Geometry of numbers: old and new problems

Jacques Martinet

Université de Bordeaux, IMB

Aachen, September 26th, 2010

Workshop, Aachen, September 26–30, 20011

FIRST PART

Geometry of Numbers and Algebraic Number Theory

We briefly discuss some problems in the "classical" geometry of numbers, on which we hope that progress may be made nowadays.
More details can be read on my homepage
http://math.u-bordeaux.fr/~martinet/, *On the Minkowski Constants for Class Groups*, Section "other texts".

In 2009, one century and a few days after Minkowski's death, I delivered a talk on his life in Besançon. The corresponding slides can be downloaded from my homepage, *Hermann Minkowski, 1864-1909*, Section "a few slides".

From positive definite quadratic forms to Euclidean lattices

Let $q := X\,A\,X^{tr}$ be a positive definite quadratic form on $\mathbb{R}^n$ ($X = (x_1, \dots, x_n)$, $A \in \mathrm{Sym}_n(\mathbb{R})$), of *discriminant* and *minimum*

$$\mathrm{disc}(q) = \det(A) \quad \text{and} \quad \min\, q = \min_{X \neq 0}\, q(X)\,.$$

We would like to bound from above quotient $\gamma(q) = \dfrac{\min q}{\mathrm{disc}(q)^{1/n}}$

(the future *Hermite invariant*). The history begins with

LAGRANGE (1770) : $n = 2$. Then:

GAUSS (1831) : $n = 3$.

HERMITE (1845) : Explicit bound (*exponential in n*).

KORKINE–ZOLOTAREFF (1873, 1877) : $n = 4, 5$.

From positive definite quadratic forms to Euclidean lattices

Let $q := X\,A\,X^{tr}$ be a positive definite quadratic form on $\mathbb{R}^n$ ($X = (x_1, \dots, x_n)$, $A \in \mathrm{Sym}_n(\mathbb{R})$), of *discriminant* and *minimum*

$$\mathrm{disc}(q) = \det(A) \quad \text{and} \quad \min\, q = \min_{X \neq 0}\, q(X)\,.$$

We would like to bound from above quotient $\gamma(q) = \dfrac{\min q}{\mathrm{disc}(q)^{1/n}}$

(the future *Hermite invariant*). The history begins with

LAGRANGE (1770) : $n = 2$. Then:

GAUSS (1831) : $n = 3$.

HERMITE (1845) : Explicit bound (*exponential in n*).

KORKINE–ZOLOTAREFF (1873, 1877) : $n = 4, 5$.

Then MINKOWSKI found a completely new point of view, introducing lattices in the problem.

The birth of Geometry of Numbers

A *lattice* in an n-dimensional Euclidean space E ($\simeq \mathbb{R}^n$) is a subgroup of E having a basis over $\mathbb{Z}$ which is a basis for E.

Basic idea.
it amounts to the same to consider the minimum

- of all forms on **the** lattice $\mathbb{Z}^n$;
- of all forms on **all** lattices ;
- of **one** form — the Euclidean structure — on all lattices ;

and one gets a bound for the *Hermite constant* $\gamma_n = \sup_q \gamma(q)$ by writing that the lattice packs the balls of radius half the minimal distance of two points of the lattice,

then bounding by 1 the density of a lattice packing of spheres.

... Geometry of Numbers (continuation)

MINKOWSKI soon discovered that spheres could be replaced by any symmetric convex body, proving his famous theorem, which allowed him to deduce important inequalities from volume computations.

His theory was later christened and developed in his book **Geometrie der Zahlen** (Teubner, Leipzig, 1896).

... Geometry of Numbers (continuation)

MINKOWSKI soon discovered that spheres could be replaced by any symmetric convex body, proving his famous theorem, which allowed him to deduce important inequalities from volume computations.

His theory was later christened and developed in his book **Geometrie der Zahlen** (Teubner, Leipzig, 1896).

