ON MAXIMAL EMBEDDINGS OF FINITE QUASISIMPLE GROUPS

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DEDICATION

This talk is dedicated to the memory of my friend and colleague Kay Magaard (1962 – 2018).

Contents

MAXIMAL SUBGROUPS OF FINITE GROUPS

Given a finite group G, determine its maximal subgroups; these correspond to the primitive permutation representations of G; moreover, every subgroup of G is contained in a maximal one.

Given a series of finite groups, describe their maximal subgroups in a uniform way.

Determine the maximal subgroups of the finite simple groups.

A large portion of these are closely related to classical groups.

THE CLASSIFICATION OF THE FINITE SIMPLE GROUPS

Groups describe symmetry.

The finite simple groups constitute the elements of symmetry.

THEOREM

Every finite simple group is one of

1. *26 sporadic simple groups; or*
2. *an alternating group A_m with $m \geq 5$; or*
3. *a finite group of Lie type; or*
4. *a cyclic group of prime order.*

Only the groups in 4 are abelian.

QUASISIMPLE GROUPS

Along with the simple groups come the quasisimple groups.

DEFINITION

A finite group G is quasisimple, if

1. G is perfect, i.e. $G = [G, G]$, and
2. $G/Z(G)$ is simple (and then nonabelian).

REMARK

Let S be a nonabelian finite simple group.

1. There is a largest quasisimple group $\widehat{S}$ with $\widehat{S}/Z(\widehat{S}) \cong S$.
2. $\widehat{S}$ is uniquely determined by S (up to isomorphism).
3. $\widehat{S}$ is a universal central extension of S.
4. $\widehat{S}$ is called the Schur covering group of S.

THE FINITE CLASSICAL GROUPS, I

Let k be a finite field of characteristic p, V an n-dimensional k-vector space, and X a finite classical group on V.

To be more specific, $V = \mathbb{F}_q^n$ (i.e. $k = \mathbb{F}_q$ for some $q = p^f$), and

1. $X = \mathrm{SL}(V) = \mathrm{SL}_n(q) \quad (n \geq 2)$, or
2. $X = \mathrm{Sp}(V) = \mathrm{Sp}_n(q) \quad (n \geq 4 \text{ even})$, or
3. $X = \Omega(V) = \Omega_n(q) \quad (n \geq 7 \text{ odd})$, or
4. $X = \Omega(V)^{\pm} = \Omega_n^{\pm}(q) \quad (n \geq 8 \text{ even})$, or

$V = \mathbb{F}_{q^2}^n$ (i.e. $k = \mathbb{F}_{q^2}$), and

5. $X = \mathrm{SU}(V) = \mathrm{SU}_n(q) \quad (n \geq 3)$.

In Cases 2–5, the group X is the stabilizer of a non-degenerate form (symplectic, quadratic or hermitian) on V.

The finite classical groups, II

Let p, n, V and X be as above;
n is called the degree of X and p its characteristic.

Remark

X is quasisimple, except for finitely many cases.

If X is quasisimple, $S := X/Z(X)$, then $X = \widehat{S}$, the Schur covering group of S, except for finitely many cases.

Example (The orthogonal groups)

Let Q be a non-degenerate quadratic form on V. Set

$$O(V, Q) := \{g \in \mathrm{GL}(V) \mid Q(gv) = Q(v) \text{ for all } v \in V\}$$

(the elements of $\mathrm{GL}(V)$ that preserve the form Q);

$$\Omega(V, Q) := [O(V, Q), O(V, Q)]; \quad \rightsquigarrow \quad \Omega_{2m+1}(q), \Omega^{\pm}_{2m}(q).$$

Analogous definitions for the symplectic and unitary groups.

MAXIMAL SUBGROUPS OF CLASSICAL GROUPS

Let X be a finite classical group as above.

Overall objective: Determine the maximal subgroups of X.

If $H \leq X$, then the embedding

$$\varphi : H \to X \to \mathrm{SL}(V)$$

is a representation of H on V.

If φ is reducible, then $H \leq_X K$ with

$$K = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in X \mid A \in GL_a(k), B \in k^{a \times b}, C \in \mathrm{GL}_b(k) \right\}$$

for some $1 < a, b < n$.

