FIRST INTEGRALS OF LOCAL ANALYTIC DIFFERENTIAL SYSTEMS

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Abstract. We investigate formal and analytic first integrals of local analytic ordinary differential equations near a stationary point. A natural approach is via the Poincaré-Dulac normal forms: If there exists a formal first integral for a system in normal form then it is also a first integral for the semisimple part of the linearization, which may be seen as "conserved" by the normal form. We discuss the maximal setting in which all such first integrals are conserved, and show that all first integrals are conserved for certain classes of reversible systems. Moreover we investigate the case of linearization with zero eigenvalues, and we consider a three-dimensional generalization of the quadratic Dulac-Frommer center problem.

1. Introduction

The question whether a differential equation admits nonconstant first integrals in the neighborhood of a stationary point is of considerable interest for its qualitative analysis. Thus, for real planar systems the existence of a local analytic first integral in a neigborhood of a center with eigenvalues $\pm\beta i$ with real $\beta \neq 0$ was shown by Poincaré and Lyapunov, see [16, 12, 14]. The existence of local analytic or smooth first integrals in a neigborhood of a nilpotent center was studied in [4], and the existence of local smooth first integral in a neigborhood of a center having zero linear part was proved by Mazzi and Sabatini [13].

On the other hand, to detect the existence of a local first integral is frequently a difficult problem. In the present paper we discuss differential systems with analytic right-hand side, and approach the problem via Poincaré-Dulac normal forms: Thus we will obtain a clear picture of formal first integrals, and some nontrivial results about local analytic first integrals.

Let U be an open neighborhood of 0 in $\mathbb{C}^n$ and let $f : U \longrightarrow \mathbb{C}^n$ be an analytic vector valued function with $f(0) = 0$. In this work we discuss autonomous differential systems of the form

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in U. \tag{1}$$

Since real-analytic systems may be extended to $\mathbb{C}^n$, our results will also be applicable to the real setting. To f there corresponds a vector field, i.e. a derivation of local analytic functions given by

$$X_f = \sum f_i \frac{\partial}{\partial x_i}.$$

We will employ differential equation and vector field interpretations simultaneously. To the commutator of two derivations there corresponds the Lie bracket of vector valued functions. If ϕ is a local analytic function then we call $X_f(\phi)$ the Lie derivative of ϕ. If $X_f(\phi) = 0$ then we call ϕ a first integral of the differential equation. (Note that we include constant functions in this definition.)

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We denote by A the Jacobian matrix of f evaluated at the origin, i.e. $A = Df(0)$, and consider the Taylor expansion

$$f(x) = Ax + \sum_{k\geq 2} g^{(k)}(x); \quad g^{(k)} \text{ homogeneous of degree } k. \tag{2}$$

In addition to local analytic functions and vector fields, we will also discuss their formal power series counterparts.

We will study the question of formal and analytic first integrals following ideas initiated in Poincaré's work [16], using what is now kno wn as Poincaré-Dulac normal forms (briefly, PD normal forms); see Dulac [6], Bibikov [2], Bruno [3], Arnold [1], and [17], among others. Every formal vector field can be transformed to a PD normal form by an invertible formal transformation, and formal first integrals are thereby mapped to formal first integrals. Thus normal forms will provide satisfactory answers in the formal power series setting, and also some results for the analytic case.

The existence question for formal first integrals will be discussed in Section 2. All arguments are based on the fact that a formal first integral of a system (2) in PD normal form is also a first integral of the semisimple part A_s of the linearization A. Thus, the normal form may *conserve* certain first integrals of A_s, and every first integral of a system in normal form arises in this way. We will proceed in Section 3 to characterize and discuss the maximal scenario when all first integrals of A_s are conserved by the system transformed to normal form. We show that conservation of first integrals may even imply the existence of a convergent normalizing transformation for analytic systems. Then we present, in Section 4, a class of generalized reversible systems that conserve all first integrals of the linear part. In Section 5 we discuss the case when some of the eigenvalues are zero in the linearization (with no other resonances), and we generalize known theorems relating local manifolds of stationary points to the existence of analytic first integrals, for more details see Remark 17. Our results also include a stability criterion, see Theorem 15. We finish the paper by introducing, and to some extent solving, a three-dimensional generalization of the quadratic Dulac-Frommer center problem.

2. Normal forms and local first integrals

Recall that A is the Jacobian matrix of f evaluated at the origin. Let $A = A_s + A_n$ be the decomposition of A into semisimple and nilpotent part; thus A_s is diagonizable, A_n is nilpotent and $[A_s, A_n] = 0$. (If A is given in Jordan canonical form then A_s is just the diagonal part and A_n is the strictly upper triangular part.) System (1) is in *Poincaré-Dulac normal form* if $[A_s, f] = 0$. Equivalently, in the Taylor expansion (2) one has all $\left[A_s, g^{(k)}\right] = 0$.

There is an explicit description of PD normal forms if A is given in Jordan canonical form, with eigenvalues $\lambda_1, \ldots, \lambda_n$ (eigenvalues will be always be counted according to their multiplicity). Let $\mathbb{Z}^+$ be the set of all non–negative integers, and $e_1, \ldots, e_n$ the standard basis of $\mathbb{C}^n$. Then a vector monomial

$$x_1^{m_1} \cdots x_n^{m_n} e_j \quad \text{with } m_i \in \mathbb{Z}^+$$

is called *resonant* if

$$m_1\lambda_1 + \cdots m_n\lambda_n - \lambda_j = 0.$$

In these coordinates, a differential equation (1) is in PD normal form if and only if all vector monomials occuring in the Taylor expansion (2) are resonant.

From the work of Poincaré [16] and Dulac [6] one knows that every formal power series system (2) admits a formal transformation to a formal power series system in normal form. Thus there exist an invertible formal power series

$$\Gamma(x) = x + \text{h.o.t.}$$

and a series

$$\tilde{f}(x) = Ax + \text{h.o.t.}$$

in PD normal form such that the identity

$$D\Gamma(x)f(x) = \tilde{f}(\Gamma(x)),$$

is satisfied. (See also Bibikov [2], Bruno [3], and [18]). This transformation induces a bijection between the formal first integrals of f and those of $\tilde{f}$. However, due to convergence problems, there may not exist an analytic transformation to normal form for analytic systems (1), and consequently the question concerning local analytic first integrals is much harder.