HILBERT (*Gedächtnis Rede*):
Dieser Beweis eines tiefliegenden zahlentheoretischen Satzes
ohne rechnerische Hilfmittel
wesentlich auf Grund einer geometrisch anschaulichen Betrachtung
ist eine perle Minkowskischer Erfindungskunst.

... Geometry of Numbers (continuation)

MINKOWSKI soon discovered that spheres could be replaced by any symmetric convex body, proving his famous theorem, which allowed him to deduce important inequalities from volume computations.

His theory was later christened and developed in his book **Geometrie der Zahlen** (Teubner, Leipzig, 1896).

HILBERT (*Gedächtnis Rede*):
Dieser Beweis eines tiefliegenden zahlentheoretischen Satzes
ohne rechnerische Hilfmittel
wesentlich auf Grund einer geometrisch anschaulichen Betrachtung
ist eine perle Minkowskischer Erfindungskunst.

Actually Minkowski's proof needs the evaluation of a density. The really simple proof we know, replacing this evaluation by an easy argument of measure theory, was discovered by BLICHFELDT a few years after Minkowski's death.

Lattice constants for homogeneous problems

A lattice Λ is *admissible* for $A \subset E$ if $\Lambda \cap A = \{0\}$ (or $= \emptyset$).
The *lattice constant* of A is

$$\kappa(A) = \inf_{\Lambda \text{ admissible}} \det(\Lambda)$$

($+\infty$ if admissible lattices do not exist). [**Warning.** The *determinant* $\det(\Lambda)$ is the square of Minkowski's *discriminant* $\Delta(\Lambda)$.]

Finding good lower bounds for suitably chosen subsets A of E can be used to prove useful inequalities in various domains of number theory, e.g., diophantine approximations, algebraic number theory, ...

Lattice constants for homogeneous problems

A lattice Λ is *admissible* for $A \subset E$ if $\Lambda \cap A = \{0\}$ (or $= \emptyset$).
The *lattice constant* of A is

$$\kappa(A) = \inf_{\Lambda \text{ admissible}} \det(\Lambda)$$

($+\infty$ if admissible lattices do not exist). [**Warning.** The *determinant* $\det(\Lambda)$ is the square of Minkowski's *discriminant* $\Delta(\Lambda)$.]

Finding good lower bounds for suitably chosen subsets A of E can be used to prove useful inequalities in various domains of number theory, e.g., diophantine approximations, algebraic number theory, ...

We now restrict ourselves to *homogeneous problems*, those for which

$$A = A_F = \{x \in E \mid F(x) < 1\}\,,$$

where F is a *distance function*, that is, satisfies a "homogeneity" condition of the form

$$F(\lambda\, x) = |\lambda|^{\delta} F(x)$$

for some strictly positive *degree* δ.

Basic examples

1. Take for F a non-degenerate quadratic form q.
When q is positive definite, it suffices to consider $q(x) = x \cdot x$. This is the problem of spheres.

Basic examples

1. Take for F a non-degenerate quadratic form q.
When q is positive definite, it suffices to consider $q(x) = x \cdot x$. This is the problem of spheres.

2. Chose a decomposition $n = r_1 + 2r_2$, and consider on $E = \mathbb{R}^n$ the function

$$F_{r_1,r_2}(x) = \frac{1}{2^{r_2}} \left| x_1 \cdots x_{r_1} \right| (y_1^2 + z_1^2) \cdots (y_{r_2}^2 + z_{r_2}^2) \,.$$

Set $\kappa_{r_1,r_2} = \kappa(A_F)$.

The fundamental theorem of Minkowski on class groups can be stated, bounding from below κ_{r_1,r_2} by a lower bound of the lattice constant of the largest convex body it contains, obtained by a volume computation.

