right\}.$$

(Modular forms.)

$R_L^G(\mathcal{Y})$ is a Harish-Chandra induced representation.

It is independent of the choice of P with $P \to L$.

With L and $\mathcal{Y}$ as before, put

$$\mathcal{H}(L, \mathcal{Y}) := \mathrm{End}_G(R_L^G(\mathcal{Y})).$$

$\mathcal{H}(L, \mathcal{Y})$ is the centraliser algebra (or Hecke algebra) of the representation $R_L^G(\mathcal{Y})$, i.e., $\mathcal{H}(L, \mathcal{Y}) =$

$$\left\{C \in \mathrm{End}_F(R_L^G(V)) \mid C \cdot R_L^G(\mathcal{Y})(g) = R_L^G(\mathcal{Y})(g) \cdot C \text{ for all } g \in G\right\}.$$

$\mathcal{H}(L, \mathcal{Y})$ is used to analyse (the sub-representations of) $R_L^G(\mathcal{Y})$.

Let $G = \mathrm{GL}_n(q)$, $F = \mathbb{C}$, $L = T$, the group of diagonal matrices of G, $\mathcal{Y}$ the trivial representation of L. Then $\mathcal{H}(L, \mathcal{Y}) = \mathcal{H}_{\mathbb{C},q}(S_n)$, the Iwahori-Hecke algebra over $\mathbb{C}$ with parameter q associated to the Weyl group S_n of G (Iwahori).

Presentation of S_n (as group):

$$\left\langle s_1, \ldots, s_{n-1} \mid \text{ braid relations }, s_i^2 = 1 \right\rangle_{\text{group}}.$$

Here, $s_i = (i, i+1)$, and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i < n, \qquad s_i s_j = s_j s_i, |i-j| \geq 2$$

are the braid relations.

Presentation of $\mathcal{H}_{\mathbb{C},q}(S_n)$ (as $\mathbb{C}$-algebra):

$$\left\langle T_1, \ldots, T_{n-1} \mid \text{ braid relations }, T_i^2 = q1_{\mathbb{C}} + (q-1)\, T_i \right\rangle_{\mathbb{C}\text{-algebra}}.$$

Harish-Chandra Classification

Let $\mathcal{X}$ be an irreducible F-representation of G.

$\mathcal{X}$ is called cuspidal, if $\mathcal{X}$ is **not** a sub-representation of $R_L^G(\mathcal{Y})$ for some **proper** Levi subgroup L of G.

Harish-Chandra theory (HC-induction, cuspidality) yields to the following classification.

Theorem (Harish-Chandra, Lusztig, Geck-H.-Malle)

$$\{\mathcal{X} \mid \mathcal{X} \text{ irreducible } F\text{-representation of } G\} / \text{equivalence}$$

$$\updownarrow$$

$$\left\{ (L, \mathcal{Y}, \theta) \;\middle|\; \begin{array}{c} L \text{ Levi subgroup of } G \\ \mathcal{Y} \text{ irred. cuspidal } F\text{-repr'n of } L \\ \theta \text{ irred. } F\text{-rep'n of } \mathcal{H}(L, \mathcal{Y}) \end{array} \right\} / \text{conjugacy}$$

The above theorem leads to the three tasks:

1. Determine the cuspidal pairs $(L, \mathcal{Y})$.
2. For each of these, "compute" $\mathcal{H}(L, \mathcal{Y})$.
3. Classify irreducible F-representations of $\mathcal{H}(L, \mathcal{Y})$.

State of the art:

- Lusztig completed this program in case $F = \mathbb{C}$. He constructs cuspidal irreducible representations from ℓ-adic cohomology groups of Deligne-Lusztig varieties.
- $\mathcal{H}(L, \mathcal{Y})$ is an Iwahori-Hecke algebra corresponding to a Coxeter group (Lusztig, Howlett-Lehrer, Geck-H.-Malle); parameters of $\mathcal{H}(L, \mathcal{Y})$ not known in general if $p > 0$
- $G = \mathrm{GL}_n(q)$; everything known (Dipper-James)
- G classical group, p "linear"; everything known (Gruber-H.)
- In general, classification of cuspidal pairs open.

This is a model classification for Case 1 (and, perhaps, Case 3).

Let $g \in G$. Then $g = us = su$ with u unipotent, s semisimple (Jordan decomposition of elements). This yields labelling

$$\{\text{conjugacy classes of } G\}$$
$$\updownarrow$$
$$\{C_{s,u} \mid s \text{ semisimple}, u \in C_G(s) \text{ unipotent}\}$$

(In the above, the labels s and u may be taken modulo conjugacy in G and $C_G(s)$, respectively.)

Moreover, $|C_{s,u}| = |G : C_G(s)||Cl_{(u)}^{C_G(s)}|$.

