The State of the Modular Atlas Project

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What is the Modular Atlas Project?

The program to compute the modular (= Brauer) character tables of the ATLAS groups.

ATLAS := *Atlas of Finite Groups*, Conway et al., 1986

Start: PhD-thesis of Gordon James on Mathieu groups (1973)

For Atlas groups up to McL (i.e., order $\leq 10^9$):
An Atlas of Brauer Characters, Jansen et al., 1995

These and more tables are collected on the web site of the Modular Atlas Project (http://www.math.rwth-aachen.de/˜MOC/)

The Players

Wilson
Waki
Thackray
Ryba
Parker
Noeske
Neunhöffer
Müller
Lux
Lübeck
Jansen
James
H.
and many others

Motivation: Classifying Irreducible Representations

Let G be a finite group and F a field.

Fact

There are only finitely many irreducible F-representations of G up to equivalence.

Aims

- *Classify all irreducible representations for a given group G and a given field F.*
- *Describe all irreducible representations of all finite simple groups.*
- *Use a computer for sporadic simple groups.*

Characters: A Simplification

The character afforded by the representation $\mathfrak{X} : G \to \mathrm{GL}(V)$ is the map:

$$\chi_{\mathfrak{X}} : G \to F, \quad g \mapsto \mathrm{Trace}(\mathfrak{X}(g)).$$

It is constant on conjugacy classes: a class function on G.

Equivalent representations have the same character.

Fact

If F has characteristic 0, then two F-representations of G are equivalent if and only if their characters are equal.

Brauer Characters

Assume that F has prime characteristic p, and let $\mathfrak{X}$ be an F-representation of G.

The character $\chi_{\mathfrak{X}}$ has some deficiencies, e.g., $\chi_{\mathfrak{X}}(1)$ only gives the degree of $\mathfrak{X}$ modulo p.

Instead one considers the Brauer (= p-modular) character of $\mathfrak{X}$.

This is obtained by consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{p'}$ to $\mathbb{C}$, where $G_{p'}$ is the set of p-regular elements of G.

Fact

Two irreducible F-representations are equivalent if and only if their Brauer characters are equal.

Example (The 3-Modular Character Table of M_{11}, (James, '73))

	$1a$	$2a$	$4a$	$5a$	$8a$	$8b$	$11a$	$11b$
φ_1	1	1	1	1	1	1	1	1
φ_2	5	1	-1	.	α	$\bar{\alpha}$	γ	$\bar{\gamma}$
φ_3	5	1	-1	.	$\bar{\alpha}$	α	$\bar{\gamma}$	γ
φ_4	10	2	2	.	.	.	-1	-1
φ_5	10	-2	.	.	β	$-\beta$	-1	-1
φ_6	10	-2	.	.	$-\beta$	β	-1	-1
φ_7	24	.	.	-1	2	2	2	2
φ_8	45	-3	1	.	-1	-1	1	1

$$(\alpha = -1 + \sqrt{-2},\ \beta = \sqrt{-2},\ \gamma = (-1 + \sqrt{-11})/2)$$

State of the Art (March 2008) for Sporadic Groups

Grp	Characteristic	
	Known	Not Completely Known
He	all	
Ru	all	
Suz	2–11	13^*
$O'N$	all	
Co_3	all	
Co_2	all	
Fi_{22}	all	
HN	3–19	$2^\dagger$

$*$: *Someone should finish this now* says Jürgen

$\dagger$: Only **one** irreducible character missing

State of the Art, cont.

Grp	Characteristic	
	Known	Not Completely Known
Ly	7, 11, 31, 37, 67	2, 3*, 5
Th	19	2–7, 13, 31
Fi_{23}	2, 5–17, 23	3
Co_1	7–13, 23	2, 3, 5
J_4	5, 7, 37	2, 3, 11, $23^\dagger$, $29^\dagger$, $31^\dagger$, $43^\dagger$
Fi_{24}'	11, 23	2–7, 13, 17, 29
B	11, 23	2–7, 13, 17, 19, 31, 47
M	17, 19, 23, 31	2–13, 29, 41, 47, 59, 71

*: Proved by Jon Thackray “up to condensation”

†: Cooperman, Müller, Robinson (in progress)

Partial Character Tables

Remaining Problems for *Th*

p	No. irr. char's	No. known char's	missing
2	21	8	13
3	16	14	2
5	41	33	8
7	44	30	14

Partial character tables for *HN* (mod 2), *Th* (mod 3, 5, 7), and Co_1 (mod 5) are now on the Modular Atlas Homepage.

These contain bounds for the degress of the missing irreducibles.

Recently Solved Problems, Work in Progress

1. 2-modular table of Fi_{23}, H., Neunhöffer, Noeske, 2006
2. 17-modular table of Fi_{23}, (H., Lux, 1989), Cooperman, Müller, Robinson, 2007
3. 5-modular table of HN, Lux, Noeske, Ryba, 2008
4. 2-modular (**up to one irreducible**) and 3-modular table of HN, H., Müller, Noeske, Thackray, 2008

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5. 2-modular tables of Ly, Th, Co_1, and J_4, Thackray
6. 3-modular table of Ly, Thackray
7. 3-modular and 11-modular tables of J_4, Waki

Idea of Condensation

Let G be a finite group, F a field, and let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be an F-representation of G. (V is an FG-module.)