The Minkowski theorem on class groups (1)

Theorem

Let K be a number field of signature (r_1, r_2) (and degree $n = r_1 + 2r_2$). Then any class of ideal of K contains an integral ideal $\mathfrak{a}$ such that

$$\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a}) \leq \left(\frac{|d_K|}{\kappa_{r_1,r_2}}\right)^{1/2}.$$

The Minkowski theorem on class groups (1)

Theorem

Let K be a number field of signature (r_1, r_2) (and degree $n = r_1 + 2r_2$). Then any class of ideal of K contains an integral ideal $\mathfrak{a}$ such that

$$\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a}) \leq \left(\frac{|d_K|}{\kappa_{r_1,r_2}}\right)^{1/2}.$$

Set

$$B_{r_1,r_2} = \left\{x \in \mathbb{E} \,\middle|\, |x_1| + \cdots + |x_{r_1}| + 2|z_1| + \cdots + 2|z_{r_2}| < 1\right\}.$$

This is a convex set, and the arithmetico-geometric inequality shows that A_F contains $n\,2^{r_2/n} B_{r_1,r_2}$. Calculating the volume of B, one obtains the famous bound

$$\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a}) \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|d_K|}.$$

The Minkowski theorem on class groups (2)

The result is announced in a letter to HILBERT (December 22nd, 1890) and proved in a letter to Hermite (January 15th, 1891), in its simplified form which only asserts that one has $|d_K| > 1$ if $K \neq \mathbb{Q}$ and thus solves a 1874 conjecture of KRONECKER ; Hermite extracted from Minkowski's letter a *Notes aux Comptes Rendus Acad. Sc. Paris*.

The main problem is *to find good lower bounds for* κ_{r_1,r_2}.

The Minkowski theorem on class groups (2)

The result is announced in a letter to HILBERT (December 22nd, 1890) and proved in a letter to Hermite (January 15th, 1891), in its simplified form which only asserts that one has $|d_K| > 1$ if $K \neq \mathbb{Q}$ and thus solves a 1874 conjecture of KRONECKER ; Hermite extracted from Minkowski's letter a *Notes aux Comptes Rendus Acad. Sc. Paris*.

The main problem is *to find good lower bounds for* κ_{r_1,r_2}.

A first trick is suggested by Minkowski in his letter to Hermite: to include balls inside A_F. The result solely depends on n, and is thus interesting only for small r_1 (Γ_n is the lattice constant of the unit ball):

$$\kappa_{r_1,r_2} \geq n^{n/2}\Gamma_n = \left(\frac{n}{\gamma_n}\right)^{n/2} .$$

Lower bounds for κ ($r_1 \leq 1$).

n	3	4	5	6	8
Minkowski	12	43	258	985	25067
Sphere	13	64	390	2187	65536
Conjectural	23	117	1609	9747	1257728

The Minkowski theorem on class groups (3)

Just some data for totally real domains:

n	2	3	4	5
Minkowski	4	20.25	113.7...	678.16...
Known	5	49	≥ 500	$\geq 3251.2...$
Conjectural			725	14641

Known. $n = 4$: P. Noordzij, 1967; $n = 5$: H.J. Godwin, 1950.
Conjectural. $n = 4$: $\mathbb{Q}(\sqrt{7+2\sqrt{5}})$; $n = 5$: $\mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$.

Warning. For larger n, better use R. ZIMMERT's analytic bounds (*Ideale kleiner Norm in Idealklassen und eine Regulator Abschätzung*, Invent. Math. **62** (1981), 367–380).

Isolation phenomena and successive minima

With F, $A = A_F$ as above, we say that an admissible lattice Λ for A is *isolated* if in a small enough neighbourhood of Λ, the only admissible lattices are of the form $\Lambda' = \lambda\, u(\Lambda)$ with $u \in \mathrm{Aut}(A)$ and $\lambda \geq 1$. We may restrict ourselves to *minimal-admissible* lattices, those for which $\lambda\Lambda$ is not admissible if $\lambda < 1$.

This notion looks pertinent for domains associated with totally real fields; in general, one should consider less restrictive isolation conditions (CASSELS for $n = 3$; SKUBENKO, AKRAMOV for all $n \geq 3$).

The set of determinants of minimal-admissible lattices we shall call the *spectrum of F* (or of A). Isolation phenomena imply the existence of discrete subsets in the spectrum.