We recall three results that are common knowledge or were proven in [17], considering first polynomial and formal first integrals of the linear system $\dot{x} = A_s x$.

Lemma 1. *Assume that A is in Jordan canonical form, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then a monomial $x_1^{k_1} \cdots x_n^{k_n}$ with $k_i \in \mathbb{Z}^+$ (and $\sum k_i \geq 1$) is a first integral of A_s if and only if the monomial is* resonant*, that is,*

$$\sum_{i=1}^{n} k_i \lambda_i = 0.$$

Moreover, any formal first integral of $\dot{x} = A_s x$ is a formal power series in resonant monomials.

Lemma 2. *The algebra of polynomial first integrals of $\dot{x} = A_s x$ is finitely generated. If A_s has eigenvalues $\lambda_1, \ldots, \lambda_n$, and the $\mathbb{Z}$-module spanned by all nonnegative integer solutions $(k_1, \ldots, k_n)$ of $\sum k_i \lambda_i = 0$ has rank d, then there are exactly d independent polynomial first integrals (and also exactly d independent formal first integrals) for A_s.*

Proof. See [17], Proposition 1.6 and proof of Theorem 3.2. □

Proposition 3. *Let the formal power series* (2) *be in PD normal form. Then every formal power series first integral of system* (1) *with expansion* (2) *is also a first integral of A_s.*

Proof. Since this fact is crucial, we repeat an outline of the argument in [17], Proposition 1.8: The derivation X_A restricts to any subspace of homogeneous polynomials of fixed degree, say degree m, and by [17], Proposition 1.4, the semisimple-nilpotent decomposition is given by $X_A = X_{A_s} + X_{A_n}$. Let ϕ be a homogeneous polynomial. By linear algebra, $X_A(\phi) = 0$ implies $X_{A_s}(\phi) = 0$, and moreover $X_{A_s} X_A(\phi) = 0$ implies $X_{A_s}(\phi) = 0$.

Now let (2) be in normal form, and let

$$\psi = \psi_r + \psi_{r+1} + \cdots, \qquad \text{with } \psi_j \text{ homogeneous of degree } j,\ \psi_r \neq 0$$

be a formal first integral. Evaluating terms of smallest degree one finds

$$X_A(\psi_r) = 0 \Rightarrow X_{A_s}(\psi_r) = 0.$$

Proceed by induction. For homogeneous terms of degree $r + j$ one has

$$0 = X_A(\psi_{r+j}) + \sum_{k=2}^{j} X_{g^{(k)}}(\psi_{r+j-k}).$$

Application of X_{A_s}, the normal form property and the induction hypothesis imply

$$\begin{aligned} X_{A_s} X_A(\psi_{r+j}) &= -\sum_{k=2}^{j} X_{A_s} X_{g^{(k)}}(\psi_{r+j-k}) \\ &= -\sum_{k=2}^{j} X_{g^{(k)}} X_{A_s}(\psi_{r+j-k}) \\ &= 0, \end{aligned}$$

and therefore $X_{A_s}(\psi_{r+j}) = 0$. □

Corollary 4. *If A_s admits only constant polynomial first integrals then the only formal first integrals of system* (1) *with f given by* (2) *- whether or not in normal form - are the constants.*

But even if nontrivial first integrals for A_s exist, system (1) will in general have only constant first integrals. The following sections will deal with criteria and settings when nontrivial first integrals exist for (2), hence when some first integrals of A_s are conserved in a corresponding PD normal form.

3. The maximal scenario

We discuss the case when all first integrals of A_s are conserved by a normal form.

Proposition 5. *Let the formal power series* (2) *be in PD normal form, and assume that A_s admits d independent polynomial first integrals. Then $\dot{x} = f(x)$ admits d independent formal first integrals if and only if this differential equation admits every polynomial first integral of $\dot{x} = A_s x$.*

Proof. The polynomial invariant algebra of A_s (i.e. the algebra of all the polynomial first integrals of system $\dot{x} = A_s x$) has a finite set of (monomial) generators $\psi_1, \ldots, \psi_r$. Let $\theta_1, \ldots, \theta_d$ be formal power series in r indeterminates such that

$$\theta_1(\psi_1, \ldots, \psi_r), \cdots, \theta_d(\psi_1, \ldots, \psi_r),$$

form a set of d independent first integrals for f. Setting

$$\Theta := \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_d \end{pmatrix}; \quad \Psi := \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_r \end{pmatrix},$$

the rank of the derivative $D(\Theta \circ \Psi)$ therefore is equal to d, and the identity

$$D(\Theta \circ \Psi)(x) f(x) = 0,$$

follows from the first integral property. The chain rule implies

$$D(\Theta \circ \Psi)(x) = D\Theta(\Psi(x)) D\Psi(x),$$

and by linear algebra both matrices on the right-hand side must have rank $\geq d$. In particular $D\Theta(\Psi(x))$ is invertible, and $D\Theta(\Psi(x))D\Psi(x)f(x) = 0$ implies $D\Psi(x)f(x) = 0$, whence every ψ_j is a first integral of f. □

Remark 6. *For real systems of dimension 2 with an A semisimple matrix with eigenvalues $\pm i\omega$, $\omega > 0$, conservation of first integrals (or just of some nonconstant first integral) is equivalent to the existence of a center (see Dulac [7], Frommer [8]). In this sense the question under what circumstances there is a maximal number of independent first integrals is an algebraic generalization of the center problem.*

Remark 7. *In a two-dimensional system having a matrix A with a unique zero eigenvalue, conservation of first integrals is equivalent to non-isolatedness of the stationary point. This is a consequence of a more general statement proven in Li et al. [11]; see also Bibikov [2], Theorem 12.2 and its proof.*

One knows that the existence of a center for planar polynomial vector fields, with linearization as above, is equivalent to a finite set of polynomial conditions for the coefficients. A generalization of this property to the "maximal scenario" is as follows:

Proposition 8. *Assume that system* (1) *with f given by* (2) *has polynomial right-hand side, A_s admits d independent polynomial first integrals, and denote by $(c_i)_{i\in I}$ the finite system of coefficients of the nonlinear terms in* (1)*. There exist finitely many polynomials $\sigma_1, \ldots, \sigma_p$ in the c_i such that* (1) *admits d independent first integrals if and only if all the σ_j vanish.*

Proof. We denote by $\tilde{f}$ a PD normal form of f. The coefficients of the nonlinear terms in $\tilde{f}$ are polynomials in the c_j, as is seen from [18], Theorem 2.4 and Algorithm 2.6, or from analyzing the "distinguished transformation" to normal form in Bibikov [2], Ch. 2, as well as in Bruno [3]. Let $\rho_1, \ldots, \rho_q$ be a set of (homogeneous) generators for the polynomial invariant algebra of A_s. Evaluating the relations $X_{\tilde{f}}(\rho_k) = 0$ we obtain conditions of the type that certain polynomials in the c_j must be zero. A priori we may have infinitely many such polynomials, but Hilbert's Basissatz (see e.g. Kunz [10], Prop. 2.3) shows that finitely many suffice. □

Conservation of all formal first integrals may ensure convergence. The following result was first proven by Zhang [19], [20], whose approach is also based on Proposition 3 and works with the theory developed in Bibikov [2].

Theorem 9. *Let the n-dimensional system* (1) *with f given by* (2) *be analytic, and assume that A_s is not zero and admits $n-1$ independent polynomial first integrals. If some formal PD normal form for f admits $n-1$ independent formal first integrals, then $\dot{x} = f(x)$ admits a convergent transformation to PD normal form, and there exist $n-1$ independent analytic first integrals.*

Proof. By Lemma 2 there are linearly independent integer vectors $(k_{i,1}, \ldots, k_{i,n})$, with $1 \le i \le n-1$, such that the eigenvalues of A_s satisfy

$$\sum_j k_{i,j}\lambda_j = 0, \quad 1 \le i \le n-1.$$

This means that $(\lambda_1, \ldots, \lambda_n)$ lies in the one-dimensional solution space of a homogeneous linear system of equations with integer coefficients, hence $(\lambda_1, \ldots, \lambda_n)$ is a complex multiple of an integer solution. This implies the existence of some $\delta > 0$ such that

$$\sum m_i\lambda_i \neq 0, \quad m_i \in \mathbb{Z}^+ \Rightarrow \left|\sum m_i\lambda_i\right| > \delta,$$

and this, in turn, implies Bruno's "Condition ω"; see [3], Part I, Ch. III, Subsection 3.2. On the other hand, every nonconstant homogeneous polynomial vector field p admitting the monomial first integrals ϕ_i which correspond to the above integer vectors (as we may assume from Proposition 5) will be a polynomial multiple of $A_s x$: If $y \in \mathbb{C}^n$ is such that the Jacobian matrix $\partial(\phi_1, \ldots, \phi_{n-1})/\partial(x_1, \cdots, x_n)$ has rank $n-1$ at y, then $p(y)$ and $A_s y$ are linearly dependent. Since the set of these y is nonempty and Zariski-open, one has linear dependence for all $y \in \mathbb{C}^n$. For the components p_i of p this implies

$$\lambda_i x_i p_j = \lambda_j x_j p_i \quad (1 \le i,\, j \le n).$$

Since at least one eigenvalue is nonzero, we have that $p_i = 0$ whenever $\lambda_i = 0$. If $i \neq j$ and both $\lambda_i \neq 0$ and $\lambda_j \neq 0$, we see that x_i divides p_i and x_j divides p_j. Invoking the linear dependence condition again we have that $p(x) = \sigma(x)A_s x$ with a polynomial σ (which is necessarily first integral of A_s since the Lie bracket equals zero). Thus any normal form $\tilde{f}$ of f will have the form $\tau \cdot A_s$ with a formal first integral τ, and therefore Bruno's "Condition A" (see [3], as above) is satisfied. The convergence assertion follows from Theorem 1 in [3], as above. □

4. Reversibility and conserved first integrals

For some classes of reversible systems, such as the two-dimensional center problem with nondegenerate linearization, one knows that nontrivial first integrals exist. Here we will discuss such systems in some detail, considering a generalized version of reversibility, with certain primitive roots of unity taking the role of the familiar -1. This makes notation and statements more cumbersome, but the greater generality seems worth the effort.

Lemma 10. *Let p be a prime number, m a positive integer, ζ a p-th root of unity, and $\omega_1, \ldots, \omega_m$ complex numbers that are linearly independent over the algebraic number field $\mathbb{Q}[\zeta]$. Moreover let C be a diagonal matrix either of size $mp \times mp$ with entries*

$$\begin{array}{c} \omega_1, \zeta\omega_1, \ldots, \zeta^{p-1}\omega_1, \\ \omega_2, \zeta\omega_2, \ldots, \zeta^{p-1}\omega_2, \\ \vdots \\ \omega_m, \zeta\omega_m, \ldots, \zeta^{p-1}\omega_m, \end{array}$$

in this order, or of size $(mp+1) \times (mp+1)$ with the above entries and a final entry 0. Then there exists an invertible matrix T such that

$$T^{-1}CT = \zeta \cdot C,$$

and T permutes cyclically the elements of each set $\{x_1, \ldots, x_p\}$, $\{x_{p+1}, \ldots, x_{2p}\}$, etc.

The invariant algebra of C is generated by the following algebraically independent polynomials.

$$\begin{array}{rcl} \phi_1 & = & x_1 \cdot x_2 \cdots x_p \\ \phi_2 & = & x_{p+1} \cdot x_{p+2} \cdots x_{2p} \\ \vdots & & \\ \phi_m & = & x_{(m-1)p+1} \cdot x_{(m-1)p+2} \cdots x_{mp} \end{array}$$

and, depending on the case, $\phi_{m+1} = x_{mp+1}$.