Thus H can only be maximal if K is maximal and $H =_X K$.

A FURTHER EXAMPLE

Let $G = \mathrm{SL}_2(\mathbb{F}_r)$ be quasisimple (r a prime power).

Let $k = \mathbb{F}_q$ such that $r - 1 \mid q - 1$ and $r \neq q$.
(Given r, there are infinitely many primes q satisfying this.)

Let $M \leq \mathrm{SL}_{r+1}(k)$ denote the subgroup of monomial matrices.

FACT

There are irreducible representations $\varphi : G \to \mathrm{SL}_{r+1}(k)$, with $\varphi(G) \leq M$.

Put $H := \varphi(G)$ and let $X \leq \mathrm{SL}_{r+1}(k)$ denote the smallest classical group containing H.

Then H is not maximal in X
(otherwise $H = M \cap X$, but $M \cap X$ is not quasisimple).

Still, $N_X(H)$ could be maximal; this depends on …

The Aschbacher classification

Let X be a finite classical group as above.

Aschbacher defines nine classes of subgroups $\mathcal{C}_1(X), \ldots, \mathcal{C}_8(X)$ and $\mathcal{S}(X)$ of X.

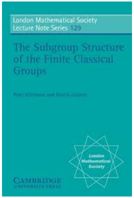
Theorem (Aschbacher, 1984)

Let $H \leq X$ be a maximal subgroup of X. Then

$$H \in \cup_{i=1}^{8} \mathcal{C}_i(X) \cup \mathcal{S}(X).$$

But: An element in $\cup_{i=1}^{8} \mathcal{C}_i(X) \cup \mathcal{S}(X)$ is not necessarily a maximal subgroup of X.

Kleidman-Liebeck/Bray-Holt-Roney-Dougal



[KL] Kleidman-Liebeck (1990): Determine the maximal subgroups among the members of $\cup_{i=1}^{8}\mathcal{C}_i(X)$ for $n \geq 13$ (amot).

[BHRD] Bray-Holt-Roney-Dougal (2013): Determine the maximal subgroups for $n \leq 12$ (amot).

SOME ASCHBACHER CLASSES, I

Let $X \leq \mathrm{SL}(V)$ be a classical group, e.g., $X = \mathrm{Sp}_n(q), \mathrm{SU}_n(q)$.

Let $K \leq X$ and $\varphi : K \to \mathrm{SL}(V)$ the corresponding representation of K.

$\mathcal{C}_1(X)$: K acts reducibly on V

$$K \leq_X \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL_a(k), B \in k^{a \times b}, C \in \mathrm{GL}_b(k) \right\} \leq \mathrm{GL}(V)$$

with $1 < a, b < n$

$\rightsquigarrow$ maximal overgroups of K known [KL,BHRD]

Some Aschbacher classes, II

$\mathcal{C}_2(X)$: K acts irreducibly but imprimitively on V

$$K \leq_X \left\{ \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{A_1} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \boxed{A_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \boxed{A_2} & \vdots & \vdots & \vdots \end{pmatrix} \;\middle|\; A_i \in GL_a(k), 1 \leq i \leq m \right\}$$

$\rightsquigarrow$ maximal overgroups of K known [KL,BHRD]

Some Aschbacher classes, III

$\mathcal{C}_4(X)$, $\mathcal{C}_7(X)$: K preserves a tensor product decomposition of V

φ is tensor decomposable, i.e.,

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_t, \tag{1}$$

such that

φ is equivalent to $\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_t$ (with $\varphi_i : K \to \mathrm{SL}(V_i)$).

$\rightsquigarrow$ maximal overgroups of K known [KL,BHRD]

SOME ASCHBACHER CLASSES, IV

$\mathcal{C}_5(X)$: K is realizable over a smaller field

$\varphi : K \to \mathrm{SL}(V)$ is realizable over a smaller field, if φ factors as

$$\begin{array}{ccc} K & \xrightarrow{\ \varphi\ } & \mathrm{SL}(V) \\ & \searrow^{\varphi_0} & \uparrow \nu \\ & & \mathrm{SL}(V_0) \end{array}$$

for some proper subfield $k_0 \lneq k$, a k_0-vector space V_0 with $V = k \otimes_{k_0} V_0$, and a representation $\varphi_0 : K \to \mathrm{SL}(V_0)$.

