IWAHORI-HECKE ALGEBRAS AND KAZHDAN-LUSZTIG POLYNOMIALS

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Summer school on
Representations of Algebraic Groups and Lie Algebras in Characteristic p
Marienheide, Sep 07 – 11, 2015

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Coxeter Groups

Let S be a finite set and $M = [m_{st}]_{s,t\in S}$ a symmetric matrix with $m_{st} \in \mathbb{Z} \cup \{\infty\}$ satisfying $m_{ss} = 1$ and $m_{st} > 1$ for $s \neq t$.

The group $W := W(M)$ with presentation

$$W := \left\langle s \in S \mid s^2 = 1, (st)^{m_{st}} = 1 (s \neq t, m_{st} \neq \infty) \right\rangle_{\text{group}},$$

is called the Coxeter group defined by M;

the (images of the) elements of S (in W) are the Coxeter generators of W.

The pair (W, S) is called a Coxeter system of rank $|S|$.

The relations $(st)^{m_{st}} = 1$ $(s \neq t)$ are called the braid relations.

In view of $s^2 = 1$, they can be written as

$$sts \cdots = tst \cdots \quad (m_{st} \text{ factors on each side}).$$

Examples

Example

The finite symmetric group S_n on n-letters is a Coxeter group: If s_i denotes the transposition $(i, i+1)$, $1 \leq i \leq n-1$, then

$$S_n = \left\langle s_i \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 (|i-j| \geq 2) \right\rangle .$$

Example

$\mathrm{PGL}(2, \mathbb{Z})$ *is a Coxeter group: Let r, s, t denote the images of* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *in* $\mathrm{PGL}(2, \mathbb{Z})$*, respectively. Then*

$$\mathrm{PGL}(2, \mathbb{Z}) = \left\langle r, s, t \mid r^2 = s^2 = t^2 = 1, (rs)^3 = (rt)^2 = 1 \right\rangle .$$

THE COXETER DIAGRAM

The matrix M is usually encoded in a Coxeter diagram, e.g.

$$M := B_n := \begin{bmatrix} 1 & 4 & 2 & 2 & \cdots & 2 \\ 4 & 1 & 3 & 2 & \cdots & 2 \\ 2 & 3 & 1 & 3 & & 2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 2 & \cdots & 2 & 3 & 1 & 3 \\ 2 & \cdots & 2 & 2 & 3 & 1 \end{bmatrix}$$

is displayed as

B_n:

with $m_{s_i s_j} - 2$ edges between s_i, s_j ($i \neq j$).

Weyl Groups and Finite Reflection Groups

A Weyl group of a reductive algebraic group is a Coxeter group with Coxeter generators s_α, where α runs through a set of simple roots.

Let (W, S) be a Coxeter system with $W = W(M)$, $M = [m_{st}]$. Let V be an $\mathbb{R}$-vector space of dimension $|S|$, and B the symmetric bilinear form on V with Gram matrix $[-\cos(\pi/m_{st})]$.

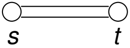
Facts

1. *W acts faithfully on V, preserves B, and the elements of S act as reflections (w.r.t. B).*
2. *W is finite if and only if B is positive definite.*

Thus W is finite if and only if it is a finite reflection group, i.e. a finite group generated by reflections in a finite-dimensional Euclidean vector space.

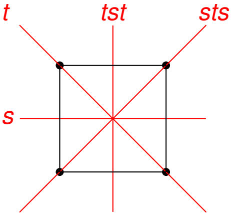
EXAMPLE

Consider the Coxeter diagram of type B_2:



$$B_2 := \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}.$$

$W := W(B_2) = \{1, s, t, st, ts, sts, tst, stst\}$ is the symmetry group of the unit square, a dihedral group.



s, t, sts and tst are reflections of the square in the indicated lines, st, $stst$ and ts are rotations by 90, 180 and 270 degrees, respectively.

Parabolic Subgroups

Let (W, S) be a Coxeter system. For a subset $J \subseteq S$ put

$$W_J := \langle s \mid s \in J \rangle \leq W.$$

Then (W_J, J) is a Coxeter system, more precisely, W_J is the Coxeter group defined by the matrix $M_J := [m_{st}]_{s,t \in J}$.

A subgroup of W of this form is called a parabolic subgroup.

THE LENGTH FUNCTION

Let (W, S) be a Coxeter system, and let $w \in W$.