(Jordan decomposition of conjugacy classes.)

EXAMPLE: THE GENERAL LINEAR GROUP

$G = \mathrm{GL}_n(q)$, $s \in G$ semisimple. Then

$$C_G(s) \cong \mathrm{GL}_{n_1}(q^{d_1}) \times \mathrm{GL}_{n_2}(q^{d_2}) \times \cdots \times \mathrm{GL}_{n_k}(q^{d_k})$$

with $\sum_{i=1}^{k} n_i d_i = n$.

Suffices to classify set of unipotent conjugacy classes $\mathcal{U}$ of G.

$$\mathcal{U} \longleftrightarrow \mathcal{P}_n = \{\text{partitions of } n\}$$

$$Cl^G_{(u)} \longleftrightarrow (\text{sizes of Jordan blocks of } u)$$

This classification is generic, i.e., independent of q.

In general, it slightly depends on q.

MAIN THEOREM (LUSZTIG; JORDAN DEC. OF CHAR'S)

Let $F = \mathbb{C}$. Then there is $\mathrm{Irr}^u(G) \subset \mathrm{Irr}(G)$ *(unipotent characters), s.t.:*

$$\mathrm{Irr}(G) \longleftrightarrow \{\chi_{s,\lambda} \mid s \in G^* \textit{ semisimple }, \lambda \in \mathrm{Irr}^u(C_{G^*}(s))\}$$

Moreover, $\chi_{s,\lambda}(1) = |G^* : C_{G^*}(s)|\lambda(1)$.

Lusztig classified $\mathrm{Irr}^u(G)$ in all cases, independently of q.

Definition of unipotent characters via ℓ-adic cohomology groups. (So far, no elementary description known, except for $\mathrm{GL}_n(q)$.)

By Harish-Chandra theory, it suffices to construct the cuspidal unipotent characters.

Example: The General Linear Group

Example

$G = GL_n(q)$. Then $\mathrm{Irr}^u(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi$ occurs in $R_T^G(1)\}$. Moreover, there is bijection $\mathcal{P}_n \leftrightarrow \mathrm{Irr}^u(G)$, $\lambda \leftrightarrow \chi_\lambda$.

The degrees of the unipotent characters are "polynomials in q".

λ	χ_λ
(5)	1
$(4, 1)$	$q(q+1)(q^2+1)$
$(3, 2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3, 1^2)$	$q^3(q^2+1)(q^2+q+1)$
$(2^2, 1)$	$q^4(q^4+q^3+q^2+q+1)$
$(2, 1^3)$	$q^6(q+1)(q^2+1)$
(1^5)	q^{10}

Assume from now on that $\mathrm{char}(F) = p > 0$.

Passage from $\mathbb{C}$-representations to F-representations.

Put $\zeta := \exp(2\pi i/|G|)$, and choose $\bar{\zeta} \in F^*$ of order $|G|_{p'}$.
Obtain homomorphism $\mathbb{Z}[\zeta] \to F$, $\zeta \mapsto \bar{\zeta}$.
Since $\chi(g) \in \mathbb{Z}[\zeta]$ for $\chi \in \mathrm{Irr}(G)$, get $\bar{\chi} : G \to F$.

Fact: $\bar{\chi}$ is the character of some F-representation of G.

Thus there are integers $d_{\chi,\varphi} \geq 0$, $\chi \in \mathrm{Irr}(G)$, $\varphi \in \mathrm{Irr}_F(G)$ s.t.

$$\bar{\chi} = \sum_{\varphi \in \mathrm{Irr}_F(G)} d_{\chi,\varphi}\, \varphi.$$

The $d_{\chi,\varphi}$ are the decomposition numbers of G modulo p.

The matrix $D = [d_{\chi,\varphi}]$ is the decomposition matrix of G.

This yields a definition of $\mathrm{Irr}_F^u(G)$.

Definition (Unipotent F-characters)

$\mathrm{Irr}_F^u(G) = \{\varphi \in \mathrm{Irr}_F(G) \mid d_{\chi,\varphi} \neq 0$ *for some* $\chi \in \mathrm{Irr}^u(G)\}$.

$D^u =$ *restriction of* D *to* $\mathrm{Irr}^u(G) \times \mathrm{Irr}_F^u(G)$.

Theorem (Geck-H.)

(Some conditions apply.)

$|\mathrm{Irr}^u(G)| = |\mathrm{Irr}_F^u(G)|$ *and* D^u *is invertible.*

Conjecture (Geck)

(Some conditions apply.)

There is a natural bijection $\mathrm{Irr}^u(G) \longleftrightarrow \mathrm{Irr}_F^u(G)$.