Real quadratic domains

Up to a normalization, we recover the notion of
the *Markoff spectrum* in the theory of approximation of real numbers by rational numbers;
dictionary in Cassels's *Cambridge tract*.
All determinants $d < 9$ of minimal-admissible lattices are isolated (the *Markoff chain*).

$\Longrightarrow$ (**e.g.**)

if the discriminant of a quadratic field has a prime factor $p \equiv 3 \mod 4$, one may replace the class bound

$$\mathrm{N}(\mathfrak{a}) \le \left(\frac{|d_K|}{5}\right)^{1/2} \quad \text{by} \quad \mathrm{N}(\mathfrak{a}) < \left(\frac{|d_K|}{9}\right)^{1/2}.$$

The spectrum is very complicated near 9 on the right;
it contains an interval $[M, +\infty)$ (the *Marshall Hall bound* M has been determined by Freiman); it is very chaotic in $[9, M]$, except for a few easily determined gaps, such as $(12, 13)$.

Totally real cubic domains (results)

A remarkably short proof that $\kappa_{3,0} = 49$, attained uniquely on $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, was found by DAVENPORT in 1941. He later proved that this value is isolated, as well as the next value, namely 81, also isolated, attained uniquely on $K = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$. This time Davenport's proof is extremely difficult.

Totally real cubic domains (results)

A remarkably short proof that $\kappa_{3,0} = 49$, attained uniquely on $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, was found by DAVENPORT in 1941. He later proved that this value is isolated, as well as the next value, namely 81, also isolated, attained uniquely on $K = \mathbb{Q}(\zeta_9 + \zeta_9^{-1})$. This time Davenport's proof is extremely difficult.

In 1971, SWINNERTON-DYER wrote a computer program to deal with totally real cubic domains. He was able to list **all** equivalence classes of minimal-admissible lattices for $A_{3,0}$ up to the determinant $289 = 17^2$.

He proved the existence of 19 rank-3 submodules $M_1, \dots, M_{19}$ of real cubic fields such that the minimal-admissible lattices of determinant $d \leq 289$ are "*algebraic lattices*" associated with one of the M_i.

It would be interesting to push further these 1971 computations, taking into account the improvements of both the computers and the convexity programs. This could give new support for the following conjecture:

Totally real cubic domains (conjecture)

Swinnerton-Dyer did not put forward any conjecture, but it seems clear that he believed that what follows might well be true.

Conjecture

There exists a sequence $M_1, \dots, M_k, \dots$ of modules in real cubic fields with increasing discriminants such that every admissible lattice for $A_{3,0}$ is of the form $\Lambda_k = \lambda\, u(M_k)$ for some $\lambda > 1$ and some $u \in \mathrm{Aut}(A_{3,0})$.

Thanks to a 1955 theorem of Cassels and Swinnerton-Dyer, the conjecture above implies that $\det(\Lambda_k) \to \infty$,

which would show that there exists a Minkowski class bound in $o(|d_K|^{1/2})$.

Totally real cubic domains (conjecture)

Swinnerton-Dyer did not put forward any conjecture, but it seems clear that he believed that what follows might well be true.

Conjecture

There exists a sequence $M_1, \dots, M_k, \dots$ of modules in real cubic fields with increasing discriminants such that every admissible lattice for $A_{3,0}$ is of the form $\Lambda_k = \lambda\, u(M_k)$ for some $\lambda > 1$ and some $u \in \mathrm{Aut}(A_{3,0})$.

Thanks to a 1955 theorem of Cassels and Swinnerton-Dyer, the conjecture above implies that $\det(\Lambda_k) \to \infty$,

which would show that there exists a Minkowski class bound in $o(|d_K|^{1/2})$.

But the conjecture must be very difficult, since it implies

Littlewood's conjecture: For any $(\alpha, \beta) \in \mathbb{R}$, we have

$$\liminf q\, ||q\alpha||\, ||q\beta|| = 0\,.$$

Totally real cubic domains (conjecture)

Swinnerton-Dyer did not put forward any conjecture, but it seems clear that he believed that what follows might well be true.

