HARISH-CHANDRA SERIES IN FINITE UNITARY GROUPS AND CRYSTAL GRAPHS

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May 22, 2014

Contents and Acknowledgements

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A preprint with Thomas Gerber and Nicolas Jacon is in preparation.

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Later in this talk I will concentrate on the unitary groups.

The General Unitary Groups

Let $G = \mathrm{GU}_n(q) = \{A \in \mathrm{GL}_n(q^2) \mid A^{tr} J \overline{A} = J\}$, with

$$J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \in \mathbb{F}_q^{n \times n}, \text{ and } \overline{[a_{ij}]} := [a_{ij}^q].$$

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$$\left\{ \begin{bmatrix} A & & \\ & B & \\ & & A^\dagger \end{bmatrix} \mid A \in \mathrm{GL}_m(q^2), B \in \mathrm{GU}_r(q) \right\}$$

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Choosing all possible r, m with $r + 2m = n$, and in $\mathrm{GL}_m(q^2)$ all Levi subgroups, we obtain all Levi subgroups of G.

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The functor

$$R_L^G : kL\text{-mod} \to kG\text{-mod}$$

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For $Y \in kL$-mod, we put $H(L, Y) := \mathrm{End}_{kG}(R_L^G(Y))$ for the Hecke algebra of the pair (L, Y).

Harish-Chandra Classification, I

A simple $X \in kG$-mod is called cuspidal, if $X \not\leq R_L^G(Y)$ for all **proper** Levi subgroups $L \leq G$ and all $Y \in kL$-mod.

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THEOREM (HARISH-CHANDRA ('70), GECK-H.-MALLE ('96))

There is a bijection

$$\{X \mid X \in kG\text{-mod } \textit{simple}\} \,/\, \textit{iso.}$$
$$\updownarrow$$
$$\left\{(L, Y, \theta) \mid \begin{array}{c} (L, Y) \textit{ a cuspidal pair} \\ \theta \in H(L, Y)\text{-mod } \textit{simple} \end{array}\right\} /\, \textit{conj.}$$

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Let $R_L^G(Y) = Z_1 \oplus \cdots \oplus Z_m$ with Z_i indecomposable.

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$$\begin{tikzcd}
\{\text{simple } H(L, Y)\text{-mods}\}/\text{iso.} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, dotted] & \{\text{PIMs of } H(L, Y)\}/\text{iso.} \arrow[d, leftrightarrow] \\
\{X_1, \ldots, X_m\}/\text{iso.} \arrow[r, leftrightarrow] & \{Z_1, \ldots, Z_m\}/\text{iso.}
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*The **Harish-Chandra series** $\mathcal{E}(G; L, Y)$ defined by (L, Y) equals*

$$\{X \leftrightarrow (L, Y, \theta) \mid \theta \in H(L, Y)\text{-mod } \textit{simple}\}/\textit{iso.}$$

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The set of simple kG-modules (upt to isom.) is partitioned into Harish-Chandra series.

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   - $\{Y_\lambda \mid \lambda \vdash n\}$, $\{X_\lambda \mid \lambda \vdash n\}$ unions of Harish-Chandra series

HARISH-CHANDRA SERIES IN CHARACTERISTIC 0, I

Task: Given $\lambda \vdash n$, determine Harish-Chandra series of Y_λ.

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   - *Y_λ and Y_μ are in the same Harish-Chandra series, if and only if $\lambda_{(2)} = \mu_{(2)}$.*
   - *Let $\lambda_{(2)} = \Delta_t$, $r := |\Delta_t|$. Then Y_λ lies in $\mathcal{E}(G; L, Y_{\Delta_t})$, where $L = \mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$ with $n = r + 2m$ (and Y_{Δ_t} viewed as a kL-module).*

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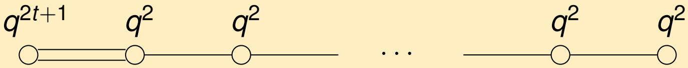
   $H(L, Y) \cong \mathcal{H}_{k,q^{2t+1},q^2}(B_m)$ (an Iwahori-Hecke algebra).