Proof. Only the assertion about the generators of the invariant algebra requires a proof. If a nonnegative integer linear combination of the eigenvalues of C is equal to zero then every partial linear combination involving only

$$\omega_j, \zeta\omega_j, \ldots, \zeta^{p-1}\omega_j, \quad 1 \le j \le m,$$

will also be zero, due to the linear independence requirement on the ω_j. Thus the problem is reduced to relations of the type

$$\sum_{i=0}^{p-1} m_i \zeta^i = 0$$

with (nonnegative) integers m_i. There is a distinguished relation

$$1 + \zeta + \cdots + \zeta^{p-1} = 0,$$

and every other integer linear combination of $1, \ldots, \zeta^{p-1}$ yielding 0 is an integer multiple of this distinguished one. (Note that a linearly independent integer linear combination of the ζ^i that yields zero would imply the existence of a polynomial with rational coefficients that annihilates ζ and has smaller degree than $p-1$; this is a contradiction to the irreducibility of the p-th cyclotomic polynomial.)

From this we see that $\phi_1, \ldots, \phi_m$ are invariants of C, and that they generate the invariant algebra. □

Proposition 11. *Let p, m, ζ be as in Lemma 10, let A be conjugate to the matrix C in Lemma 10, and assume that $f = A + \cdots$ in (2) satisfies $T^{-1} \circ f \circ T = \zeta \cdot f$ for some linear invertible T. Then all first integrals of A are conserved in a PD normal form of f.*

Proof. We may assume that A is diagonal. Moreover, due to [18], Proposition 2.8, we may assume that f is in PD normal form. By Lemma 10 one has $\psi \circ T = \psi$ for every polynomial (as well as every formal) first integral ψ of A, and $X_f(\psi)$ is again a formal first integral of A, because X_f and X_A commute. Now let ρ be a formal first integral of A. Evaluating the (universally valid) identity

$$X_{T^{-1} \circ f \circ T}(\rho \circ T) = X_f(\rho) \circ T,$$

we obtain $\zeta X_f(\rho)\circ T = X_f(\rho)\circ T$. So

$$\zeta X_f(\rho) = X_f(\rho),$$

and therefore $X_f(\rho) = 0$. □

Example 1. *The three-dimensional system*

$$\dot{x} = f(x) := \begin{pmatrix} 0 \\ i\omega x_2 \\ -i\omega x_3 \end{pmatrix} + \begin{pmatrix} a_2x_1x_2 + 2a_3x_2^2 - 2a_3x_3^2 - a_2x_1x_3 \\ b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + b_4x_2x_3 + 3b_5x_3^2 + 2b_6x_1x_3 \\ -b_1x_1^2 - 2b_6x_1x_2 - 3b_5x_2^2 - b_4x_2x_3 - b_3x_3^2 - b_2x_1x_3 \end{pmatrix},$$

with complex parameters $\omega \neq 0$, a_j and b_j satisfies

$$T^{-1}\circ f\circ T = -f,$$

for the permutation matrix T that exchanges x_2 and x_3, and therefore both first integrals of the linear part are conserved. In this case, Theorem 9 also shows that a convergent transformation to normal form exists, and therefore the system admits two independent analytic first integrals. (The labelling of the parameters has been chosen in view of Section 6.)

Example 2. *Let*

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and consider the differential system

$$\dot{x} = f(x) := A\cdot x + \begin{pmatrix} a_1x_1x_2 + a_2x_2^2 \\ a_3x_2x_3 + a_4x_1^2 \\ a_5x_1x_2 + a_6x_3^2 \end{pmatrix},$$

with real parameters a_j. If ζ is a primitive third root of unity and $T = \mathrm{diag}\,(1, \zeta^2, \zeta)$, one verifies $T^{-1}\circ f\circ T = \zeta\cdot T$. Therefore the cubic first integral ψ of A is also a first integral of every formal normal form of this system. This indicates that the generalized notion of reversibility is also relevant if one is only interested in real systems. (Note: The quadratic part is the most general admitting reversibility. In the example following Proposition 2.8 in [18]*, a more special system was discussed, with the additional property that Bruno's* [3] *"Condition A" holds for the normal form).*

The algebraic independence assumption for the ω_j in Proposition 11 cannot be discarded in general, as is shown by the example following Corollary 3.17 in [17]: Given the (m,p) resonance in dimension 4 (with relatively prime $m,p\in\mathbb{Z}^+$), a reversible system of this type generically admits two independent first integrals, but not three, and Proposition 11 does not hold.

5. The case of zero eigenvalues

Throughout this section we assume that there is an integer m, $1\le m<n$, such that

$$\lambda_1 = \cdots = \lambda_m = 0, \tag{3}$$

(so 0 is an m-fold eigenvalue of A_s) and moreover

$$\sum_{i=m+1}^{n} k_i\lambda_i \neq 0 \text{ whenever } k_{m+1},\dots,k_n\in\mathbb{Z}^+ \text{ with } \sum k_i > 0. \tag{4}$$

We recall the familiar characterization of the normal forms in this case.

Lemma 12. *Assume that conditions* (3) *and* (4) *are satisfied, that A is in Jordan canonical form, and that $\tilde{f}(x) = Ax+\cdots$ is in PD normal form. Then $\tilde{f}_1,\dots,\tilde{f}_m$ depend only on $x_1,\dots,x_m$.*

Proof. A monomial $x_1^{\ell_1}\cdots x_n^{\ell_n}e_j$, with $j\le m$, is resonant if and only if $\sum_{i>m}\ell_i\lambda_i = 0$. According to (4), this implies $\ell_{m+1}=\cdots=\ell_n=0$. □

Theorem 13. *Assume that conditions* (3) *and* (4) *are satisfied, and that* A *is in Jordan canonical form. Then there is a bijective correspondence between:*

(i) *The formal first integrals of* (2).
(ii) *The formal first integrals of a corresponding* $\tilde{f}$ *in PD normal form.*
(iii) *The formal first integrals of the* m*-dimensional subsystem*

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(x_1,\ldots,x_m) \\ \vdots \\ \tilde{f}_m(x_1,\ldots,x_m) \end{pmatrix},$$

of $\tilde{f}$ *in PD normal form.*

Proof. The correspondence between (i) and (ii) follows from the fact that (formal) coordinate transformations map first integrals to first integrals. Moreover, Proposition 3 shows that every formal first integral of $\tilde{f}$ depends on $x_1,\ldots,x_m$ only, and therefore is also a first integral of the subsystem, whence (ii) implies (iii). The converse is obvious. □

Corollary 14. *A system* $\tilde{f}$ *in PD normal form admits* m *independent formal first integrals if and only if* $\tilde{f}_1 = \cdots = \tilde{f}_m = 0$.