$\rightsquigarrow$ maximal overgroups of K known [KL,BHRD]

$\mathcal{C}_8(X)$: $K = X$, i.e. K is a classical group.

$\rightsquigarrow$ maximal overgroups of K known [KL,BHRD]

Some Aschbacher classes: Summary

1. K acts reducibly on V ($\mathcal{C}_1$-case)
   $\rightsquigarrow$ maximal overgroups of K known
2. K acts irreducibly but imprimitively on V ($\mathcal{C}_2$-case)
   $\rightsquigarrow$ maximal overgroups of K are known
3. ...
4. action of K respects a tensor decomposition $V = U \otimes_K W$ ($\mathcal{C}_4$-case)
   $\rightsquigarrow$ maximal overgroups of K are known

$\vdots$

8. $K \in \{\mathrm{Sp}(V), \Omega(V), \Omega^{\pm}(V), \mathrm{SU}(V)\}$ (only if $X = \mathrm{SL}(V)$)
   $\rightsquigarrow$ maximal overgroups of K are known
9. crucial case $K \in \mathcal{S}(X)$: next slide

The class $\mathcal{S}(X)$

Let $H \leq X$.

Definition

$H \in \mathcal{S}(X)$, if $H = N_X(G)$ for some $G \leq X$ with:

1. G is quasisimple,
2. $\varphi : G \to X \hookrightarrow \mathrm{SL}(V)$ is absolutely irreducible,
3. not realizable over a smaller field,
4. G is not a classical group.

The structure of $H \in \mathcal{S}(X)$

Let $H = N_X(G) \in \mathcal{S}(X)$.

Put $Z := Z(X)$.

Then

$$C_X(G) = Z = C_X(H) = Z(H),$$

as $\varphi : G \to X$ is absolutely irreducible.

Also, $H/ZG \leq \mathrm{Out}(G)$ is solvable by Schreier's conjecture.

Hence $G = H^\infty$, the last term in the derived series of H.

Moreover, H/Z is almost simple, i.e. if $S := G/Z(G)$ (recall S is nonabelian simple), then there is a short exact sequence

$$1 \to S \to H/Z \to \mathrm{Aut}(S) \to 1$$

ON THE MAXIMALITY OF THE ELEMENTS OF $\mathcal{S}(X)$

Let $H = N_X(G) \in \mathcal{S}(X)$.

QUESTION

Is H a maximal subgroup of X?

If not, there is a maximal subgroup L of X with

$$H \lneq L \lneq X.$$

Write $\varphi : L \to X \hookrightarrow \mathrm{SL}(V)$ for the embedding.

By the definition of the classes $\mathcal{C}_i(X)$ and $\mathcal{S}(X)$, we have

$$L \in \mathcal{C}_2(X) \cup \mathcal{C}_4(X) \cup \mathcal{C}_6(X) \cup \mathcal{C}_7(X) \cup \mathcal{S}(X).$$

If $L \in \mathcal{C}_i(X)$ (resp. $\mathcal{S}(X)$), we call this a $\mathcal{C}_i$- (resp. $\mathcal{S}$-)*obstruction* to the maximality of H.

SOME OBSTRUCTIONS

$\mathcal{C}_2$-obstruction: $\varphi : L \to X \hookrightarrow \mathrm{SL}(V)$ is imprimitive.

In particular, $\mathrm{Res}^L_G(\varphi) : G \to X \hookrightarrow \mathrm{SL}(V)$ is imprimitive.

Joint project with Kay Magaard:
Classify (G, V, φ), with G quasisimple, $\varphi : G \to \mathrm{SL}(V)$ absolutely irreducible and imprimitive.

$\mathcal{S}$-obstruction: There is a quasisimple group K with $G \lneq K \lneq X$ (take $K = L^\infty$).

In particular, $\mathrm{Res}^K_G(\varphi)$ is absolutely irreducible.

Project of Donna Testerman and Kay Magaard: Classify (G, K, V, φ) with $G \lneq K$ both quasisimple, $\varphi : K \to \mathrm{SL}(V)$ absolutely irreducible, and $\mathrm{Res}^K_G(\varphi)$ absolutely irreducible.

