A Note on Global Asymptotic Stability of Nonautonomous Master Equations

Christian Lax, Sebastian Walcher
Lehrstuhl A für Mathematik
RWTH Aachen
D-52056 Aachen, Germany

March 5, 2013

Abstract

We present sufficient conditions to either preclude or guarantee global asymptotic stability of linear differential equations for time-dependent $\mathbb{W}$-matrices. These conditions are concerned with integrability or non-integrability of the matrix entries. The proofs employ differential inequalities.

AMS Subject Classification (2010): 34D05, 34D23, 37B55

1 Introduction and statement of the main results

A recent paper by Earnshaw and Keener [EK] comprises a discussion of nonautonomous linear ordinary differential equations of the special type

$$\dot{x} = A(t)x, \tag{1}$$

with a continuous map $[0, \infty) \to \mathbb{R}^{n\times n}$, $t \mapsto A(t)$ such that every $A(t)$ is a $\mathbb{W}$-matrix (see e.g. [vK] for the terminology). Thus the following two conditions hold for all $t \geq 0$:

1. $a_{ij}(t) \geq 0$ for all $i \neq j$,
2. $\sum_{i=1}^{n} a_{ij}(t) = 0$ for all $j \in \underline{n} := \{1, \ldots, n\}$.

(In the terminology used e.g. by Berman and Plemmons [BP], Ch. 6, negative $\mathbb{W}$-matrices form a special class of M-matrices.)

Equations of this type arise as master equations for non-stationary Markovian jump processes, which are of relevance in the mathematical modelling and analysis of various phenomena in chemistry, biology and other sciences. In view of applications, one is mainly interested in probability distribution solutions $x(t)$ of (1), i.e. solutions in the standard simplex. In contrast to the autonomous case, the long-time behavior of solutions of (1) is less well understood. Earnshaw and Keener [EK] proved a number of results on the asymptotic behavior of solutions, given certain asymptotic properties of the matrix entries. In the present note we add a few more results and observations, using different methods of proof.

We fix notation, introducing the standard simplex

$$\Sigma := \{x \in \mathbb{R}^n;\ \sum_{i=1}^{n} x_i = 1,\ x_j \geq 0,\ j = 1, \ldots, n\}$$

and the hyperplane

$$H := \{x \in \mathbb{R}^n;\ \sum_{i=1}^{n} x_i = 0\}.$$

Clearly these sets are positively invariant for (1).

If A is a constant $\mathbb{W}$-matrix then $\exp(tA)$ is a stochastic matrix for all t, and the asymptotic behavior of solutions of (1) is known. For irreducible A (for the notion see [BP] or [vK]) there is a unique stationary point in Σ, and every solution starting in Σ converges toward this stationary point. In van Kampen's monograph [vK] one finds a refinement of this result: Call a reducible $\mathbb{W}$-matrix decomposable or split if, up to a simultaneous permutation of rows and columns, an upper left minor has the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

(Decomposability means that the minor can be chosen as the matrix itself.) Then, given a reducible matrix A, every solution in Σ converges to the unique stationary point if and only if A is neither decomposable nor split.

In the nonautonomous case, discussing the asymptotic behavior of solutions is more complicated. First, one has to establish an appropriate nonautonomous generalization of convergence to a unique equilibrium. The following is taken from [EK]:

Definition 1. One calls (1) globally asymptotically stable (briefly GAS) if all solutions p, q of (1) in Σ satisfy $p(t) - q(t) \to 0$ as $t \to \infty$.

Clearly, the condition $x(t) \to 0$ as $t \to \infty$ for all solutions $x(t) \in H$ is necessary and sufficient for global asymptotic stability of (1).

Only non-autonomous systems of dimension two seem amenable to an elementary analysis: Given a solution $x(t)$ in the standard simplex, one has $x_1(t) + x_2(t) = 1$, hence

$$\dot{x}_1 = -\left(a_{12}(t) + a_{21}(t)\right) \cdot x_1 + a_{12}(t).$$

Therefore the system is GAS if and only if $\int_0^\infty (a_{12}(t) + a_{21}(t))\,\mathrm{d}t = \infty$. Moreover, the following is obvious: If, up to a fixed permutation of indices, a time-dependent $\mathbb{W}$-matrix A has for all t the form

$$A = \begin{pmatrix} A_1 & 0 & * \\ 0 & A_2 & * \\ 0 & 0 & * \end{pmatrix}$$

with matrices of fixed sizes, then the system cannot be GAS.