Conjecture

There exists a sequence $M_1, \dots, M_k, \dots$ of modules in real cubic fields with increasing discriminants such that every admissible lattice for $A_{3,0}$ is of the form $\Lambda_k = \lambda\, u(M_k)$ for some $\lambda > 1$ and some $u \in \mathrm{Aut}(A_{3,0})$.

Thanks to a 1955 theorem of Cassels and Swinnerton-Dyer, the conjecture above implies that $\det(\Lambda_k) \to \infty$,

which would show that there exists a Minkowski class bound in $o(|d_K|^{1/2})$.

But the conjecture must be very difficult, since it implies

Littlewood's conjecture: For any $(\alpha,\beta) \in \mathbb{R}$, we have

$$\liminf q\,||q\alpha||\,||q\beta|| = 0\,.$$

What about totally real domains in dimension 4 ?

Connection with diophantine approximations (1)

The sequence of determinants $d_1 = 5$, $d_2 = 8$, $d_3 = \frac{221}{25}, \dots$, with $\lim d_n = 9$ (the Markoff chain) is well known in the theory of continued fractions: for every irrational θ, there are infinitely many rationals $\frac{p}{q}$ such that $\theta - \frac{p}{q} < \frac{1}{\sqrt{5}q^2}$; and if θ is not equivalent to $\frac{1+\sqrt{5}}{2}$, then such approximations exist in $\frac{1}{\sqrt{8}q^2}$; and if ...

Connection with diophantine approximations (1)

The sequence of determinants $d_1 = 5$, $d_2 = 8$, $d_3 = \frac{221}{25}, \dots$, with $\lim d_n = 9$ (the Markoff chain) is well known in the theory of continued fractions: for every irrational θ, there are infinitely many rationals $\frac{p}{q}$ such that $\theta - \frac{p}{q} < \frac{1}{\sqrt{5}q^2}$; and if θ is not equivalent to $\frac{1+\sqrt{5}}{2}$, then such approximations exist in $\frac{1}{\sqrt{8}q^2}$; and if ...

An analogue: given an imaginary quadratic field K_0, one wishes to approximate an irrational complex number θ by elements of K_0. This problem was considered in detail by DESCOMBES and POITOU in the fifties; there has not been much work since.

These approximations are connected with minimal-admissible lattices Λ for the totally imaginary quartic domain $A_{0,2}$ such that $\mathbb{R} \otimes \Lambda$ contains the image of K_0, corresponding to quadratic extensions K/K_0.

.../...

Connection with diophantine approximations (2)

Now K_0 is quadratic real; and we consider the totally real quartic domain $A_{4,0}$; and more precisely minimal-admissible lattices for $A_{4,0}$ such that $\mathbb{R} \otimes \Lambda$ contains the image of K_0.

Connection with diophantine approximations (2)

Now K_0 is quadratic real; and we consider the totally real quartic domain $A_{4,0}$; and more precisely minimal-admissible lattices for $A_{4,0}$ such that $\mathbb{R} \otimes \Lambda$ contains the image of K_0.

What about approximations of pairs (θ_1, θ_2) of real numbers by pairs $(\alpha, \overline{\alpha})$ of conjugate elements in K_0 ?

SECOND PART

Some problems related to spheres,

with emphasis on minimal classes.

The notion of a minimal class is the subject of Section 9 of my Springer book Perfect Lattices in Euclidean Spaces.

The papers with ACHILL SCHÜRMANN and/or WOLFGANG KELLER can be downloaded from Arxiv or from my homepage.