EXAMPLE: MATHIEU GROUP M_{11} (K. MAGAARD)

Let $\varphi : M_{11} \to X$ be absolutely irreducible, faithful, and not realizable over a smaller field. (All such (φ, X) are known.)

Put $G := \varphi(M_{11})$. Then $N_X(G) = Z(X) \times G$.

Is $Z(X) \times G$ maximal in X?

NO, except for $\varphi : M_{11} \to \mathrm{SL}_5(3)$.

EXAMPLES

(1) $M_{11} \to A_{11} \to \Omega_{10}^{+}(3)$ ($\mathcal{S}$-obstruction).

(2) $M_{11} \to \Omega_{55}(q)$ is imprimitive, $q \geq 5$ prime ($\mathcal{C}_2$-obstruction).

(3) Also: $M_{11} \to M_{12} \to A_{12} \to \Omega_{11}(q) \to \Omega_{55}(q)$, $q \geq 5$ prime.

(4) $M_{11} \to 2.M_{12} \to \mathrm{SL}_{10}(3)$ ($\mathcal{S}$-obstruction).

(5) $M_{11} \to \mathrm{SL}_5(3) \to \Omega_{24}^{-}(3)$ ($\mathcal{S}$-obstruction).

EXAMPLE: THE MONSTER GROUP M

Let M denote the Monster group.

There is an absolutely irreducible representation $\varphi : M \to \mathrm{SL}_{196\,882}(2)$.

In fact, $\varphi(M) \leq \Omega^-_{196\,882}(2)$ (Rob Wilson).

Is $G := \varphi(M)$ maximal in $X := \Omega^-_{196\,882}(2)$?

No $\mathcal{C}_i$ obstruction for $i \in \{2, 4, 6, 7\}$.

Is there an $\mathcal{S}$-obstruction L?

No: $K := L^\infty$ is quasisimple; as every non-trivial representation of K would have degree at least 196 882, K would not fit into X.

N.B.: $|M| \sim 8 \cdot 10^{53}$; $|\Omega^-_{196\,882}(2)| \sim 10^{5\,814\,378\,288}$.

The finite quasisimple groups

Recall

A finite quasisimple group is of the form $\widehat{S}/Z$, where $\widehat{S}$ is the Schur covering group of a simple group S and $Z \leq Z(\widehat{S})$.

Moreover, S is one of

1. a sporadic simple group;
2. an alternating group A_m, $m \geq 5$;
3. a simple group of Lie type.

Consequence: All finite quasisimple groups are known.

The invariant $R(G)$

Let G, H be groups, with G finite.
Write $G \preceq H$, if G is isomorphic to a subgroup of H.

Definition (Kleidman-Liebeck)

(a) If p is a prime, set

$$R_p(G) := \min\{0 \neq n \in \mathbb{N} \mid G \preceq \mathrm{PGL}_n(\overline{\mathbb{F}}_p)\}.$$

(b) We also set

$$R(G) := \min\{R_p(G) \mid p \text{ a prime}\}.$$

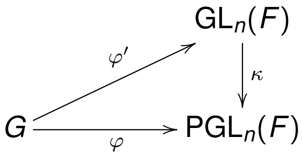
Remark (Kleidman-Liebeck)

$$R_p(G) = \min\{0 \neq n \in \mathbb{N} \mid G \preceq \mathrm{PGL}_n(F), F \text{ a field}, \mathrm{char}(F) = p\}$$

PROJECTIVE REPRESENTATIONS

Let G be a group and F a field.

If $G \preceq \mathrm{PGL}_n(F)$ we get a diagram



where κ denotes the canonical epimorphism.

Choose a section for κ to complete to a commutative diagram.

A projective representation of G is a map $\varphi' : G \to \mathrm{GL}_n(F)$, that fits into such a commutative diagram.

LINEARIZING PROJECTIVE REPRESENTATIONS

Let G be a nonabelian simple group.

Suppose we have a commutative diagram as above, where φ' is absolutely irreducible.

