Algebraic Groups I

Summer School
Representations of Algebraic Groups and Lie Algebras in Characteristic p

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Experimental and constructive algebra

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Affine algebraic sets

Let K be a field.

Definition

- $\mathbb{A}^n(K) := \{(\alpha_1, ..., \alpha_n) \in K^n\}$ **affine n-space**.
- $f \in K[x_1, ..., x_n]$, $V(f) := \{\alpha \in \mathbb{A}^n(K) \mid f(\alpha) = 0\}$ **zero locus** of f.
- $S \subset K[x_1, ..., x_n]$, $V(S) := \bigcap_{f \in S} V(f)$ an **affine algebraic set**.

Examples

Let $K = \mathbb{R}$.

- $f = x_1^2 + x_2^2 - 1$, $V(f)$ is a circle in $\mathbb{A}^2(\mathbb{R})$.
- $f = x_1 x_4 - x_2 x_3 - 1$, $V(f) = \mathrm{SL}_2(\mathbb{R})$.

Remark

- $S \subseteq K[x_1, ..., x_n]$, $V(\langle S \rangle) = V(S)$.
- Hilbert's Basis Theorem: $K[x_1, ..., x_n]$ is Noetherian.

Definition

$\mathfrak{X} \subseteq \mathbb{A}^n(K)$, R comm. ring, $A \trianglelefteq R$.

- $I(\mathfrak{X}) := \{f \in K[x_1, \dots, x_n] \mid f(\alpha) = 0 \ \forall \ \alpha \in \mathfrak{X}\} \trianglelefteq K[x_1, \dots, x_n]$.
- $K[\mathfrak{X}] := K[x_1, \dots, x_n]/I(\mathfrak{X})$ **coordinate ring** of $\mathfrak{X}$.

Zariski topology

The affine algebraic sets are the closed sets of the **Zariski topology**.

Regular functions

$\mathfrak{X} \subseteq \mathbb{A}^n$ affine, $\mathcal{Y} \subseteq \mathfrak{X}$ open.
$f : \mathcal{Y} \to K$ is called regular in $\alpha \in \mathcal{Y}$ if:
$\exists \ \mathcal{U} \subseteq \mathcal{Y}$ open s.t. $\alpha \in \mathcal{U}$,
$\exists \ g, h \in K[\mathfrak{X}] \ : \ \forall \ \beta \in \mathcal{U} \ : \ f(\beta) = \frac{g(\beta)}{h(\beta)}, \ h(\beta) \neq 0$.
$\mathcal{O}(\mathcal{Y}) := \{f : \mathcal{Y} \to K \mid f \text{ regular in } \mathcal{Y}\}$.
$\mathcal{O}(\mathfrak{X}) \cong K[\mathfrak{X}]$.

Affine varieties

- $\mathfrak{X} \subseteq \mathbb{A}^n$ affine, $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(\ ))$ is called an **affine variety**.
- $\mathfrak{X}$, $\mathcal{Y}$ affine varieties, $\phi : \mathfrak{X} \to \mathcal{Y}$ is a **morphism** of affine varieties, if
  1. ϕ is Zariski-continuous.
  2. $\forall\, \mathcal{U} \subseteq \mathcal{Y}$ open $\forall\, f \in \mathcal{O}_{\mathcal{Y}}(\mathcal{U}) : f \circ \phi \in \mathcal{O}_{\mathfrak{X}}(\phi^{-1}(\mathcal{U}))$.

Remark

Affine varieties form a category which is anti-equivalent to the category of finitely generated reduced K-algebras if $K = \overline{K}$.
The category of affine varieties has finite products defined in the usual category-theoretic sense.

Linear algebraic groups

A linear algebraic group is an affine variety G with morphisms

$$m : G \times G \to G, \quad i : G \to G$$

such that G becomes a group by putting $\sigma \cdot \tau := m(\sigma, \tau)$ and $\sigma^{-1} := i(\sigma)$.

Theorem

Any linear algebraic group admits a faithful linear representation $\rho : G \to \mathrm{GL}_n(K)$ for some $n \in \mathbb{N}$ which is a morphism of algebraic groups.

Remark

A linear algebraic group G can be considered as a functor $K - \mathrm{CommAlg} \to \mathrm{Grp}$ such that

$$K - \mathrm{CommAlg} \xrightarrow{G} \mathrm{Grp} \xrightarrow{\text{forget}} \mathrm{Set}$$

is affine, i.e. representable by a finitely generated commutative K-algebra.
R-**rational points:** $G(R)$.