Earnshaw and Keener [EK] stated and proved sufficient criteria for global asymptotic stability of non-autonomous master equations. On the other hand they showed that irreducibiliy of all $A(t)$ is not sufficient for global asymptotic stability of (1); the asymptotic behaviour of $A(t)$ is also relevant. Among the sufficient conditions for global asymptotic stability of (1) is the following (see [EK], Theorem 3.2): The system is GAS if the entries of A are C^1 and bounded, with bounded derivatives, and the omega limit set of A contains no decomposable or splitting matrix.

Most proofs in [EK] are based on an expression for the time derivative of $\|x(t)\|_1$, with $x(t) \in H$ a solution of (1). (Earnshaw and Keener note that this is also used by van Kampen [vK] for the autonomous case.) We will recall the expression in Lemma 2 below, and derive some estimates to prove or disprove GAS for certain classes of systems (1). The essential tool in our proofs, which are relatively straightforward, will be differential inequalities.

Our first result generalizes a statement in [EK], Theorem 3.1, and it shows that integrability of all matrix entries precludes GAS.

Theorem 1. *Let $A\colon [0, \infty) \to \mathbb{R}^{n\times n}$, $t \mapsto A(t)$ be continuous such that $A(t)$ is a $\mathbb{W}$-matrix for all $t \geq 0$. Asssume that $\int_0^\infty a_{ij}(s)\,\mathrm{d}s < \infty$ for $1 \leq i,\, j \leq n$, $i \neq j$. Then (1) is not globally asymptotically stable.*

Our second result may be seen as complementary to [EK], Theorem 3.2.

Theorem 2. *Let $n \geq 3$ and let $A\colon [0, \infty) \to \mathbb{R}^{n\times n}$, $t \mapsto A(t)$ be continuous such that $A(t)$ is a $\mathbb{W}$-matrix for all $t \geq 0$. Define*

$$\rho(t) := \min\{a_{ij}(t) + a_{kj}(t);\ i,\ j,\ k \textit{ pairwise different}\}.$$

If $\int_0^\infty \rho(t)\,\mathrm{d}t = \infty$ then (1) is globally asymptotically stable.

The hypotheses of this theorem hold, in particular, if the minimum of all off-diagonal entries of $A(t)$ is non-integrable over $[0, \infty)$. In any case, Theorem 2 allows at most one integrable off-diagonal entry (thus at most one entry that is identically zero) in every column of the matrix. These hypotheses are somewhat restrictive, and this is in part due to the method of proof. But on the other hand, neither differentiability nor boundedness of entries are required, and clearly the omega limit set of $A(t)$ may contain decomposable or split matrices. Thus the hypotheses are markedly different from those in [EK], Theorems 3.2 and 3.6.

Our results, like those by Earnshaw and Keener, seem to indicate that criteria for GAS which are both necessary and sufficient will be difficult to obtain.

2 Proofs

We start by recalling a standard result on differential inequalities (see e.g. Satz IX in [W], § 9).

Lemma 1. *Let $D \subseteq \mathbb{R}^2$ be an open set and suppose that $f\colon D \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ satisfies a local Lipschitz condition with respect to x. Let ψ be a solution of $\dot{x} = f(t, x)$ on the compact interval $[a, b]$ and suppose the differentiable map $\varphi\colon [a, b] \to \mathbb{R}$ satisfies*

$$\varphi(a) \leq \psi(a) \quad \textit{and} \quad \dot{\varphi}(t) \leq f(t, \varphi(t)) \quad \textit{for all } t \in [a, b].$$

Then $\varphi(t) \leq \psi(t)$ holds for all $t \in [a, b]$.
The analogous statement with all inequalities reversed also holds true.

Given $x \in \mathbb{R}^n$, define $I_+ = \{i \in \underline{n} : x_i > 0\}$; moreover define $I_- = \{i \in \underline{n} : x_i < 0\}$ and $I_0 = \{i \in \underline{n} : x_i = 0\}$. Note that both I_+ and I_- are nonempty for $x \in H \setminus \{0\}$. Our arguments are based on some observations from [EK] and [vK], which are summarized next.

