Übungen zur Algebraischen Zahlentheorie (WS 2023)

PD Dr. Jürgen Müller, **Ausgabe:** 09.11.2023

(5.1) Exercise: Invertible ideals.
Let R be an integral domain, and let $\{0\} \neq I \trianglelefteq R$. Show the following:
a) If $\{0\} \neq J \trianglelefteq R$, then I and J are isomorphic as R-modules if and only if there are $r, s \in R \setminus \{0\}$ such that $rI = sJ$.
b) If I is invertible then it is finitely generated.
c) The set of invertible ideals forms an Abelian monoid, which is free if and only if R is a principal ideal domain.

(5.2) Exercise: Dedekind domains.
Let R be a Dedekind domain. Show the following:
a) If R has only finitely many prime ideals, then R is a principal ideal domain.
b) If $\{0\} \neq \mathfrak{a} \trianglelefteq R$, then all ideals of $R/\mathfrak{a}$ are principal. (Thus if $\mathfrak{a}$ is prime then $R/\mathfrak{a}$ is a principal ideal domain.)

(5.3) Exercise: Non-factorial quadratic number rings.
For $d \in \mathbb{Z} \setminus \{0, 1\}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. Show that $\mathcal{O}_d$ is not factorial for the following d (actually these are all cases for $|d| \leq 30$), by exhibiting elements having non-unique factorisations into irreducible elements:
a) $d \in \{-5, -6, -10, -13, -14, -15, -17, -21, -22, -23, -26, -29, -30\}$,
b) $d \in \{10, 15, 26, 30\}$.

(5.4) Exercise: Non-factorial domains.
For $d \in \mathbb{Z} \setminus \{0, 1\}$ square-free let $\mathcal{O}_d$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$, and let $R_d := \mathbb{Z}[\sqrt{d}] \subseteq \mathcal{O}_d$; recall that $R_d = \mathcal{O}_d$ if and only if $d \not\equiv 1 \pmod 4$.
a) Let $d \leq -3$ be odd. Show that $2 \in R_d$ is irreducible but not prime.
b) Let $d \leq -5$ such that $d \equiv 3 \pmod 4$. Show that $d - 1$ and $2 + 2\sqrt{d}$ do not have a greatest common divisor in R_d.
c) Let $d := -3$. Show that there is a unique prime ideal $\mathfrak{p} \triangleleft R_{-3}$ containing (2), and show that $\mathfrak{p} \neq (2)$ and $\mathfrak{p}^2 = 2\mathfrak{p}$. Is (2) a product of prime ideals?

What do the above results imply for $\mathcal{O}_d$?