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\ \widehat{\varphi}\ } & \mathrm{GL}_n(F) \\ {\scriptstyle \pi}\big\downarrow & \overset{\varphi'}{\nearrow} & \big\downarrow{\scriptstyle \kappa} \\ G & \xrightarrow[\ \varphi\ ]{} & \mathrm{PGL}_n(F) \end{array}$$

Let $\widehat{G}$ be the Schur covering group of G with canonical map $\pi : \widehat{G} \to G$.

Then there is a representation $\widehat{\varphi} : \widehat{G} \to \mathrm{GL}_n(F)$, completing the diagram.

Some properties of $R(G)$

Let p be a prime, F a field with $\mathrm{char}(F) = p$.

Lemma

Let G and K be quasisimple with $G \leq K$.

Then $R_p(G/Z(G)) \leq R_p(K/Z(K))$.

In particular, $R(G/Z(G)) \leq R(K/Z(K))$.

Lemma

Let S be nonabelian simple, $\hat{S}$ the Schur covering group of S.

(a) *Let φ an F-representation of $\hat{S}$ of degree $R_p(S)$ with non-trivial image. Then φ is absolutely irreducible.*

(b) *If $U \lneq S$ is a proper subgroup, then $R(S) < [S : U]$.*

Some examples

Examples

1. Let A_m denote the alternating group of degree m.

$$R(A_m) = \begin{cases} 2, & \text{if } m = 5, 6 \\ 3, & \text{if } m = 7 \\ 4, & \text{if } m = 8 \\ m-2, & \text{if } m \geq 9 \end{cases}$$

2. Let X be a quasisimple classical group of degree n and characteristic p as on the 8th slide.

   Then $R(X) = n$, up to finitely many exceptions.

   Also $R(X) = R_p(X)$ and $R_r(X) > R_p(X)$ for primes $r \neq p$, up to finitely many exceptions.

The basic setup

Notation

1. S: a finite nonabelian simple group; $n := R(S)$
2. p a prime such that $R(S) = R_p(S)$
3. G: quasisimple covering group of S such that G has a faithful representation φ of degree n over $\overline{\mathbb{F}}_p$
4. q a power of p minimal with: φ is realizable over $\mathbb{F}_q$
5. $V := \mathbb{F}_q^n$; identify G with $\varphi(G) \leq \mathrm{SL}(V)$
6. $X \leq \mathrm{SL}(V)$: smallest classical group containing G

The smallest classical groups containing G

In the above notation, X is isomorphic to one of

1. $\Omega_n(q)$, n odd, or $\Omega_n^{\pm}(q)$, n even
2. $\mathrm{Sp}_n(q)$, n even
3. $\mathrm{SU}_n(q_0)$, $q = q_0^2$
4. $\mathrm{SL}_n(q)$

If G stabilizes a non-degenerate quadratic form on V, then X is as in 1;

else, if G stabilizes a non-degenerate symplectic or hermitian form on V, then X is as in 2 or 3, respectively;

otherwise, $X = \mathrm{SL}(V)$.

X is uniquely determined by G [BHRD].

X is quasisimple, except for finitely many cases.

THE MAIN RESULT, PART I

THEOREM

Assume the above notation. Then one of the following holds:

1. $G = X$.
2. $N_X(G)$ *is a maximal subgroup in* X.
3. $G = {}^2G_2(q)$ *with* $q = 3^{2m+1}$, $m \geq 1$ *and* $n = 7$.
   In this case, $X = \Omega_7(q)$ *and* $G \lneq G_2(q) \lneq X$ *for all* q.
4. $G = {}^2F_4(q)'$ *with* $q = 2^{2m+1}$, $m \geq 0$ *and* $n = 26$.
   In this case, $X = \Omega_{26}^{+}(q)$ *and* $G \lneq F_4(q) \lneq X$ *for all* q.

The groups G in 3 and 4 are the Ree groups (when $m \geq 1$), ${}^2F_4(2)'$ is the Tits group.