To compute $\chi_{\mathfrak{X}}(g)$ for $g \in G$, we have to compute the matrix $X(g)$ of the action of g on V.

This is not feasible, if $\dim(V)$ is too large, say $\geq 250\,000$.

For this reason, *Condensation* was invented.

Idea: Let $W \leq V$, and let $e \in \mathrm{End}_F(V)$ be the projection onto W. Note that e is an idempotent, i.e., $e^2 = e$.

Condense $X(g)$ by computing the matrix of ege on $W = eV$.

Problem: How to recover information about $\mathfrak{X}$?

Condensation in Theory . . . [Green 1980]

Let A be a F-algebra and $e \in A$ an idempotent,
i.e., $0 \neq e = e^2$ (a projection).

Get an exact functor: $A\text{-mod} \rightarrow eAe\text{-mod}$, $V \mapsto eV$.

If $S \in A$-mod is simple, then $eS = 0$ or simple (so a composition series of a module V is mapped to a composition series of eV).

If $eS \neq 0$ for all simple $S \in A$-mod, then this functor is an equivalence of categories.

(A and eAe have the same representations.)

. . . and Practice

Let $K \leq G$ with $\mathrm{char}(F) \nmid |K|$. Put

$$e := e_K := \frac{1}{|K|} \sum_{x \in K} x \in FG.$$

Let $V := F\Omega$ be the permutation module w.r.t. an action of G on the finite set Ω. Then eV is the set of K-fixed points in V.

Task: Given $g \in G$, determine action of ege on eV,

without explicit computation of action of g on V!

Theorem (Thackray and Parker, 1981)

This can be done!

...and Practice, cont.

Let V and W be two FG-modules.

Task: Given $g \in G$, determine action of ege on $e(V \otimes W)$,

without explicit computation of action of g on $V \otimes W$!

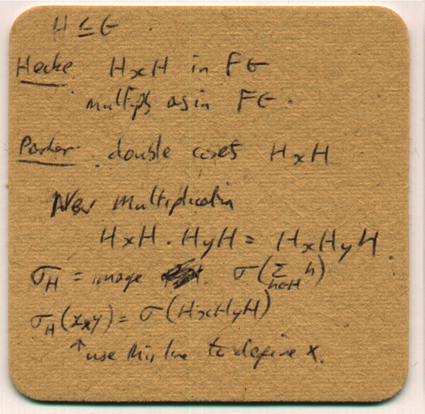
Theorem (Lux and Wiegelmann, 1997)

This can be done!

Other classes of modules, e.g., induced modules, homomorphism spaces of modules, can also be condensed (Lux, Müller, Neunhöffer, Noeske, Rosenboom).

Other idempotents can be used, e.g., $e = 1/|K| \sum_{x \in K} \lambda(x^{-1})x$, where $\lambda : K \to F$ is a homomorphims (Noeske).

Condensation: History



The Generation Problem

We investigate eV through the Meat-Axe, using matrices of generators of $eFGe$.

Question (The Generation Problem)

How can eFGe be generated with "a few" elements?

If $\mathfrak{E} \subseteq FG$ with $\langle \mathfrak{E} \rangle = FG$, then in general $\langle e\mathfrak{E}e \rangle \lneq eFGe$!

- Let $\mathfrak{C} := \langle e\mathfrak{E}e \rangle \leq eFGe$.
  Instead of eV we consider the $\mathfrak{C}$-module $eV|_{\mathfrak{C}}$.
- We can draw conclusions on V from eV, but not from $eV|_{\mathfrak{C}}$.

Generation and Matching

Theorem (F. Noeske, 2005)

Let $K \trianglelefteq N \leq G$. If $\mathfrak{T}$ is a set of double coset representatives of $N \backslash G / N$ and $\mathfrak{N}$ a set of generators of N, then we have

$$eFGe = \langle e\mathfrak{N}e, e\mathfrak{T}e \rangle$$

as F-algebras.

More sophisticated results by Noeske on generation are available, but have not found applications yet.

Matching Problem: Let $e, e' \in FG$ be idempotents. Suppose $S, S' \in FG$-are simple, and we know eS and $e'S'$. Can we decide if $S \cong S'$? Yes! (Noeske)

Condensing Projective Modules

Not a new idea, but now feasible through

- improved condensation techniques
- programs by Jon Carlson for matrix algebras

If $P = eFG$ is a projective FG-module, then $\mathrm{End}_{FG}(P) = eFGe$ decomposes in the same way as P.

Example ($G = Th$, the Thompson Group, $p = 5$)

1. *(Done in 2007 with Jon Carlson):*
   $P = e_K FG$, *for* $K = 3xG_2(3)$, $\dim(P) = 7\,124\,544\,000$,
   $\dim_F(\mathrm{End}_{FG}(P)) = 788 \rightsquigarrow$ *some progress*
2. *(Envisaged):*
   $\dim(Q) = 43\,957\,879\,875$, $\dim_F(\mathrm{End}_{FG}(Q)) = 21\,530$
   $\rightsquigarrow$ *almost finish Th modulo* 5