The Modular Atlas Project

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Representations

Let G be a finite group and F a field.

An F-representation of G of degree d is a homomorphism

$$\mathfrak{X} : G \to \mathrm{GL}(V),$$

where V is a d-dimensional F-vector space.

Choosing a basis of V, we obtain a matrix representation $G \to \mathrm{GL}_d(F)$ to compute with.

$\mathfrak{X}$ is irreducible, if V does not have any proper G-invariant subspaces.

Classification

Fact

There are only finitely many irreducible F-representations of G up to equivalence.

Aims

- *Classify all irreducible representations for a given group G and a given field F.*
- *Describe all irreducible representations of all finite simple groups.*
- *Use a computer for sporadic simple groups.*

Characters

The character afforded by the representation $\mathfrak{X}$ is the map:

$$\chi_{\mathfrak{X}} : G \to F, \quad g \mapsto \mathrm{Trace}(\mathfrak{X}(g)).$$

- It is computed via a matrix representation,
- independent of the chosen basis,
- constant on conjugacy classes: a class function on G.

Equivalent representations have the same character.

Fact

If F has characteristic 0, then two F-representations of G are equivalent if and only if their characters are equal.

The Ordinary Character Table

Let $\chi_1, \ldots, \chi_k$ be the irreducible $\mathbb{C}$-characters of G ($F = \mathbb{C}$).

Let $g_1, \ldots, g_k$ be representatives of the conjugacy classes of G (same k as above!).

The square matrix

$$\left[\chi_i(g_j)\right]_{1 \leq i,j \leq k}$$

is called the ordinary character table of G.

Example (The Ordinary Character Table of M_{11})

	$1a$	$2a$	$3a$	$4a$	$5a$	$6a$	$8a$	$8b$	$11a$	$11b$
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	2	.	-1	.	.	-1	-1
χ_3	10	-2	1	.	.	1	α	$-\alpha$	-1	-1
χ_4	10	-2	1	.	.	1	$-\alpha$	α	-1	-1
χ_5	11	3	2	-1	1	.	-1	-1	.	.
χ_6	16	.	-2	.	1	.	.	.	β	$\bar{\beta}$
χ_7	16	.	-2	.	1	.	.	.	$\bar{\beta}$	β
χ_8	44	4	-1	.	-1	1	.	.	.	.
χ_9	45	-3	.	1	.	.	-1	-1	1	1
χ_{10}	55	-1	1	-1	.	-1	1	1	.	.

$$(\alpha = \sqrt{-2},\ \beta = (-1+\sqrt{-11})/2)$$

Brauer Characters

Assume that F has prime characteristic p, and let $\mathfrak{X}$ be an F-representation of G.

The character $\chi_{\mathfrak{X}}$ of $\mathfrak{X}$ as defined above does not convey all the desired information, e.g., $\chi_{\mathfrak{X}}(1)$ only gives the degree of $\mathfrak{X}$ modulo p.

Instead one considers the Brauer character $\varphi_{\mathfrak{X}}$ of $\mathfrak{X}$.

This is obtained by consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{p'}$ to $\mathbb{C}$, where $G_{p'}$ is the set of p-regular elements of G.

Fact

Two irreducible F-representations are equivalent if and only if their Brauer characters are equal.

The Brauer Character Table

(Assume that F is large enough.) Let $\varphi_1, \ldots, \varphi_l$ be the irreducible Brauer characters of G.

Let $g_1, \ldots, g_l$ be representatives of the conjugacy classes contained in $G_{p'}$ (same l as above!).

The square matrix

$$\left[\varphi_i(g_j)\right]_{1 \leq i,j \leq l}$$

is called Brauer character table (or p-modular character table) of G.