Suppose that $l \in \mathbb{N}$ is minimal such that there exist $s_1, \ldots, s_l \in S$ with

$$w = s_1 \cdots s_l.$$

Then $s_1 \cdots s_l$ is called a reduced expression for w, and

$$\ell(w) := l$$

is called the length of w.

Thus 1 is the unique element of W of length 0, and $w \in S$ if and only if $\ell(w) = 1$.

FACT

If W is finite, there is a unique element w_0 of maximal length. It satisfies $w_0^2 = 1$.

The Bruhat Order

Let (W, S) be a Coxeter system and let $v, w \in W$.

We write

$$v \leq w,$$

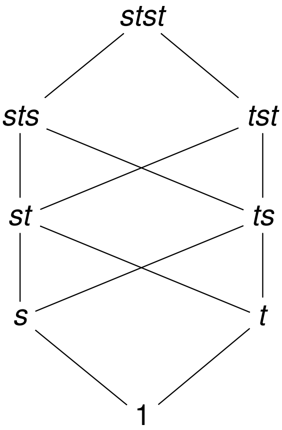
if $v = s_{i_1} \cdots s_{i_m}$ for some reduced expression $s_1 \cdots s_l$ for w.

This defines a partial order on W, the Bruhat order.

We have $1 \leq w$ for all $w \in W$.

If W is finite, we have $w \leq w_0$ for all $w \in W$.

EXAMPLE: THE BRUHAT ORDER OF $W(B_2)$



THE IWAHORI-HECKE ALGEBRA

Let (W, S) be a Coxeter system with $W = W(M)$, $M = [m_{st}]$.

Let A be a commutative ring and $u \in A$. The algebra

$$\mathcal{H}_{A,u}(W) := \left\langle T_s, s \in S \mid T_s^2 = u\mathbf{1} + (u-1)T_s, \text{ braid rel's } \right\rangle_{A\text{-alg.}}$$

is called the (one-parameter) Iwahori-Hecke algebra of W over A with parameter u.

(The symbol $\mathbf{1}$ denotes the unit element of $\mathcal{H}_{A,u}(W)$.
Braid rel's: $T_s T_t T_s \cdots = T_t T_s T_t \cdots$ (m_{st} factors on each side))

$T_s \mapsto -1$ and $T_s \mapsto u$ define two A-algebra homomorphisms $\mathcal{H}_{A,u}(W) \to A$ (from defining relations).

The Theorem of Iwahori

Let $G := \mathrm{GL}_n(q)$ denote the general linear group over the finite field with q elements.

Let $B \leq G$ denote the group of upper triangular matrices.

Let $\mathbb{C}[G/B]$ denote the permutation module of $\mathbb{C}G$ on the set G/B of left B-cosets in G.

Put $E := \mathrm{End}_{\mathbb{C}G}(\mathbb{C}[G/B])$.

Theorem (Iwahori ('64))

$E \cong \mathcal{H}_{\mathbb{C},q}(S_n)$.

Notice that the Coxeter group S_n is the Weyl group of G.

A BASIS OF THE IWAHORI-HECKE ALGEBRA

Let $w \in W$. Choose a reduced expression $w = s_1 s_2 \cdots s_l$ and put

$$T_w := T_{s_1} T_{s_2} \cdots T_{s_l} \in \mathcal{H}_{A,u}(W).$$

FACTS

1. *T_w is independent of the chosen reduced expression for w.*
2. *$\mathcal{H}_{A,u}(W)$ is a free A-module with A-basis T_w, $w \in W$.*

The elements $T_w, w \in W$ are called the standard basis elements of $\mathcal{H}_{A,u}(W)$. Notice that T_1 is the identity of $\mathcal{H}_{A,u}(W)$.

INVERTIBILITY OF THE STANDARD BASIS ELEMENTS

Suppose that $u \in A$ is invertible.

Then T_s is invertible in $\mathcal{H}_{A,u}(W)$.

Indeed $T_s^{-1} = u^{-1}T_s + (u^{-1} - 1)T_1$.

In turn, the standard basis elements are invertible.

PROPOSITION

For all $w \in W$,

$$(T_{w^{-1}})^{-1} = (-u)^{-\ell(w)} \sum_{y \leq w} (-1)^{\ell(y)} R_{y,w} T_y,$$

with $R_{y,w} \in \mathbb{Z}[u] \leq A$ of degree $\ell(w) - \ell(y)$ in u, and $R_{w,w} = 1$.