THE MAIN RESULT, PART II

THEOREM

Assume the above notation. Apart from 1 – 4, the following possibilities can occur:

5. $G = J_2$, $n = 6$ *and* $q = 4$.
   In this case $X = \mathrm{Sp}_6(4)$ *and* $G \lneq G_2(4) \lneq X$.
6. $G = M_{23}$, $n = 11$ *and* $q = 2$.
   In this case, $X = \mathrm{SL}_{11}(2)$ *and* $G \lneq M_{24} \lneq X$.
7. $G = 3.\mathrm{Fi}_{22}$, $n = 27$ *and* $q = 4$.
   In this case, $X = \mathrm{SU}_{27}(2)$, *and* $G \lneq 3.{}^2E_6(2) \lneq X$.
8. $G = \mathrm{Th}$, $n = 248$ *and* $q = 3$.
   In this case, $X = \Omega^+_{248}(3)$, *and* $G \times 2 \lneq E_8(3) \times 2 \lneq X$.

The first case of the theorem

Suppose that $G = X \lneq \mathrm{SL}(V)$ in the above theorem.

Then G is in Aschbacher class $\mathcal{C}_8$.

In this case, replace X by the smallest classical group Y properly containing G.

In matrix notation, $Y = \mathrm{Sp}_n(q)$ if n and q are even and $G = \Omega_n^{\pm}(q)$.

In all other cases $Y = \mathrm{SL}_n(q) = \mathrm{SL}(V)$.

Then $N_Y(G)$ is a maximal subgroup of Y [KL, BHRD].

Maximal Overgroups

Let $G \lneq K \lneq X$ be as in one of the cases 3 – 8 of the main theorem.

Here K denotes the quasisimple group (given in the theorem) disproving the maximality of $N_X(G)$.

Then $N_X(K)$ is maximal in X.

Moreover, $N_X(K) = K$, except in cases 7 and 8.

In case 7, we have $N_X(K) = 3.{}^2E_6(2).3$.

In case 8, we have $N_X(K) = E_8(3) \times 2$.

SOME REMARKS ON THE PROOFS, I

Recall that $n = R(S)$. For $n \leq 4$, use [BHRD].

Assume that $n \geq 5$.

If S is classical, we get $G = X$ (with one exception), hence Conclusion 1 of the main theorem.

If $G \lneq X$, we get $N_X(G) \lneq X$ (as X is perfect).

Choose maximal subgroup $L \lneq X$ with $N_X(G) \leq L$ and let $\varphi : L \to X \hookrightarrow \mathrm{SL}(V)$ denote the embedding.

What are the possible Aschbacher classes of L?

SOME REMARKS ON THE PROOFS, II

φ is irreducible, as $\mathrm{Res}_G^L(\varphi)$ is, i.e. $L \notin \mathcal{C}_1(X)$.

φ is tensor indecomposable, as $\dim(V) = R_p(G)$, i.e. $L \notin \mathcal{C}_4(X), \mathcal{C}_7(X)$.

φ is primitive, i.e. $L \notin \mathcal{C}_2(X)$.

By definition, G and hence L do not lie in Aschbacher class $\mathcal{C}_5$.

One can also rule out Aschbacher classes $\mathcal{C}_3$, $\mathcal{C}_6$, $\mathcal{C}_8$ for L.

Conclusion: L lies in Aschbacher class $\mathcal{S}(X)$.

Some remarks on the proofs, III

Put $K := L^\infty$. Then K is quasisimple and $G \leq K \lneq X$.

If $G = K$, then $N_X(G) = N_X(K) = L$, hence Conclusion 2.

Assume $G \lneq K$ and put $T := K/Z(K)$. Recall $S = G/Z(G)$.

Then $R_p(T) = R_p(S)$ and $R(T) = R(S)$.

If T is a group of Lie type, its characteristic equals p (with 2 exceptions).

None of S, T is a classical group, as $K \lneq X$ (with 1 exception).

If S and T are exceptional groups of Lie type, we get (G, K) as in Conclusions 3, 4 of the main theorem.

SOME REMARKS ON THE PROOFS, IV

Suppose T is an exceptional groups of Lie type.

Then $R(T) \leq 248$.

Use results of Lübeck and Malle-H., classifying all quasisimple groups with representations of degree ≤ 250.

Yields Conclusions 5, 7, 8 of the main theorem.

If $T = A_m$, then $m = n + 2$, and $G \to K$ yields a 2-transitive permutation representation of G on $n + 2$ points.

Then S known, implying $R(S) < n$, a contradiction.

If T is sporadic, use explicit knowledge of $R(T)$ [Jansen].

This yields Conclusion 6 of the main theorem.

Thank you for your attention!