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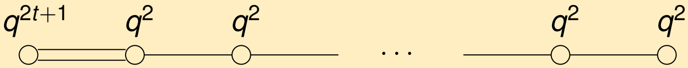

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4. *The bijection*

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   is given by the 2-quotient of a partition.

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2. *Write $\lambda = \lambda_1 + 2\lambda_2$ such that λ_1' is 2-regular, similarly for μ.*

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If e is **even**, division of $\{X_\lambda \mid \lambda \vdash n\}$ into Harish-Chandra series is known (Dipper-James ('86), Fong-Srinivasan ('89), Geck-H.-Malle ('94), Gruber-H. ('97)).

THEOREM (GECK-H.-MALLE, '96)

Suppose that $e = 1$, $\ell > n$ and let $\lambda, \mu \vdash n$. Then

1. *X_λ is cuspidal if and only if λ' is 2-regular.*
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Want: Similar combinatorial description of Harish-Chandra series for **odd** $e > 1$.

EXAMPLE: $\mathrm{GU}_7(q)$, $e = 3$, $\ell > 7$ (DUDAS-MALLE, '13)

7	1														
61	.	1													
52	1	.	1												
51^2	.	.	.	1											
43	.	.	.	.	1										
421	1	.	.	.	.	1									
41^3	.	.	.	.	1	.	1								
3^21	.	.	.	1	.	.	.	1							
32^2	.	.	.	.	.	.	.	.	1						
321^2	.	.	1	.	.	.	.	.	.	1					
31^4	.	.	.	.	.	.	.	.	1	.	1				
2^31	.	.	.	.	.	.	1	.	.	.	.	1			
2^21^3	1	.	1	.	.	1	.	.	.	.	.	.	1		
21^5	.	.	.	.	.	.	.	.	.	.	.	.	.	1	
1^7	.	.	1	.	.	.	.	.	.	1	.	2	1	.	1
	ps	21	ps	ps	21	1^3	21	1^3	ps	B	1^3	c	1^3	21	c

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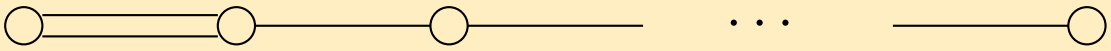
A Levi subgroup of G is called pure*, if it corresponds to a left connected subset of the Dynkin diagram of G.*



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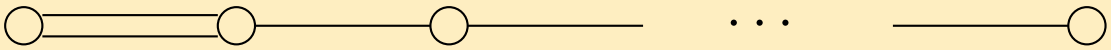

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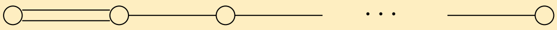

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$$L = \left\{ \begin{bmatrix} A & & \\ & B & \\ & & A^\dagger \end{bmatrix} \mid A \text{ a diagonal matrix}, B \in \mathrm{GU}_r(q) \right\}.$$

HC-CLASSIFICATION WITH PURE LEVI SUBGROUPS

A simple $X \in kG$-mod is called weakly cuspidal, if $X \not\leq R_L^G(Y)$ for all **proper pure** Levi subgroups $L \leq G$ and all $Y \in kL$-mod.

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Harish-Chandra theory yields the following classification.

Theorem (Various authors)

$$\{X \mid X \in kG\text{-mod } \textit{simple}\} \,/\textit{iso.}$$
$$\updownarrow$$
$$\left\{(L, Y, \theta) \;\middle|\; \begin{array}{c} (L, Y) \textit{ a weak cuspidal pair} \\ \theta \in H(L, Y)\text{-mod } \textit{simple} \end{array}\right\} /\textit{conj.}$$

Why it Works

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Remark

Let $L, M \leq G$ be pure Levi subgroups and let $x \in N$. (The N form the BN-pair.) Then

$$L^x \cap M$$

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Proposition

$H(L, Y)$ is a symmetric algebra.