Lemma 2. *Let $x(t) \in H$, $t \in [0, \infty)$ be a solution of (1). Then the one-sided derivatives of $\|x(t)\|_1 := \sum_{i=1}^n |x_i(t)|$ exist for all $t \geq 0$, and there is a* *discrete subset M of $[0,\,\infty)$ such that the (two-sided) derivative of $\|x(t)\|_1$ exists for all $t \in [0,\,\infty) \setminus M$. If the derivative exists, then*

$$\begin{aligned}\tfrac{d}{dt}\|x(t)\|_1 \;&=\; -\textstyle\sum_{i\in\underline{n}\setminus I_+}\sum_{j\in I_+} a_{ij}(t)x_j(t) - \sum_{i\in I_-}\sum_{j\in I_+} a_{ij}(t)x_j(t)\\ &\quad -\textstyle\sum_{i\in\underline{n}\setminus I_-}\sum_{j\in I_-} a_{ij}(t)|x_j(t)| - \sum_{i\in I_+}\sum_{j\in I_-} a_{ij}(t)|x_j(t)|.\end{aligned} \tag{2}$$

For the following we will assume

$$M = \{m_0,\, m_1,\, \ldots\} \text{ with } m_0 = 0 \text{ and } m_k < m_{k+1} \text{ for all } k \geq 0.$$

Proof of Theorem 1. The proof will be given for the case of infinite M; the case of finite M is a simpler variant. We set

$$\widetilde{\rho}(t) := \max\{a_{ij}(t);\, i \neq j\},$$

noting that $\widetilde{\rho}$ is integrable on $[0,\,\infty)$. Let $x(t) \in H$ be a solution of (1). We may assume, in addition, that I_+, I_- and I_0 are constant on each interval $(m_k,\, m_{k+1})$. Let $t \geq 0$ in the complement of M. Using (2) we find

$$\begin{aligned}\frac{d}{dt}\|x(t)\|_1 \;&\geq\; -\sum_{i\in\underline{n}\setminus I_+}\sum_{j\in I_+}\widetilde{\rho}(t)x_j(t) - \sum_{i\in I_-}\sum_{j\in I_+}\widetilde{\rho}(t)x_j(t)\\ &\quad\; -\sum_{i\in\underline{n}\setminus I_-}\sum_{j\in I_-}\widetilde{\rho}(t)|x_j(t)| - \sum_{i\in I_+}\sum_{j\in I_-}\widetilde{\rho}(t)|x_j(t)|\\ &\geq\; -2\widetilde{\rho}(t)\|x(t)\|_1.\end{aligned}$$

Define

$$\psi\colon\ [0,\,\infty) \to \mathbb{R}, \quad t \mapsto \|x(0)\|_1 \cdot \exp\left(-2\int_0^t \widetilde{\rho}(s)\, ds\right)$$

By our estimate and Lemma 1 we have $\|x(t)\|_1 \geq \psi(t)$ for $m_0 \leq t \leq m_1$. Now induction, using $\|x(m_k)\|_1 \geq \psi(m_k)$ and Lemma 1 on $[m_k,\, m_{k+1}]$, $k \geq 1$, shows that

$$\|x(t)\|_1 \geq \psi(t) \text{ for all } t \geq 0.$$

Since $\widetilde{\rho}$ is integrable, we see that $\psi(t)$, and hence $\|x(t)\|_1$, does not converge to zero whenever $x(0) \neq 0$. $\square$

Proof of Theorem 2 . The essential argument is similar to the proof of Theorem 1. Given a nonzero solution $x(t) \in H$ of (1), showing the inequality

$$\frac{d}{dt}\|x(t)\|_1 \leq -\rho(t)\|x(t)\|_1$$

will prove the assertion, by Lemma 1. We may assume without loss of generality that $I_+ = \{1, \dots, k\}$, $I_- = \{k+1, \dots, l\}$ and $I_0 = \{l+1, \dots, n\}$ are constant on a given interval (m_p, m_{p+1}), and we have $1 \le k < l \le n$ due to $x(t) \in H$.