Minimal classes (1)

Minimal classes are the classes for the equivalence relation

$$\Lambda \sim \Lambda' \iff \exists u \in \mathrm{GL}(E) \mid u(\Lambda) = \Lambda' \text{ and } u(S(\Lambda)) = S(\Lambda')\,;$$

they are equipped with the ordering

$$\mathcal{C} \prec \mathcal{C}' \iff \exists \Lambda \in \mathcal{C},\, \exists \Lambda' \in \mathcal{C}' \mid S(\Lambda) \subset S(\Lambda')\,.$$

Besides s, the number of pairs $\pm x$ of minimal vectors
[the (*half*-)*kissing number*],
the most important invariant of a class $\mathcal{C}$ is its *perfection rank* r,
the rank in $\mathrm{End}^s(E)$ of the set of orthogonal projections p_x to the minimal vectors of any lattice $\Lambda \in \mathcal{C}$.

In practice we may restrict ourselves to *well-rounded lattices* (those which have n independent minimal vectors)
and to the corresponding *well-rounded classes*.

Minimal classes (2).

For well-rounded classes, we have $$n \le r \le \frac{n(n+1)}{2}\,.$$

Lattices (and classes) with $s = \frac{n(n+1)}{2}$ are called *perfect*.

The dimension of a class $\mathcal{C}$ is its *perfection co-rank* $\dfrac{n(n+1)}{2} - r$; a class of dimension k consists of isometry classes of lattices depending affinely on k parameters. Thus classes of dimension zero are the similarity classes of perfect lattices. In general classes can be viewed as convex polytopes, the extremal points of which correspond to perfect classes in their closure.

Two classes $\mathcal{C}$ and $\mathcal{C}' \succ \mathcal{C}$ may be connected by a chain
$\mathcal{C}_0 = \mathcal{C} \prec \mathcal{C}_1 \prec \cdots \prec \mathcal{C}_m = \mathcal{C}'$
such that perf rank of $\mathcal{C}_{i+1} =$ perf rank of $\mathcal{C}_i + 1$.

Minimal classes (3).

On the side of (positive, definite) quadratic forms, the space of forms having a fixed minimum carries a structure of an (infinite) cell complex, and minimal classes appear as equivalence classes of cells. This correspondence preserves dimensions, which can be viewed as the number of parameters on which a minimal class depends.

The 0-cells are the perfect forms.

The 1-cells are the Voronoi paths connecting perfect forms.

One can extract (Ash, Mumford, Rapoport, Tai; 1975) a finite complex from the infinite one, which allows the computation of various K-groups. This motivated Elbaz-Vincent, Gangl and Soulé to construct explicitly such a complex in dimensions 6 and 7, and in particular to list all classes.

(Dimension $n \leq 4$: easy; dimension 5: Batut.)

n	2	3	4	5	6	7	8	9
Perfect	1	1	2	3	7	33	10 916	$>$ 530 000
Edges	1	1	2	4	18	357	83 092	
Total	2	5	18	136	5 634	10 722 899		

Table: Number of classes

Minimal classes: identification.

Attaching to a well-rounded lattice the matrix $A = S\,S^{tr}$ induces an injective map from

minimal classes to equivalence classes of positive, definite, integral matrices;

otherwise stated, to isometry classes of integral lattices.
The matrix $\mathrm{Bc}(S) = SS^{tr}$ is called the Bacher or barycenter matrix.

Problem

Can one deduce invariants of a minimal class from those of this matrix (or lattice) ?
Can one find restrictions on the invariants of such a lattice ?

I have not a lot to say about it, except what concerns spherical 3-designs.

Weak eutaxy.

A eutaxy relation is an equality $\mathrm{Id} = \sum_{x \in S/\pm 1} \rho_x\, p_x$ with real coefficients ρ_x.

Given a basis $\mathcal{B} = (e_1, \dots, e_n)$ for Λ, and a column X of components of x, one has

$$\mathrm{Gram}(\mathcal{B}) = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}, \mathcal{B}^*) \text{ and } XX^{tr} = \mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B})\,.$$

Hence in terms of quadratic forms, a eutaxy relation reads

$$A^{-1} = \sum_{X \in S/\pm 1} \rho'_x\, XX^{tr}\,.$$

Weak eutaxy.

A eutaxy relation is an equality $\mathrm{Id} = \sum_{x \in S/\pm 1} \rho_x\, p_x$ with real coefficients ρ_x.