Proof. By Proposition 5 there exist m independent first integrals for $\tilde{f}$ if and only if $x_1,\ldots,x_m$ are first integrals of the system given in Theorem 13 (iii). □

Turning to the analytic case, we replace (4) by a stronger condition: We require the existence of a $\delta > 0$ such that

$$\left|\sum_{i=m+1}^{n} k_i\lambda_i\right| > \delta \text{ whenever } k_{m+1},\ldots,k_n \in \mathbb{Z}^+ \text{ with } \sum k_i > 0. \tag{5}$$

Even if there is no convergent transformation to PD normal form, convergence may be ascertained for certain transformations to less restrictive types of normal form, such as *quasi-normal form (QNF)*, as introduced in Bibikov [2], Section 10. In the special case of eigenvalue conditions (3) and (4), with A in Jordan form, a system (2) is in QNF if and only if $f_1,\ldots,f_m$ depend only on $x_1,\ldots,x_m$. Since PD normal forms are special QNF's, an arbitrary system admits a formal transformation to QNF.

Theorem 15. *Assume that conditions* (3) *and* (5) *are satisfied, and that* A *is in Jordan canonical form. Assume further that* (2) *admits* m *independent formal first integrals. Then this system admits* m *independent analytic first integrals, and moreover there is an* m*-dimensional local analytic manifold of stationary points of* f*. Moreover, if system* (2) *is real and all nonzero eigenvalues have real part* < 0 *then the stationary point* 0 *is stable in the sense of Lyapunov.*

Proof. By Corollary 14 we have $\tilde{f}_1 = \cdots = \tilde{f}_m = 0$ for any formal PD normal form. Due to this and to condition (5), the hypotheses (i) and (ii) of Bibikov [2], Theorem 10.2 are satisfied; in particular hypothesis (ii) holds because the left hand side of the required inequality is equal to zero. Therefore an analytic transformation to QNF f^* exists. Since $f_1^*,\ldots,f_n^*$ contain only resonant vector monomials, there exists a formal transformation to a PD normal form $\tilde{f}$ which sends x_i to x_i for $1 \le i \le m$. (This holds e.g. for the transformation given in [18], Theorem 2.4). In particular, $f_i^* = \tilde{f}_i = 0$ for $1 \le i \le m$, and both assertions follow, since the transformation to QNF is convergent.

The stability assertion follows from the QNF: Every affine subpace given by $x_1 = y_1,\ldots,x_m = y_m$ (with $|y_1|,\ldots,|y_m|$ sufficiently small) is invariant. Stability of any stationary point

$$(y_1,\ldots,y_m,0,\ldots,0),$$

is therefore determined by the eigenvalues of

$$C(y_1,\dots,y_m) := \left(\frac{\partial f_i^*}{\partial x_j}(y_1,\dots,y_m,0,\dots,0)\right)_{m<i,j\le n}$$

All eigenvalues of $C(0,\dots,0)$ have negative real parts, and this property continues to hold if all $|y_i|$ are sufficiently small. □

Corollary 16. *Assume that conditions* (3) *and* (5) *are satisfied, and that A is in Jordan canonical form. Then the analytic system* (2) *admits m independent formal first integrals (and also m independent analytic first integrals) if and only if it admits an m-dimensional local analytic manifold of stationary points.*

Proof. By Theorem 15 the "only if" part is proven. Now we will show the "if" part.
(i) Assume that $\tilde f$ is some normal form of f and $(\tilde f_1,\dots,\tilde f_m)^{\mathrm{tr}}$ not identically zero. If

$$\sum \alpha_{\ell_1,\dots,\ell_m} x_1^{\ell_1}\cdots x_m^{\ell_m} \ne 0,$$

is the homogeneous lowest order nonvanishing term in $\tilde f_j$ then we may assume that the same holds for f_j, because there exist analytic transformations into normal form up to any finite degree.
(ii) There is an m-dimensional local manifold of stationary points for f, which may be parameterized by

$$\Theta: \begin{pmatrix} t_1\\ \vdots\\ t_m\end{pmatrix} \mapsto \begin{pmatrix}\theta_1(t_1,\dots,t_m)\\ \vdots\\ \theta_n(t_1,\dots,t_m)\end{pmatrix},$$

with $\Theta(0)=0$. We may assume that

$$A = \begin{pmatrix} 0 & 0\\ 0 & B\end{pmatrix},$$

with block matrices of suitable size and invertible B. The identity $f(\Theta(t))=0$ implies

$$Df(0)\cdot D\Theta(0) = A\cdot D\Theta(0) = 0,$$

and therefore

$$D\theta_{m+1}(0) = \cdots = D\theta_n(0) = 0$$

which forces the rank of $(D\theta_1(0),\dots,D\theta_m(0))^{\mathrm{tr}}$ to equal m. Therefore the map

$$\begin{pmatrix} t_1\\ \vdots\\ t_m\end{pmatrix} \mapsto \begin{pmatrix}\theta_1(t_1,\dots,t_m)\\ \vdots\\ \theta_m(t_1,\dots,t_m)\end{pmatrix},$$

is a local analytic diffeomorphism, and we may assume that $\theta_1 = t_1,\dots,\theta_m = t_m$ and furthermore that $\theta_{m+1},\dots,\theta_m$ are series of order ≥ 2 in $t_1,\dots,t_m$.
(iii) By parts (i) and (ii) we have

$$0 = \sum \alpha_{\ell_1,\dots,\ell_m} t_1^{\ell_1}\cdots t_m^{\ell_m} + \text{h.o.t.},$$

because the higher order terms in x of f_j become higher order terms in t of $f_j\circ\Theta$. But this is a contradiction. □

Remark 17. *Variants of the arguments in the proofs of Theorem 15 and Corollary 16 were also employed (in more restrictive settings) in* [5]*, Theorem 1 and* [15]*, Proposition 2. Moreover, the result in Li et al.* [11] *corresponds to the case $m=1$ above.*

There remains the obvious question whether Condition (5) *can be replaced by a weaker condition (e.g. analogous to Bruno's "Condition ω") such that Theorem 15 and Corollary 16 still hold. The statements and results in Bruno* [3]*, Part II, on sets of analyticity are highly relevant in this context, and strongly indicate that substantially weaker conditons suffice. However, most* *of these statements are given without proofs, or with references to partial proofs only. Condition* (5) *was chosen because proofs of all the results we require are readily available in* [2].