This shows the relevance of the Bruhat order.

SPECIALIZATION

Let B be a commutative ring and

$$\varphi : A \to B$$

a ring homomorphism.

Consider B as an A-module via φ and put

$$B\mathcal{H}_{A,u}(W) := B \otimes_A \mathcal{H}_{A,u}(W).$$

$B\mathcal{H}_{A,u}(W)$ is called the specialization of $\mathcal{H}_{A,u}(W)$ via φ.

FACT

$B\mathcal{H}_{A,u}(W) \cong \mathcal{H}_{B,\varphi(u)}(W)$.

This allows to construct all Iwahori-Hecke algebras of W from a generic one.

THE GENERIC IWAHORI-HECKE ALGEBRA

Let $\mathbf{v}$ be an indeterminate, put $\mathbb{A} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$ and $\mathbf{u} := \mathbf{v}^2$.

Then $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$ is called the generic (one-parameter) Iwahori-Hecke algebra of W.

Assume that u has a square root $v \in A$.

By the specialization $\varphi : \mathbb{A} \to A, \mathbf{v} \mapsto v$, we obtain $\mathcal{H}_{A,u}(W)$.

Note that $\mathcal{H}_{A,1}(W) \cong AW$, so that the group algebra AW is a specialization of $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$,
or $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$ is a (generic) deformation of AW.

Kazhdan-Lusztig Polynomials

Let (W, S) be a Coxeter system and let $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$ be the generic Iwahori-Hecke algebra of W.

There is an involution ι on $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$ determined by $\iota(\mathbf{v}) = \mathbf{v}^{-1}$ and $\iota(T_w) = (T_{w^{-1}})^{-1}$ for all $w \in W$.

Let $\leq$ denote the Bruhat order on W.

Theorem (Kazhdan-Lusztig, 1979)

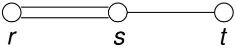
There is a unique basis C'_w, $w \in W$ of $\mathcal{H}_{\mathbb{A},\mathbf{u}}(W)$ such that

1. $\iota(C'_w) = C'_w$ *for all* $w \in W$*;*
2. $C'_w = \mathbf{v}^{-\ell(w)} \sum_{y \leq w} P_{y,w} T_w$ *with* $P_{w,w} = 1$, $P_{y,w} \in \mathbb{Z}[\mathbf{u}]$, $\deg P_{y,w} \leq (\ell(w) - \ell(y) - 1)/2$ *for all* $y < w \in W$.

The $P_{y,w} \in \mathbb{Z}[\mathbf{u}]$, $y \leq w \in W$, are called the Kazhdan-Lusztig polynomials of W.

Example

Let $W := W(B_3)$ with Coxeter diagram



The following table gives all $P_{1,w}$ with $w \in W$ and $P_{1,w} \neq 1$.

w	$P_{1,w}$
$strs$	$\mathbf{u} + 1$
$rstrs$	$\mathbf{u} + 1$
$rstsr$	$\mathbf{u} + 1$
$strsr$	$\mathbf{u} + 1$
$tsrst$	$\mathbf{u} + 1$
$rstrsr$	$\mathbf{u}^2 + 1$
$rstsrs$	$\mathbf{u} + 1$

w	$P_{1,w}$
$tsrstr$	$\mathbf{u} + 1$
$srstsr$	$\mathbf{u} + 1$
$tsrstrs$	$\mathbf{u} + 1$
$srstsrs$	$\mathbf{u}^2 + \mathbf{u} + 1$
$tsrstsr$	$\mathbf{u} + 1$
$tsrstrsr$	$\mathbf{u} + 1$
$tsrstsrs$	$\mathbf{u}^2 + 1$

Some Properties of Kazhdan-Lusztig-Polynomials

Let (W, S) be a Coxeter system and let $y, w \in W$ with $y \leq w$.

1. $P_{y,w}$ can be computed recursively (by inducing on the Bruhat order); see Frank's talk.
2. $P_{y,w}(0) = 1$ (this follows from the recursion formulae).
3. If W is finite, then

$$\sum_{y \leq z \leq w} (-1)^{\ell(z)+\ell(w)} P_{y,z} P_{w_0 w, w_0 z} = \delta_{y,w}.$$

Theorem (Kazhdan-Lusztig ('79), …, Elias-Williamson ('14))

The coefficients of $P_{y,w}$ are non-negative integers.

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Thank you for your listening!