From (2) we obtain

$$
\begin{aligned}
\frac{d}{dt}\|x(t)\|_1 &= -\sum_{i=k+1}^{n}\sum_{j=1}^{k} a_{ij}(t)x_j(t) - \sum_{i=k+1}^{l}\sum_{j=1}^{k} a_{ij}(t)x_j(t) \\
&\quad -2\sum_{i=1}^{k}\sum_{j=k+1}^{l} a_{ij}(t)|x_j(t)| - \sum_{i=l+1}^{n}\sum_{j=k+1}^{l} a_{ij}(t)|x_j(t)| \\
&= -\sum_{j=1}^{k}\left(2\sum_{i=k+1}^{l} a_{ij}(t) + \sum_{i=l+1}^{n} a_{ij}(t)\right) x_j(t) \\
&\quad -\sum_{j=k+1}^{l}\left(2\sum_{i=1}^{k} a_{ij}(t) + \sum_{i=l+1}^{n} a_{ij}(t)\right) |x_j(t)|.
\end{aligned}
$$

We distinguish four cases.

1. Assume $n > l$, thus $I_0 \neq \emptyset$. Then for every $j \in \underline{l}$, every sum $2\sum_{i=k+1}^{l} a_{ij}(t) + \sum_{i=l+1}^{n} a_{ij}(t)$, as well as every sum $2\sum_{i=1}^{k} a_{ij}(t) + \sum_{i=l+1}^{n} a_{ij}(t)$, contains at least two summands and therefore may be estimated from below by $\rho(t)$. Altogether, this implies

$$\frac{d}{dt}\|x(t)\|_1 \le -\sum_{j=1}^{k} \rho(t)x_j(t) - \sum_{j=k+1}^{l} \rho(t)|x_j(t)| = -\rho(t)\|x(t)\|_1.$$

2. Assume $n = l$ and $k > 1$, $l > k+1$, thus $I_0 = \emptyset$, $|I_+| \ge 2$ and $|I_-| \ge 2$. Then, with slight modifications, the same arguments as in the first case are applicable, and we obtain the desired inequality.

3. Assume $l = n$ and $k = 1$, thus $|I_+| = 1$. Then $l > k+1$, since $n \ge 3$, and (2) has the form

$$\frac{d}{dt}\|x(t)\|_1 = -2\left(\sum_{i=2}^{n} a_{i1}(t)\right) x_1(t) - 2\sum_{j=2}^{n} a_{1j}(t)|x_j(t)|.$$

From $x(t) \in H$, we have $x_1(t) = \sum_{j=2}^{n} |x_j(t)|$, and rearranging yields

$$
\begin{aligned}
\tfrac{d}{dt}\|x(t)\|_1 &= -\left(\textstyle\sum_{i=2}^{n} a_{i1}(t)\right) x_1(t) \\
&\quad -\textstyle\sum_{j=2}^{n}\left(2a_{1j}(t) + \sum_{i=2}^{n} a_{i1}(t)\right)|x_j(t)| \\
&\le -\rho(t)\|x(t)\|_1,
\end{aligned}
$$

as desired.

4. The remaining case $n = l$, and $k = n-1$ is proven in the same fashion.

$\square$

3 Examples

Theorem 2 provides a relatively tight criterion for global asymptotic stability of 3×3-matrices, as the following two examples indicate.

Example 1. Consider

$$A(t) = \begin{pmatrix} -|\cos(t)|/t & 0 & |\cos(t)|/t \\ |\cos(t)|/t & -|\sin(t)|/t & 0 \\ 0 & |\sin(t)|/t & -|\cos(t)|/t \end{pmatrix}$$

(a non-periodic variant of [EK], p. 222). For this matrix the hypotheses of Theorems 3.2 or 3.6 in [EK] are not satisfied, and the limit set of A consists only of the zero matrix. Theorem 2, due to non-integrability of

$$\rho(t) = \min\{|\cos(t)|/t,\, |\sin(t)|/t\},$$

shows global asymptotic stability of (1).