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$\mathrm{char}(k) \neq 0$:

Harish-Chandra series = union of weak Harish-Chandra series
(since a cuspidal is weakly cuspidal)

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7	1														
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2^31	.	.	.	.	.	.	1	.	.	.	.	1			
2^21^3	1	.	1	.	.	1	.	.	.	.	.	.	1		
21^5	.	.	.	.	.	.	.	.	.	.	.	.	.	1	
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	ps	21	ps	ps	21	1^3	21	1^3	ps	wc	1^3	c	1^3	21	c

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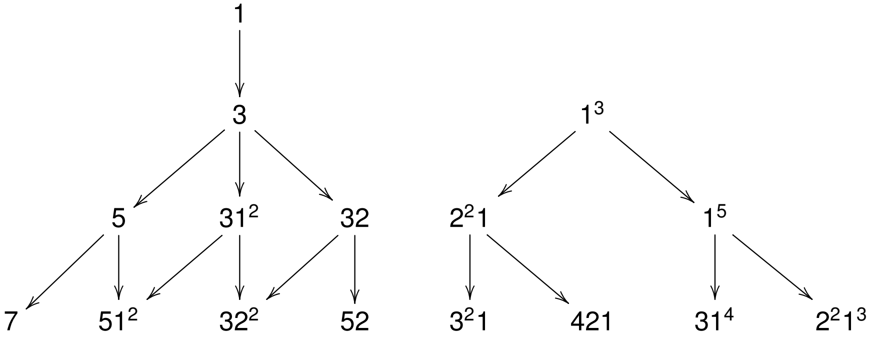
A Truncated Harish-Chandra Branching Graph

Let $\iota = 1$, $\ell \mid q^2 - q + 1$ $(e = 3)$, $n \leq 7$.



Two further root vertices: 1^7, 321^2

A Truncated Harish-Chandra Branching Graph, Continued

$$\begin{tikzcd}
 & & 21 \arrow[dl] \arrow[dr] & & & \\
 & 41 \arrow[dl] \arrow[d] & & 21^3 \arrow[d] \arrow[dr] & & \\
61 & 43 & & 41^3 & 21^5 & 2^3 1
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3. $|(\emptyset, \emptyset), \mathbf{c}\rangle$ is a highest weight vector and

$$\mathcal{U}_v(\widehat{\mathfrak{sl}_e}).|(\emptyset, \emptyset), \mathbf{c}\rangle \cong V(\Lambda(\mathbf{c})),$$

   the simple highest weight module with weight $\Lambda(\mathbf{c})$ (computable from $\mathbf{c}$).

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(Jimbo, Misra, Miwa, Okada ('91); Uglov ('99))

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$$\begin{tikzcd}
 & & \emptyset.\emptyset \arrow[d] & & & & & \\
 & & 1.\emptyset \arrow[dl] \arrow[d] \arrow[dr] & & & \emptyset.1 \arrow[dl] \arrow[dr] & & \\
 & 2.\emptyset \arrow[dl] \arrow[d] & 1.1 \arrow[dl] \arrow[d] & 1^2.\emptyset \arrow[dl] \arrow[d] & \emptyset.2 \arrow[d] \arrow[dr] & & \emptyset.1^2 \arrow[d] \arrow[dr] & \\
3.\emptyset & 2.1 & 1^2.1 & 21.\emptyset & 1.2 & \emptyset.3 & 1.1^2 & \emptyset.21
\end{tikzcd}$$

Two further root vertices: $\emptyset.1^3$, $1^3.\emptyset$

A TRUNCATED HARISH-CHANDRA BRANCHING GRAPH

$$\begin{tikzcd}
 & & 1 \arrow[d] & & & & & \\
 & & 3 \arrow[dl] \arrow[d] \arrow[dr] & & & 1^3 \arrow[dl] \arrow[dr] & & \\
 & 5 \arrow[dl] \arrow[d] & 31^2 \arrow[dl] \arrow[d] & 32 \arrow[dl] \arrow[d] & 2^21 \arrow[d] \arrow[dr] & & 1^5 \arrow[d] \arrow[dr] & \\
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Suppose that e is odd.