Example (The 3-Modular Character Table of M_{11})

	$1a$	$2a$	$4a$	$5a$	$8a$	$8b$	$11a$	$11b$
φ_1	1	1	1	1	1	1	1	1
φ_2	5	1	-1	.	α	$\bar{\alpha}$	γ	$\bar{\gamma}$
φ_3	5	1	-1	.	$\bar{\alpha}$	α	$\bar{\gamma}$	γ
φ_4	10	2	2	.	.	.	-1	-1
φ_5	10	-2	.	.	β	$-\beta$	-1	-1
φ_6	10	-2	.	.	$-\beta$	β	-1	-1
φ_7	24	.	.	-1	2	2	2	2
φ_8	45	-3	1	.	-1	-1	1	1

$$(\alpha = -1 + \sqrt{-2},\ \beta = \sqrt{-2},\ \gamma = (-1 + \sqrt{-11})/2)$$

Goals and Results, I

Aim (I)

Describe all ordinary character tables of all finite simple groups.

Almost done:

- For alternating groups: Frobenius
- For groups of Lie type: Green, Deligne, Lusztig, Shoji, ...
- For sporadic groups and other "small" groups: *Atlas of Finite Groups* (Conway, Curtis, Norton, Parker, Wilson)

The character tables of the Atlas are also contained in GAP (http://www.gap-system.org/) and in Magma (http://magma.maths.usyd.edu.au/magma/).

Goals and Results, II

Aim (II)

Describe all Brauer character tables of all finite simple groups.

Wide open.

For Atlas groups up to McL (i.e., order $\leq 10^9$): *An Atlas of Brauer Characters* (Jansen, Lux, Parker, Wilson)

More information is available from the Web site of the Modular Atlas Project (http://www.math.rwth-aachen.de/~MOC/)

Methods: GAP, MOC, MeatAxe, Condensation

Persons: Wilson, Waki, Thackray, Parker, Noeske, Neunhöffer, Müller, Lux, Jansen, James, H., . . .

State of Art for Sporadic Groups (as of Nov. 2005), I

Grp	Characteristic	
	Known	Not Completely Known
He	all	
Ru	all	
Suz	2–11	13
$O'N$	all	
Co_3	all	
Co_2	all	
Fi_{22}	all	
HN	7, 11, 19	2*, 3*, 5

*: proved by Jon Thackray "up to condensation"

State of Art for Sporadic Groups (as of Nov. 2005), II

Grp	Characteristic	
	Known	Not Completely Known
Ly	7, 11, 31, 37, 67	2, 3*, 5
Th	19	2–7, 13, 31
Fi_{23}	2, 5–13, 23	3, 17
Co_1	7–13, 23	2, 3, 5
J_4	5, 7, 37	2, 3, 11, 23, 29, 31, 43
Fi'_{24}	11, 23	2–7, 13, 17, 29
B	11, 23	2–7, 13, 17, 19, 31, 47
M	17, 19, 23, 31	2–13, 29, 41, 47, 59, 71

*: proved by Jon Thackray “up to condensation”

Constructions

Representations can be constructed

- from permutation representations,
- from two representations through their tensor product,
- from representations through invariant subspaces,
- in various other ways.

Permutation Representations

A permutation representation of G on the finite set Ω is a homomorphism

$$\kappa : G \to S_\Omega,$$

where S_Ω denotes the symmetric group on Ω.

Let $F\Omega$ denote an F-vector space with basis Ω.

Replacing each $\kappa(g) \in S_\Omega$ by the corr. linear map $\mathfrak{X}(g)$ of $F\Omega$ (permuting its basis as $\kappa(g)$),

we obtain an F-representation of G.

$F\Omega$ is called the corresponding permutation module.

Invariant Subspaces

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be an F-representation of G.

For $v \in V$ and $g \in G$, write $g.v := \mathfrak{X}(g)(v)$.
(V is a left FG-module.)

Let W be a G-invariant subspace of V, i.e.:

$$g.w \in W \quad \text{for all } w \in W, g \in G.$$

We obtain F-representations

$\mathfrak{X}_W : G \to \mathrm{GL}(W)$ and $\mathfrak{X}_{V/W} : G \to \mathrm{GL}(V/W)$
in the natural way.

All Irreducible Representations

Iterating the constructions, e.g.,

- F-representations from permutation representations,
- tensor products,
- various others,

and reductions via invariant subspaces,

one obtains all irreducible representations of G.