For $\varphi \in \mathrm{Irr}_F(G)$, we write $\deg(\varphi)$ for the degree of an F-representation with character φ.
($\varphi(1)$ only gives $\deg(\varphi)$ modulo p.)

CONJECTURE

There is a labelling

$$\mathrm{Irr}_F(G) \leftrightarrow \{\varphi_{s,\mu} \mid s \in G^* \textit{ semisimple }, p \nmid |s|, \lambda \in \mathrm{Irr}_F^u(C_{G^*}(s))\},$$

such that $\deg(\varphi_{s,\mu}) = |G^* \colon C_{G^*}(s)| \deg(\mu)$.

Moreover, D can be computed from the various $D^u_{C_{G^*}(s)}$.

Known to be true for $\mathrm{GL}_n(q)$ (Dipper-James) and in many other cases (Bonnafé-Rouquier).

A conjecture of Geck specialises to the following for D^u.

Conjecture (Geck)

(Some conditions apply.)

With respect to suitable orderings of $\mathrm{Irr}^u(G)$ *and* $\mathrm{Irr}^u_F(G)$, D^u *has shape*

$$\begin{bmatrix} 1 & & & & \\ \star & 1 & & & \\ \star & \star & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \star & \star & \star & \star & 1 \end{bmatrix}$$

This would give a canonical bijection $\mathrm{Irr}^u(G) \longleftrightarrow \mathrm{Irr}^u_F(G)$.

About Geck's Conjecture

Geck's conjecture on D^u is known to hold for

- $\mathrm{GL}_n(q)$ (Dipper-James)
- $\mathrm{GU}_n(q)$ (Geck)
- G a classical group and p "linear" (Gruber-H.)
- $\mathrm{Sp}_4(q)$ (White)
- $\mathrm{Sp}_6(q)$ (An-H.)
- $G_2(q)$ (H.)
- $F_4(q)$ (Köhler)
- $E_6(q)$, some cases (Geck-H., Miyachi)
- Steinberg triality groups (Geck)
- Suzuki groups
- Ree groups (Himstedt)

Put $e := \min\{i \mid p \text{ divides } q^i - 1\}$, the order of q in $\mathbb{F}_p^*$.

If G is classical and e is odd, p is linear for G.

EXAMPLE

$G = Sp_{2n}(q)$, $|G| = q^{n^2}(q^2-1)(q^4-1)\cdots(q^{2n}-1)$.
If $p \mid |G|$ and $p \nmid q$, then $p \mid q^{2d} - 1$ for some minimal d.
Thus $p \mid q^d - 1$ (p linear) or $p \mid q^d + 1$.

CONJECTURE (JAMES)

If $G = GL_n(q)$ and $pe > n$, then D^u only depends on e.

THEOREM

(1) Conjecture is true for $n \leq 10$ (James).

(2) If p is "large enough", D^u only depends on e (Geck).

EXAMPLE: THE GENERAL LINEAR GROUP

Let $G = \mathrm{GL}_5(q)$, $e = 2$ (i.e., $p \mid q + 1$ but $p \nmid q - 1$), and assume $p > 2$. Then D^u equals

(5)	1						
$(4, 1)$		1					
$(3, 2)$			1				
$(3, 1^2)$	1		1	1			
$(2^2, 1)$			1	1	1		
$(2, 1^3)$		1				1	
(1^5)	1			1			1

The triangular shape defines φ_λ, $\lambda \in \mathcal{P}_5$.

Example: The General Linear Group

The degrees of the ϕ_λ are "polynomials in q".

λ	$\deg(\varphi_\lambda)$
(5)	1
$(4, 1)$	$q(q+1)(q^2+1)$
$(3, 2)$	$q^2(q^4+q^3+q^2+q+1)$
$(3, 1^2)$	$(q^2+1)(q^5-1)$
$(2^2, 1)$	$(q^3-1)(q^5-1)$
$(2, 1^3)$	$q(q+1)(q^2+1)(q^5-1)$
(1^5)	$q^2(q^3-1)(q^5-1)$

Theorem (Brundan-Dipper-Kleshchev)

The degrees of $\chi_\lambda(1)$ and of $\deg(\varphi_\lambda)$ as polynomials in q are the same.

Boundedness

Let $G = G_n(q)$ be a series of groups of Lie type (n fixed, q variable).

Question

Is an analogue of James' conjecture true in general?

If **yes**, only finitely many matrices D^u to compute (finitely many e's and finitely many "small" p's).

The following is a weaker form.

Conjecture

The entries of D^u are bounded independently of q.

This conjecture is known to be true for $\mathrm{GL}_n(q)$ (Dipper-James), G classical and p linear (Gruber-H.), $\mathrm{GU}_3(q)$, $\mathrm{Sp}_4(q)$ (Okuyama-Waki).

Thank you for your attention!