Example 3. *Consider the four-dimensional system*

$$\dot{x} = f(x) := (-x_1 + x_2x_3)\begin{pmatrix} 1 - x_1 \\ x_3 + x_1x_4 \\ x_2 \\ x_1x_2x_3 \end{pmatrix} + (-2x_4 + x_1x_2x_3)\begin{pmatrix} x_3 \\ x_4 \\ x_1 + x_2x_3 \\ 1 + x_2 \end{pmatrix}.$$

This system has a stationary point in 0 *with linearization* $A = \mathrm{diag}\,(-1,\,0,\,0,\,-2)$ *and by construction* 0 *is contained in a two-dimensional local manifold of stationary points. By Corollary 16 there exist two independent local first integrals.*

6. A three-dimensional center problem

In this section we discuss three-dimensional real systems

$$\dot{x} = Bx + \text{ h.o.t.}$$

with

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix}, \quad \omega > 0.$$

Specifically we ask for necessary and sufficient conditions on the higher order terms such that the system admits a maximal number of independent formal first integrals. Since B admits two independent polynomial first integrals, this maximal number is equal to two. By Theorem 9, the existence of two independent formal first integrals implies the existence of two independent analytic first integrals

$$\phi_1 = x_1 + \cdots, \quad \phi_2 = x_2^2 + x_3^2 + \cdots,$$

and moreover there exists a convergent transformation to PD normal form. Roughly speaking, real differential systems of the above type with two independent conserved first integrals are distinguished by the property that all trajectories near 0 are stationary or closed. In this sense one may speak of a "three-dimensional center".

We will consider the problem for quadratic right-hand side. A direct evaluation of the conservation conditions and use e.g. of Gröbner bases does not seem feasible (at least with standard algorithms and implementations) in view of the high number of coefficients. We will therefore follow a strategy employed already by Dulac [7] and Frommer [8] in dimension two: Normalize the system by suitable transformations, determine necessary conditions for the existence of two independent first integrals (using normal forms and Proposition 8), and relate to classes of equations which are known to admit two independent first integrals. Only elementary algebraic arguments will be used.

A linear transformation to diagonalize B will yield

$$\dot{x} = f(x) := \begin{pmatrix} 0 \\ i\omega x_2 \\ -i\omega x_3 \end{pmatrix} + \begin{pmatrix} a_1x_1^2 + a_2x_1x_2 + 2a_3x_2^2 + a_4x_2x_3 + 2a_5x_3^2 + a_6x_1x_3 \\ b_1x_1^2 + b_2x_1x_2 + b_3x_2^2 + b_4x_2x_3 + 3b_5x_3^2 + 2b_6x_1x_3 \\ c_1x_1^2 + 2c_2x_1x_2 + 3c_3x_2^2 + c_4x_2x_3 + c_5x_3^2 + c_6x_1x_3 \end{pmatrix}. \tag{6}$$

Since (6) stems from a real quadratic system, the following relations hold between the coefficients:

$$\begin{array}{llllll} a_1 = \overline{a}_1, & a_6 = \overline{a}_2, & a_5 = \overline{a}_3, & a_4 = \overline{a}_4, & & \\ c_1 = \overline{b}_1, & c_6 = \overline{b}_2, & c_5 = \overline{b}_3, & c_4 = \overline{b}_4, & c_3 = \overline{b}_5, & c_2 = \overline{b}_6. \end{array} \tag{7}$$

We first record some known cases.

Lemma 18. *System* (6) *admits two independent first integrals in the following cases:*

(i) *The system is reversible with respect to the map exchanging* x_2 *and* x_3.
(ii) *The system has divergence zero and admits the first integral* x_1.

(iii) *The system is Hamiltonian with respect to the Lie-Poisson bracket defined by the standard vector product.*

We note that the system is reversible if and only if all coefficients are purely imaginary, thus $a_1 = 0$, $a_2 + a_6 = 0$ etc.; see the example following Proposition 11. Case (iii) can be described as follows: Letting

$$\begin{array}{rcl}\psi_1 & := & x_1 + x_1^2/2 + \omega x_2 x_3 \\ \psi_2 & := & \gamma_1 x_1^2 + \gamma_2 x_1 x_2 + \gamma_3 x_2^2 + \gamma_4 x_2 x_3 + \gamma_5 x_3^2 + \gamma_6 x_3 x_1,\end{array}$$

one obtains a quadratic vector field by taking the vector product of the gradients of ψ_1 and ψ_2. By construction this vector field admits the first integrals ψ_1 and ψ_2. Explicitly the homogeneous quadratic part is as follows:

$$\begin{pmatrix} -\omega\gamma_2 x_1 x_2 - 2\omega\gamma_3 x_2^2 + 2\omega\gamma_5 x_3^2 + \omega\gamma_6 x_3 x_1 \\ -\gamma_6 x_1^2 + (2\omega\gamma_1 - \gamma_4) x_1 x_2 + \omega\gamma_2 x_2^2 + \omega\gamma_6 x_2 x_3 - 2\gamma_5 x_3 x_1 \\ \gamma_2 x_1^2 + 2\gamma_3 x_1 x_2 - \omega\gamma_2 x_2 x_3 - \omega\gamma_2 x_3^2 + (\gamma_4 - 2\omega\gamma_1) x_3 x_1 \end{pmatrix}. \quad (8)$$

For later use we record certain normalizations.

