Computational Representation Theory of Finite Groups

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Throughout my lecture, G denotes a finite group and K a field.

Representations: Definitions

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Choosing a basis of V, we obtain a matrix representation $G \rightarrow \mathsf{GL}_d(K)$ to compute with.

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– Use a computer for sporadic simple groups.

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- from two representations through their Kronecker product,
- from representations through invariant subspaces,
- in various other ways.

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we obtain a K-representation of G.

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We obtain K-representations

$\mathfrak{X}_W : G \to \mathrm{GL}(W)$ and $\mathfrak{X}_{V/W} : G \to \mathrm{GL}(V/W)$

in the natural way.

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one obtains all irreducible representations of G.

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Since then it has been improved and enhanced by many people, including Derek Holt, Gábor Ivanyos, Klaus Lux, Jürgen Müller, Sarah Rees, and Michael Ringe.

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How does one prove that $\mathfrak{X}$ is irreducible?

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Let $B \in \mathfrak{A}$.

Then one of the following occurs:

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4. $\mathfrak{A}$ acts irreducibly on $K^{1\times d}$.

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For one $0 \neq w$ in the nullspace of B^t test if $w.\mathfrak{A}^t = K^{1\times d}$. If YES, $\mathfrak{X}$ is irreducible.

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As K gets larger, it gets harder to find a suitable B by a random search.

Holt and Rees use characteristic polynomials of elements of $\mathfrak{A}$ to find suitable Bs and also to reduce the number of tests considerably.

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the representation of M of degree $196\,882$ over $\mathbb{F}_2$ by Linton, Parker, Walsh, and Wilson.

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A matrix of $M \leq \mathrm{GL}(196\,882, 2)$: 5GB memory

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– PSL$(2, 23)$, is **not** maximal (though in M)

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To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).

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(A and eAe have the same representations.)

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For $g \in G$, need to describe action of ege on Me.

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$$a_{ij} = \frac{1}{|\Omega_j|} |\Omega_i g \cap \Omega_j|.$$

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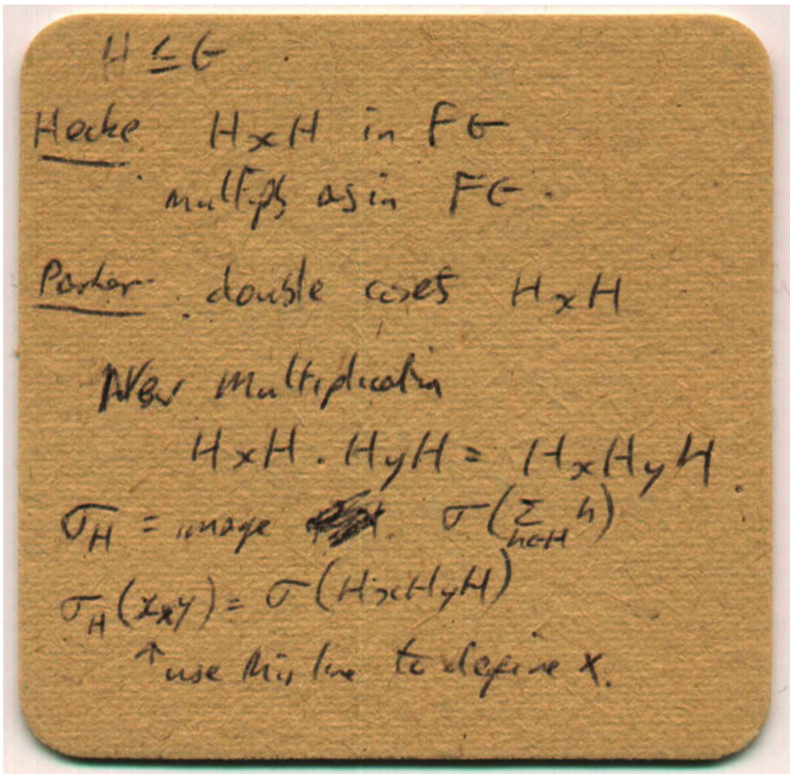

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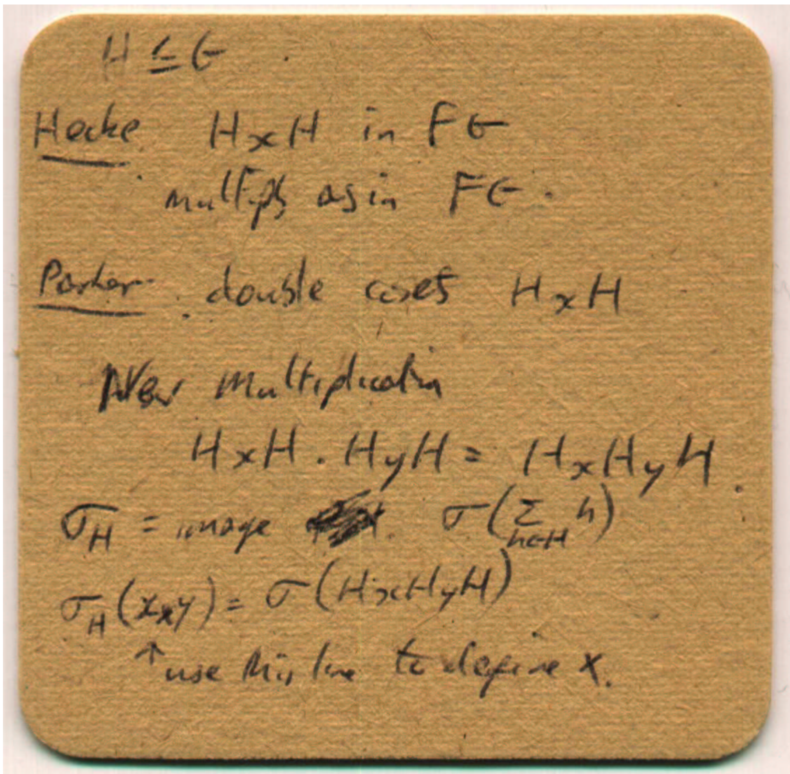

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$|\Omega_i g \cap \Omega_j|$ structure constants of $\mathfrak{B}$, the intersection numbers of $\mathfrak{S}$

(Ω_j orbits of $H :=$ Stab(ω_1) on Ω)

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$\mathbb{Q}\Omega_{m,n} \leq \mathbb{Q}\Omega_{n,m}$, as $\mathbb{Q}S_{mn}$-modules.

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Orbital Graphs as Ramanujan Graphs

Suppose G acts transitively on Ω with orbitals $\mathcal{O}_1, \ldots, \mathcal{O}_m$, adjacency matrices $A_1, \ldots, A_m$.

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If the Bose-Mesner algebra is commutative, these eigenvalues are entries of its character table.

Example: $G = J_2$

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Character table of Bose-Mesner algebra:

J_2	A_1	A_2	A_3	A_4	A_5	A_6
χ_1	1	192	96	192	12	32
χ_2	1	-18	6	2	-3	12
χ_3	1	-28	16	12	7	-8
χ_4	1	0	-12	12	0	-1
χ_5	1	10	-2	-18	5	4
χ_6	1	6	6	-6	-3	-4

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Thank you for your attention!