Given a basis $\mathcal{B} = (e_1, \dots, e_n)$ for Λ, and a column X of components of x, one has

$$\mathrm{Gram}(\mathcal{B}) = \mathrm{Mat}(\mathrm{Id}, \mathcal{B}, \mathcal{B}^*) \ \text{ and } \ XX^{tr} = \mathrm{Mat}(p_x, \mathcal{B}^*, \mathcal{B})\,.$$

Hence in terms of quadratic forms, a eutaxy relation reads

$$A^{-1} = \sum_{X \in S/\pm 1} \rho'_x\, XX^{tr}\,.$$

Weak eutaxy : existence of a eutaxy relation.
Semi-eutaxy : $\rho_x \geq 0$.
Eutaxy : $\rho_x > 0$.
Strong eutaxy : equal $\rho_x \iff S$ is a 3-design. [$\implies \rho_x > 0$.]

It is known (A.-M.B $+$ J.M.) that a class $\mathcal{C}$ contains at most **one** weakly eutactic lattice, that on which γ attains a minimum.

(Otherwise the minimum is attained on $\overline{\mathcal{C}} = \cup_{\mathcal{C}' \succ \mathcal{C}}\, \mathcal{C}'$.)

Strong eutaxy.

Obvious: $\mathcal{C}$ contains a “streut” lattice $\iff$ $(SS^{tr})^{-1}$ is streut.

Generalization: equal non-zero coefficients in case of semi-eutaxy.
The set of x with $\rho_x \neq 0$ is then a 3 -design (“strong semi-eutaxy property”).

Data of Elbaz-Vincent and Gangl allowed the classification of strongly semi-eutactic lattices up to dimension 6; see my home page, where I have also listed examples for $n = 7 — 10$.

Question

Can one forecast (weak) eutaxy of a class from the barycenter matrix ?

Other invariants are discussed in the next slide.

The index structure (1).

We consider pairs (Λ, Λ') where Λ is well rounded and Λ' has a basis of minimal vectors of Λ.
[More generally, one can consider pairs of any lattice and a "*Minkowskian sublattice*" – generated by successive minima in the sense of Minkowski; this boils down to the well-rounded case.]

One has $[\Lambda : \Lambda'] \leq \gamma_n^{n/2}$ (use the Hadamard inequality).
The *maximal index of* Λ is $\imath(\Lambda) = \max\,[\Lambda : \Lambda']$.

Problem

Given n, what are the possible structures of Λ/Λ' for $[\Lambda : \Lambda'] = \imath(\Lambda)$?

The index structure (1).

We consider pairs (Λ, Λ') where Λ is well rounded and Λ' has a basis of minimal vectors of Λ.

[More generally, one can consider pairs of any lattice and a "*Minkowskian sublattice*" – generated by successive minima in the sense of Minkowski; this boils down to the well-rounded case.]

One has $[\Lambda : \Lambda'] \leq \gamma_n^{n/2}$ (use the Hadamard inequality).
The *maximal index of* Λ is $\imath(\Lambda) = \max\,[\Lambda : \Lambda']$.

Problem

Given n, what are the possible structures of Λ/Λ' for $[\Lambda : \Lambda'] = \imath(\Lambda)$?

The *annihilator* d of Λ/Λ' is also bounded by $\gamma_n^{n/2}$. On a basis of Λ', vectors $x \in \Lambda$ are of the form

$$\frac{a_1 e_1 + \cdots + a_n e_n}{d}\,.$$

Such systems $(a_1, \ldots, a_n) \mod d$ are the words of a code over $\mathbb{Z}/d\mathbb{Z}$.

Problem

Given n, classify these codes.

The index structure (2).

The question was first considered by Watson in 1971, and his results were then extended by Ryshkov and Zahareva. I gave in 2001 a complete picture up to dimension 8, where I introduced new invariants.