Lemma 19. *Let $\alpha \in \mathbb{R}^*$ and $\beta \in \mathbb{C}$ with $|\beta| = 1$. The linear invertible transformation*

$$\begin{array}{rcl} x_1 & \mapsto & \alpha x_1 \\ x_2 & \mapsto & \beta x_2 \\ x_3 & \mapsto & \beta^{-1} x_3 \end{array}$$

sends (6) *to a quadratic system (corresponding to a real quadratic system) with the same linear part. By this transformation, b_1 is changed to $\alpha^2\beta^{-1}b_1$, c_1 is changed to $\alpha^2\beta c_1$, a_2 is changed to βa_2, and a_6 is changed to $\beta^{-1}a_6$.*

In particular, we may assume that b_1 is purely imaginary and thus $c_1 = -b_1$. In case $b_1 = c_1 = 0$ we may assume $a_6 = -a_2$ to be purely imaginary.

From now on we impose the following condition:

$$b_1 + c_1 = 0. \quad (9)$$

Now we start to use results from the computation of normal forms. (The requisite computations of normal forms, performed by a standard computer algebra system according to Algorithm 2.6 in [18], will not be recorded here; they are quite bulky.) The normal form up to degree 2 can be read off directly:

$$\tilde{f}_2(x) = \begin{pmatrix} 0 \\ i\omega x_2 \\ -i\omega x_3 \end{pmatrix} + \begin{pmatrix} a_1 x_1^2 + a_4 x_2 x_3 \\ b_2 x_1 x_2 \\ c_6 x_1 x_3 \end{pmatrix} + \cdots$$

Lemma 20. *The normal form up to degree 2 admits the first integrals x_1 and x_2x_3 if and only if*

$$a_1 = a_4 = 0 \quad \textit{and} \quad b_2 + c_6 = 0. \quad (10)$$

These conditions are therefore necessary for the conservation of two independent first integrals of $A = Df(0)$.

Computing a normal form up to degree 3 and requiring that x_1 is a first integral of the homogeneous cubic term yields the condition $b_1(a_2 + a_6) = 0$ and therefore we find, or may assume, due to Lemma 19, that

$$a_2 + a_6 = 0. \quad (11)$$

(In any case the parameter β in Lemma 19 is now fixed, unless $b_1 = a_2 = 0$). Substituting $a_6 = -a_2$ and re-calculating the normal form of degree 3 leads to the following set of conditions:

$$\begin{array}{rcl} a_2(b_4 + c_4) & = & 4(a_5 c_2 - a_3 b_6); \\ b_1(b_4 + c_4) & = & -2b_1(b_3 + c_5); \\ c_4 c_5 - b_3 b_4 & = & 2(a_5 c_2 - a_3 b_6). \end{array} \quad (12)$$

Lemma 21. *The following statements hold.*

(a) *Given* (9)*,* (10) *and* (11)*, equations* (12) *imply*

$$b_1\left(-a_2+2c_5+b_4\right)\left(b_4+c_4\right)=0. \tag{13}$$

(b) *In the case* $b_1\neq 0$ *and* $b_4+c_4=0$*, conditions* (12) *are equivalent to*

$$\begin{array}{rcl} b_4+c_4 & = & 0;\\ b_3+c_5 & = & 0;\\ a_5c_2-a_3b_6 & = & 0. \end{array} \tag{14}$$

(c) *In the case* $b_1\neq 0$ *and* $-a_2+2c_5+b_4=0$*, conditions* (12) *are equivalent to*

$$\begin{array}{rcl} a_2+2b_3+c_4 & = & 0;\\ -a_2+2c_5+b_4 & = & 0;\\ -a_2(b_3+c_5) & = & 2(a_5c_2-a_3b_6). \end{array} \tag{15}$$

The first two of these conditions are equivalent to $\operatorname{div} f=0$*.*

Proof. Combining the first and third equations in (12) yields

$$a_2(b_4+c_4)=2(c_4c_5-b_3b_4)=2c_5(b_4+c_4)-2b_4(b_3+c_5).$$

Multiplying by b_1 and using the second equation in (12) one obtains

$$b_1a_2(b_4+c_4)=2b_1c_5(b_4+c_4)+b_1b_4(b_4+c_4),$$

and hence the assertion of (a). The remaining assertions follow from straightforward computations. □

Remark 22. *The rewriting procedure in the first equation of the proof is not accidental. In principle, we are determining the ideal I in $\mathbb{C}[a_1,a_2,\ldots,c_5,c_6]$ whose zero set corresponds to conservation of the first integrals in a normal form of* (6)*. On the other hand, the example following Proposition 11 shows that zero set of the ideal*

$$J:=\langle a_1,a_2+a_6,a_3+a_5,a_4,b_1+c_1,b_2+c_6,b_3+c_5,b_4+c_4,b_5+c_3,b_6+c_2\rangle$$

lies in the zero set of I. Since J is generated by degree one polynomials, it is prime and hence radical. Hilbert's Nullstellensatz now shows $I\subseteq J$. Similar considerations have been employed for the two-dimensional center problem; see Jarrah et al. [9]*.*

From here on we assume that

$$b_1\neq 0. \tag{16}$$

Geometrically, this means that the straight line defined by $x_2=x_3=0$ is not invariant for system (6).

We proceed to further discuss the case from Lemma 21(b). The homogeneous normal form terms of degree four (resp. five) provide five (resp. six) additional conditions for the coefficients, and it seems advisable to first consider the easily accessible ones. (Some routine verifications will only be mentioned; we will outline the non-routine arguments). Recall that we have $b_1\neq 0$, and assume equations (9) (thus $c_1=-b_1$), (10) (thus $a_1=a_4=0$ and $c_6=-b_2$), (11) (thus $a_6=-a_2$) and (14), which implies $c_4=-b_4$ and $c_5=-b_3$. The remaining condition from (14) will be rewritten as

$$a_5(c_2+b_6)-b_6(a_3+a_5)=0. \tag{17}$$

From the degree four normal form one obtains:

$$\begin{array}{rcrcl} b_1(a_5+a_3) & + & a_2(b_6+c_2) & = & 0;\\ 4b_1(a_5+a_3) & + & (3a_2+b_4-2b_3)(b_6+c_2) & = & 0. \end{array} \tag{18}$$

Combining these equations one obtains

$$(a_2-b_4+2b_3)(c_2+b_6)=0. \tag{19}$$

This provides a natural distinction of two cases. If $a_2 - b_4 + 2b_3 = 0$ then the system has divergence zero, and conversely. We first dispose of the other case.