Example 2. Let $\alpha\colon\ [0, \infty) \to [0, \infty)$ be integrable and $\beta\colon\ [0, \infty) \to [0, \infty)$ non-integrable, with $\beta \geq \alpha$. The matrix

$$A(t) = \begin{pmatrix} -2\alpha(t) & \alpha(t) & \alpha(t) \\ \alpha(t) & -(\alpha(t)+\beta(t)) & \beta(t) \\ \alpha(t) & \beta(t) & -(\alpha(t)+\beta(t)) \end{pmatrix} \tag{3}$$

fails to satisfy the hypotheses of Theorem 2 only because the sum of the off-diagonal elements in the first column is integrable. But this is sufficient to preclude GAS. For a proof, note that the subspace defined by $x_2 - x_3 = 0$ is invariant for the system with matrix (3), due to

$$\frac{d}{dt}(x_2 - x_3) = -(\alpha + 2\beta) \cdot (x_2 - x_3),$$

hence for $(x_1,\, x_2,\, x_2)^{\mathrm{tr}} \in H$ one obtains the following differential equation for $y_1 = x_1$, $y_2 = 2x_2$, $y_1 + y_2 = 0$:

$$\begin{aligned} \dot{y}_1 &= -2\alpha y_1 + \alpha y_2, \\ \dot{y}_2 &= 2\alpha y_1 - \alpha y_2. \end{aligned}$$

According to Theorem 1 (for instance), only the trivial solution of this system will converge to 0 as $t \to \infty$.

The following examples illustrate the range and limitations of our results and arguments.

Example 3. A matrix satisfying the conditions of Theorem 2 may be reducible (even with a fixed pattern of zeros) for all t. For instance consider

$$A = \begin{pmatrix} * & 0 & \dots & 0 \\ * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & * & * \end{pmatrix}$$

with every asterisk representing a non-integrable function.

Example 4. The hypothesis of Theorem 2 cannot be relaxed to sums of more than two off-diagonal elements in every column, as is shown by a constant 4×4-matrix

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

with irreducible 2×2 $\mathbb{W}$-matrices M_1 and M_2.

Finally we present an example to indicate that combining all the known results will not provide necessary and sufficient conditions for GAS.

Example 5. For the matrix

$$A(t) = \begin{pmatrix} -t & e^t & 0 & 0 \\ 0 & -e^t & 1-\cos(t) & 0 \\ 0 & 0 & \cos(t)-1 & \frac{1}{(t+1)^2} \\ t & 0 & 0 & -\frac{1}{(t+1)^2} \end{pmatrix} \tag{4}$$

neither Theorem 1 nor Theorem 2, nor any of the results in Earnshaw and Keener [EK] are applicable. Numerical evidence, with initial values running through a basis of H, suggests that the system is GAS (see Figure 1 for the basis vector $(1, 0, -1, 0)^{\mathrm{tr}}$).

References

[BP] Berman, A., Plemmons, R.J., *Nonnegative matrices in the mathematical sciences*. Academic Press, New York (1979).

[EK] Earnshaw, B.A., Keener, J.P., *Global Asymptotic Stability of Solutions of Nonautonomous Master Equations*, SIAM J. Applied Dynamical Systems **9**(1), 220-237 (2010).

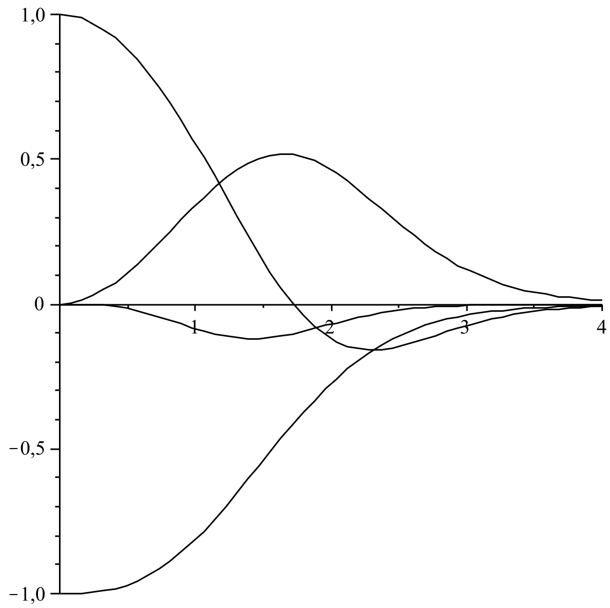

Figure 1: Solution components of $\dot{x} = A(t)x$ with matrix (4) for the initial conditions $(1, 0, -1, 0)^{\mathrm{tr}}$.

[vK] van Kampen, N.G., *Stochastic processes in physics and chemistry*, North-Holland, Amsterdam (1987).

[W] Walter, W., *Gewöhnliche Differentialgleichungen*, 6. Aufl., Springer-Verlag, Berlin (2000).