The case of dimension 9 was solved very recently (W. Keller, J. M., A. Schürmann; to appear in *Math. comp.*).

n	≤ 3	4	5	6	7	8	9
$\lfloor \gamma^{n/2} \rfloor$	1	2	2	4	8	16	30-22
bound $(*)$	1	1	2	3	4	8	
$\imath =$	1	1	2	4	8	16	16
nb. of new codes		1	1	3	7	28	134

Table: Bounds for the index; $(*)$: not $\mathbb{D}_4$, $\mathbb{D}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$

[$n = 9$: 30 by Cohn-Elkies's bound for γ_9, 22 if $\gamma_9 = \gamma(\Lambda_9)$.]

The index structure (3).

Main difficulty in large dimensions: enormous waste of time to get rid of large indices that we (almost surely know) not to exist.

Example ($n = 9$).

Proved bound for $\lfloor \gamma_n^{n/2} \rfloor$	: $\imath_{\max} \leq 30$.	
Conjectural bound for $\lfloor \gamma_n^{n/2} \rfloor$	: $\imath_{\max} \leq 22$.	[Λ_9]
Actual value	: $\imath_{\max} = 16$.	

Example ($n = 10$).

Proved bound for $\lfloor \gamma_n^{n/2} \rfloor$	: $\imath_{\max} \leq 59$.	
Conjectural bound for $\lfloor \gamma_n^{n/2} \rfloor$	: $\imath_{\max} \leq 36$.	[Λ_{10}]
Expected value	: $\imath_{\max} = 32$.	[several perfect lattices]

A complete classification in dimension $n = 10$ looks out of the today computational devices.

Remark. For $n = 24$, the bound $\gamma_{24}^{12} = 2^{24}$ is attained on the Leech lattice.

The index structure (4).

Any structure which exists for Λ/Λ' in some dimension n exists in all larger dimensions (consider orthogonal sums).
Thus it suffices to list new structures in each dimension.

- $n = 1 : \{1\}$.
- $n = 4 : \{2\}$.
- $n = 6 : \{3\}, \{2^2\}$.
- $n = 7 : \{4\}, \{2^3\}$.
- $n = 8 : \{5\}, \{6\}, \{4 \cdot 2\}, \{3^2\}, \{2^4\}$.
- $n = 9 : \{7\}, \{8\}, \{9\}, \{10\}, \{12\}, \{6 \cdot 2\}, \{4 \cdot 2^2\}, \{4^2\}$.

Many new structures are expected to exist in dimension 10; we have met

- $\{11\}, \{2^5\}, \{4^2 \cdot 2\}$ and $\{4 \cdot 2^3\}$.

Applications.

1. Bases of minimal vectors. We can answer the following

Question

Does a lattice which is generated by its minimal vectors necessarily has a basis of minimal vectors ?

Conway and Sloane (1995): NO if $n \geq 11$.

JM (2007): YES if $n \leq 8$;
Moreover a counter-example with $n < 11$ must have $\imath \geq 5$.

JM and AC: YES if $n \leq 9$; a counter-example exists with $n = 10$ and $\imath = 5$.

Applications.

1. Bases of minimal vectors. We can answer the following

Question

Does a lattice which is generated by its minimal vectors necessarily has a basis of minimal vectors ?

Conway and Sloane (1995): NO if $n \geq 11$.

JM (2007): YES if $n \leq 8$;

Moreover a counter-example with $n < 11$ must have $\imath \geq 5$.

JM and AC: YES if $n \leq 9$; a counter-example exists with $n = 10$ and $\imath = 5$.

2. Ratio Hermite / Minkowski. Obviously 1 if $n \leq 4$. Let $n \geq 5$.

van der Waerden (Acta. Math., 1956): $\dfrac{H}{M} \leq \left(\dfrac{5}{4}\right)^{n-4}$.

AC (conjecture, 2007): for $\boxed{n \leq 8}$, $\dfrac{H}{M} \leq \dfrac{n}{4}$.

JM (2007, unpublished): TRUE, attained uniquely on the centred cubic lattice.

$n \geq 10$: FALSE. n=9: **??**; at any rate, *not* attained uniquely on the centred cubic lattice.

THE END