Case b1:

$$c_2 = -b_6 \quad \Rightarrow \quad a_5 = -a_3 \quad \text{by (18).}$$

From the degree five normal form and $b_1 \neq 0$ one obtains

$$a_2(b_5 + c_3) = 0.$$

If $c_3 = -b_5$ then, by Lemma 18 the system is reversible. If $a_2 = 0$ and $c_3 + b_5 \neq 0$ then another degree four condition implies

$$a_3 b_1 = 0 \quad \Rightarrow \quad a_3 = 0 = a_5,$$

and hence x_1 is a first integral of the quadratic system. A further degree five condition forces

$$b_4 - 2b_3 = 0 \quad \Rightarrow \quad \operatorname{div} f = 0.$$

Therefore, in Case b1 every system (6) admitting two independent first integrals is among those listed in Lemma 18. This leaves

Case b2:

$$a_2 - b_4 + 2b_3 = 0; \quad \text{divergence } 0.$$

Here we obtain another degree four condition, viz.

$$a_2(c_2 + b_6) + b_1(a_3 + a_5) = 0. \tag{20}$$

Combination with (17) yields

$$\begin{array}{rcl} (c_2 + b_6) \cdot (a_2 b_6 + a_5 b_1) & = & 0; \\ (a_3 + a_5) \cdot (a_2 b_6 + a_5 b_1) & = & 0. \end{array} \tag{21}$$

These conditions suggest a further splitting into subcases.

Subcase b2.1:

$$c_2 + b_6 = 0 \quad \Rightarrow \quad a_3 + a_5 = 0 \quad \text{by (20).}$$

Here we are back in Case b1. Assume $c_2 + b_6 \neq 0$ now.

Subcase b2.2:

$$a_3 + a_5 = 0 \quad \Rightarrow \quad a_2 = a_3 = a_5 = 0 \quad \text{by (17), (20).}$$

Thus x_1 is a first integral of (6) and the system has divergence zero, so Lemma 18 applies.

Subcase b2.3:

$$a_2 b_6 + a_5 b_1 = 0. \tag{22}$$

Here we employ Lemma 19 a second time, by choosing the real parameter α. Due to $b_1 \neq 0$, we may assume

$$a_2 \in \{0,\, b_1,\, -b_1\}.$$

Subcase b2.3.1:

$$a_2 = 0 \quad \Rightarrow \quad a_3 = a_5 = 0 \quad \text{by (22).}$$

Again, Lemma 18(b) applies.

Subcase b2.3.2:

$$a_2 = b_1 \quad \Rightarrow \quad b_6 = -a_5 \text{ and } c_2 = -a_3 \text{ by (22), (17).}$$

From degree five and $b_1 \neq 0$ one has

$$b_5 + c_3 = 0,$$

and there are two more (simplified) degree four conditions:

$$\begin{array}{rcl} (b_1 + b_3 + 3b_5)(a_5 + a_3) & = & 0; \\ (5b_1 b_5 + 8b_3 b_5 + 7b_1 b_3 + 4b_3^2 + 3b_1^2)(a_5 + a_3) & = & 0. \end{array} \tag{23}$$

If $a_3 + a_5 = 0$ then one also has $c_2 + b_6 = 0$ in this case, and reversibility. If $a_3 + a_5 \neq 0$, the conditions from (23) simplify to

$$\begin{aligned} b_1 + b_3 + 3b_5 &= 0; \\ -4b_5(b_1 + b_3) &= 0; \end{aligned}$$

and therefore

$$b_1 + b_3 = b_5 = 0, \text{ and } c_3 = 0.$$

This implies that the system is Hamiltonian with respect to a Lie-Poisson bracket (Lemma 18(c)).

Subcase b2.3.3: This runs closely parallel to the previous case. If $a_2 = -b_1 \neq 0$ then $b_6 = a_5$ and $c_2 = a_3$ by (22) and (17), and evaluating the normal form of degree five one finds $b_5 + c_3 = 0$. The remaining arguments are similar to those in Case b2.3.2, and again one finds that the vector field is Hamiltonian with respect to a Lie-Poisson bracket. To summarize:

Proposition 23. *The following statements hold.*

(a) *Every system* (6)*, with* $b_1 \neq 0$*, which admits two independent first integrals and satisfies the hypotheses of Lemma 21(b) is listed in Lemma 18.*

(b) *Generally, if system* (6) *has nonzero divergence, and does not admit the invariant line given by* $x_2 = x_3 = 0$*, then it admits two independent first integrals near* 0 *if and only if it is reversible.*

Concerning the remaining cases, if $b_1 \neq 0$ and the conditions in Lemma 21(c) hold, then by similar, but more arduous, arguments and computations (using normal forms up to degree 6) one can show that only systems of the types listed in Lemma 18 admit two independent first integrals near 0. It does not seem appropriate to include these extensive computations and reasoning in the present paper.

As for the last remaining case, viz. $b_1 = 0$, the list from Lemma 18 certainly will not suffice. Consider a special system (6) such that it admits the first integral x_1 and that the second and third entries of the right-hand side are independent of x_1. Clearly, such a system admits two independent first integrals if the two-dimensional system for x_2 and x_3 admits a nonconstant first integral (and thus a center). Inspection shows that this system is of the type listed in Lemma 18 if and only if the two-dimensional equation for x_2 and x_3 is either Hamiltonian or reversible. However, as it is known from Dulac [7], there exist two-dimensional quadratic center systems of other types (e.g. admitting a Darboux first integral). It seems that the case $b_1 = 0$ is much harder to tackle, and it may require more advanced methods than we have used.

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