Example

Consider $\mu_n : \mathbb{Q} - \mathrm{CommAlg} \to \mathrm{Grp},\ R \mapsto \{r \in R \mid r^n = 1\}$ and compare $\mu_4(\mathbb{Q})$ with $\mu_4(\mathbb{C})$.

Definition

Let G be irreducible. $\dim(G) := \mathrm{Krull\text{-}dim}(K[G]) = \mathrm{top.dim}(G)$.

Example

- $\mathrm{GL}_n(K) := \{(d, X_{i,j}) \in K^{n^2+1} \mid d \cdot \det(X) = 1\}$ is an algebraic group.
- $\dim(\mathrm{GL}_n) = n^2$.
- Consider GL_n as an algebraic group defined over $\mathbb{F}_p$.
- $\mathrm{GL}_n(\overline{\mathbb{F}_p})$ is locally finite: $(x_{i,j})_{1 \le i,j, \le n} \in \mathrm{GL}_n(\mathbb{F}_p[x_{i,j}])$.
- $\mathrm{GL}_n(\overline{\mathbb{F}_p}) = \bigcup_{i=1}^{\infty} \mathrm{GL}_n(\mathbb{F}_{p^{i!}})$.

Definition

Let (G, m, i) be an algebraic group. If $U \subseteq G$ is a Zariski-closed subset of G such that $(U, m|_{U \times U}, i_U)$ is a group, then we call U an (algebraic) subgroup of G. U is an algebraic group in its own right.

Connectedness

Definition

G alg. group over K. G is **connected** if its underlying topological space is connected, i.e. $G = U_1 \cup U_2$ with U_i closed and $U_1 \cap U_2 = \emptyset$ implies $U_1 = \emptyset$ or $U_2 = \emptyset$.
G is **irreducible** if its underlying affine variety is not the union of two proper closed subsets.

Theorem

An affine algebraic group is connected if and only if it is irreducible.

Remark

G is irreducible if and only if $K[G]$ is an integral domain.

Definition: identity component

$G^0 :=$ connected component of $1 \in G$ is a closed normal subgroup of finite index.

Semisimple and unipotent elements

Let $\rho : G \hookrightarrow \mathrm{GL}(V)$ be a faithful representation with $\dim_K(V) < \infty$.

Definition

$g \in G$ is called

- **semisimple** if $\rho(g)$ is diagonalizable,
- **unipotent** if $\rho(g) - 1$ is nilpotent.

Remark

The above properties are intrinsic of g, i.e. they are independent of the choice of ρ.

Example

Consider $\mathbb{G}_m := \mathrm{GL}_1$ and $\mathbb{G}_a$, defined by $\mathbb{G}_a(k) := (k, +)$.

- $\mathbb{G}_m$ consists of semisimple elements.
- $\mathbb{G}_a$ consists of unipotent elements, cf. $x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

Jordan Decomposition

Remark

Let V be a finite dimensional vector space over a perfect field.
Let $x \in \mathrm{End}(V)$ be invertible. Then we can write

$$x = x_s x_u,$$

where x_s is semisimple, x_u is unipotent and $x_s x_u = x_u x_s$. This decomposition is **unique**.
If the underlying field is not perfect, such a decomposition may not exist.

Definition

Let $\rho : G \hookrightarrow \mathrm{GL}_n(K)$ be a faithful representation.
For every $g \in G$ there is a Jordan decomposition $g = g_s g_u$ defined by

$$\rho(g_s) = \rho(g)_s, \ \rho(g_u) = \rho(g)_u. \qquad (\dagger)$$

$(\dagger)$ is satisfied for every faithful representation.

Definition

- A maximal closed connected solvable subgroup $B \leq G$ is called a **Borel** subgroup.
- A subgroup $P \leq G$ s.t. $B \subseteq P$ is called **parabolic**.

Example

The invertible upper triangular matrices $(\triangledown)_n$ are a Borel subgroup of GL_n. $(\triangledown)_n \cap \mathrm{SL}_n$ is a Borel subgroup of SL_n.

Definition

$T \leq G$ is called a **torus** if $T(\overline{K}) \cong \mathbb{G}_m^r(\overline{K})$ for some $r \geq 1$.
T is called **split** if $T(K) \cong \mathbb{G}_m^r(K)$.

Example

SO_2 is a torus, but not a split torus: $\mathrm{SO}_2(\mathbb{C}) \cong \mathbb{C}^\times$, $\mathrm{SO}_2(\mathbb{R}) \cong S^1$.