Theorem (Burnside-Brauer)

Let V be a non-trivial faithful FG-module. Then for every irreducible FG-module W there is an $m \in \mathbb{N}$ such that W is a composition factor of $V^{\otimes m}$.

The MeatAxe

The MeatAxe is a collection of programs that perform the above tasks (for finite fields F).

It was invented and developed by Richard Parker and Jon Thackray around 1980.

Since then it has been improved and enhanced by many people, including Derek Holt, Gábor Ivanyos, Klaus Lux, Jürgen Müller, Felix Noeske, Sarah Rees, and Michael Ringe.

The MeatAxe: Basic Problems

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be an F-representation of G.

Question

How does one find a non-trivial proper G-invariant subspace of V?

- It is enough to find a vector $w \neq 0$ which lies in a proper G-invariant subspace W.
- Indeed, given $0 \neq w \in W$, the orbit $\{g.w \mid g \in G\}$ spans a G-invariant subspace contained in W.

Question

How does one prove that $\mathfrak{X}$ is irreducible?

Norton's Irreducibility Criterion

Let $A_1, \ldots, A_l$, be $(d \times d)$-matrices over F. Put $\mathfrak{A} := F[A_1, \ldots, A_l]$ (algebra span).

Write A^t for the transpose of A, and $\mathfrak{A}^t := F[A_1^t, \ldots, A_l^t]$.

Let $B \in \mathfrak{A}$. Then one of the following occurs:

1. B is invertible.
2. There is a non-trivial vector in the nullspace of B which lies in a proper $\mathfrak{A}$-invariant subspace.
3. Every non-trivial vector in the nullspace of B^t lies in a proper $\mathfrak{A}^t$-invariant subspace.
4. $\mathfrak{A}$ acts irreducibly on F^d.

The MeatAxe: Basic Strategy

If $G = \langle g_1, \ldots, g_l \rangle$, put $A_i := \mathfrak{X}(g_i)$, $1 \leq i \leq l$.

Find singular $B \in \mathfrak{A}$ (by a random search) with nullspace N of small dimension (preferably 1).

For all $0 \neq w \in N$ test if $\mathfrak{A}.w = F^d$. (Note that $\mathfrak{A}.w$ is G-invariant.)

If YES,

For one $0 \neq w$ in the nullspace of B^t test if $\mathfrak{A}^t.w = F^d$.

If YES, $\mathfrak{X}$ is irreducible.

The MeatAxe: Remarks

The above strategy works very well if F is small.

As F gets larger, it gets harder to find a suitable B by a random search.

Holt and Rees use characteristic polynomials of elements of $\mathfrak{A}$ to find suitable Bs and also to reduce the number of tests considerably.

The MeatAxe can handle representations of degree up to 50 000 over $\mathbb{F}_2$.

Over larger fields, only smaller degrees are feasible.

To overcome this problem, The Condensation is used (Thackray and Parker, 1981).

Condensation in Theory . . . [Green 1980]

Let $\mathfrak{A}$ be a F-algebra and $e \in \mathfrak{A}$ an idempotent,
i.e., $0 \neq e = e^2$ (a projection).

Get an exact functor: $\mathfrak{A}\text{-mod} \to e\mathfrak{A}e\text{-mod}$, $V \mapsto eV$.

If $S \in \mathfrak{A}$-mod is simple, then $Se = 0$ or simple (so a composition series of a module V is mapped to a composition series of eV).

If $Se \neq 0$ for all simple $S \in \mathfrak{A}$-mod,
then this functor is an equivalence of categories.

($\mathfrak{A}$ and $e\mathfrak{A}e$ have the same representations.)

... and Practice, I [Thackray and Parker, 1981]

Let $K \leq G$ with $\mathrm{char}(F) \nmid |K|$. Put

$$e := \frac{1}{|K|} \sum_{x \in K} x \in FG.$$

Let $V := F\Omega$ be the permutation module w.r.t. an action of G on the finite set Ω. Then eV is the set of K-fixed points in V.

Task: Given $g \in G$, determine action of ege on eV,

without explicit computation of action of g on V!

Theorem (Thackray and Parker, 1981)

This can be done!