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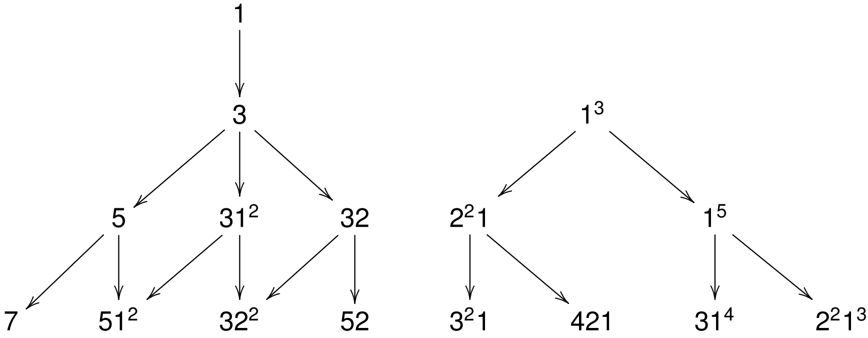
If Conjecture I is true, $\mathcal{B}_{\iota,t}$ is a union of connected components of $\mathcal{B}_\iota$.

A TRUNCATED HARISH-CHANDRA BRANCHING GRAPH

$\mathcal{B}_{1,1}$: all partitions with 2-core (1) $(\ell \mid q^2 - q + 1,\ n \leq 7)$



Two further root vertices: 1^7, 321^2

A TRUNCATED HARISH-CHANDRA BRANCHING GRAPH, CONTINUED

$\mathcal{B}_{1,2}$: all partitions with 2-core (21) $\quad (\ell \mid q^2 - q + 1,\ n \leq 7)$

$$
\begin{tikzcd}
 & & 21 \arrow[dl] \arrow[dr] & & & \\
 & 41 \arrow[dl] \arrow[d] & & 21^3 \arrow[d] \arrow[dr] & & \\
61 & 43 & & 41^3 & 21^5 & 2^31
\end{tikzcd}
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In particular, the root vertices of $\mathcal{G}_{\mathbf{c},e}$ correspond to the weakly cuspidal $k\mathrm{GU}_n(q)$-modules, if $n < \ell$,

and the vertices at distance m from a root vertex of $\mathcal{G}_{\mathbf{c},e}$ label the modules in the weak Harish-Chandra series in $\mathrm{GU}_n(q)$ corresponding to this root vertex for $n = |\Delta_t| + 2m < \ell$.

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   Either of these sets of vertices labels the simple modules of $\mathcal{H}_{k,q^{2t+1},q^2}(B_m)$.

Consequences of Conjecture II

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Let $\mathbf{c} = (t + (1-e)/2, 0)$. Then $|(\emptyset, 1^m), \mathbf{c}\rangle$ is a highest weight vertex in $\mathcal{G}_{\mathbf{c},e}$, if and only if $e \mid 2m + t$ or $e \mid 2m + t - 1$.

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Then X_μ and X_ν lie in distinct ℓ-blocks.

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This follows from a combinatorial description of the highest weight vertices of $\mathcal{G}_{\mathbf{c},e}$ by Jacon and Lecouvey ('13).

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Let $n = r + 2m$, $G = \mathrm{GU}_n(q)$ *and* $L = \mathrm{GU}_r(q) \times \mathrm{GL}_1(q^2)^m$.

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According to Conjecture II, the $(L, X_{(1^4)})$-series is also labelled by the connected component of $\mathcal{G}_{\mathbf{c},3}$ containing $|(\emptyset, 1^2), \mathbf{c}\rangle$ where $\mathbf{c} = (-1, 0)$.

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If $\lambda_{(2)} = \Delta_t$ and $\lambda^{(2)} = \mu$, then the (L, X_λ)-Harish-Chandra series is also labelled by the connected component of $\mathcal{G}_{\mathbf{c},e}$ containing $|\mu,\mathbf{c}\rangle$ with $\mathbf{c} = (t + (1-e)/2, 0)$.

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Proposition

With the notation introduced above, the connected component of $\mathcal{G}_{\mathbf{c},e}$ with highest weight vertex $|\mu,\mathbf{c}\rangle$ is isomorphic (as a directed, non-coloured graph) to the connected component of $\mathcal{G}_{\mathbf{s},e}$ with highest weight vertex $|(\emptyset,\emptyset),\mathbf{s}\rangle$.

Thank you for your attention!