. . . and Practice, II [Lux and Wiegelmann, 1997]

Let V and W be two FG-modules.

Task: Given $g \in G$, determine action of ege on $e(V \otimes W)$,

without explicit computation of action of g on $V \otimes W$!

Theorem (Lux and Wiegelmann, 1997)

This can be done!

The Fischer Group Fi_{23}

Let G denote the Fischer group Fi_{23}.

This is a sporadic simple group of order

$$4\,089\,470\,473\,293\,004\,800.$$

It was discovered and constructed by Bernd Fischer in 1971.

G has a maximal subgroup H of index 31 671, isomorphic to $2.Fi_{22}$, the double cover of the Fischer group Fi_{22}.

In joint work with Max Neunhöffer and F. Noeske we have computed the 2-modular character table of G.

Some Representations of Fi_{23}

In the following, let F denote a finite field of characteristic 2.

Let $\Omega := G/H$ and put $V = F\Omega$, the corresponding permutation module over F (thus $\dim_F(V) = 31\,671$).

Using the MeatAxe we found: V contains composition factors 1, 782, 1 494, 3 588, 19 940 (denoted by their degrees).
(This took about 4 days of CPU time in 8 GB main memory.)

Using the Condensation we analyzed the ten tensor products:

$$782 \otimes 782, 782 \otimes 1\,494, \ldots, 19\,940 \otimes 19\,940.$$

Note: $\dim_F 19\,940 \otimes 19\,940 = 367\,603\,600$. One such matrix over $\mathbb{F}_2$ would need $\approx 18\,403\,938$ GB.

The Condensation for Fi_{23}

- We took $K \leq G$, $|K| = 3^9 = 19\,683$.
- We found that $eFGe$ and FG are Morita equivalent.
- $\dim_F e(19\,940 \otimes 19\,940) = 25\,542$.

  One such matrix over $\mathbb{F}_2$ needs $\approx$ 77,8 MB.

  About 1 week of CPU time to compute the action of one element ege on $e(19\,940 \otimes 19\,940)$.
- Every irreducible FG-module (of the principal 2-block) occurs in $19\,940 \otimes 19\,940$.
- Now we are done, aren't we? Unfortunately not.

The Generation Problem

Recall: We investigate eV using matrices of generators of $eFGe$.

Question (The Generation Problem)

How can eFGe be generated with "a few" elements?

If $\mathcal{E} \subseteq FG$ with $\langle \mathcal{E} \rangle = FG$, then in general $\langle e\mathcal{E}e \rangle \lneq eFGe$!

- Let $\mathcal{C} := \langle e\mathcal{E}e \rangle \leq eFGe$.
  Instead of eV we consider the $\mathcal{C}$-module $eV|_{\mathcal{C}}$.
- Contrary to eV we can not directly draw conclusions on V from $eV|_{\mathcal{C}}$.

Generation

Let $K \trianglelefteq N \leq G$.

Theorem (F. Noeske, 2005)

If $\mathcal{T}$ is a set of double coset representatives of $N\backslash G/N$ and $\mathcal{N}$ a set of generators of N, then we have

$$eFGe = \langle e\mathcal{N}e, e\mathcal{T}e \rangle$$

as F-algebras.

- N the 7th maximal subgroup, $[G : N] = 1\,252\,451\,200$
- $|\mathcal{T}| = 36$ and $|\mathcal{N}| = 3$, i.e. 38 generators for $eFGe$.
- Computation of $\mathcal{T}$: This is a HUGE task, completed by Max Neunhöffer.

The Irreducible Brauer Characters of Fi_{23}

The results of the condensation and further computations with Brauer characters using GAP and MOC gave all the irreducible 2-modular characters of G.

Degrees of the irreducible 2-modular characters of Fi_{23}:

1,	782,	1 494,	3 588,
19 940,	57 408,	79 442,	94 588,
94 588,	583 440,	724 776,	979 132,
1 951 872,	1 997 872,	1 997 872,	5 812 860,
7 821 240,	8 280 208,	17 276 520,	34 744 192,
73 531 392,	97 976 320,	166 559 744,	504 627 200,
504 627 200.			