Algebraic Geometry (WS 2025)

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Introduction

(1.1) Example: Solving polynomial equations. Let $\mathcal{X} := \{X, Y, Z\}$ be indeterminates over $\mathbb{C}$, and let $\mathbb{C}[\mathcal{X}]$ be the associated polynomial $\mathbb{C}$-algebra. We consider the following polynomials in $f, g, h \in \mathbb{C}[\mathcal{X}]$:

$$\begin{array}{rcl} f & := & X + Y + Z - 1, \\ g & := & X^2 + Y^2 + Z^2 - 1, \\ h & := & X^3 + Y^3 + Z^3 - 1, \end{array}$$

and let $I := \langle f, g, h \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$ be the ideal generated by them. We aim at finding the complex solutions of the equations '$f = g = h = 0$', that is the set

$$\mathbf{V}(f, g, h) := \{[x, y, z] \in \mathbb{C}^3; f(x, y, z) = g(x, y, z) = h(x, y, z) = 0\}.$$

Then we have $\mathbf{V}(f, g, h) = \mathbf{V}(I) = \{v \in \mathbb{C}^3; p(v) = 0 \text{ for } p \in I\}$.

i) In order to determine $\mathbf{V}(I)$, we try to find suitable elements of I by 'eliminating indeterminates': To this end, we order monomials **lexicographically** with respect to '$X > Y > Z > 1$', and apply polynomial divisioa iteratively. (This falls short of applying the full **Buchberger algorithm**, but suffices here to actually find a **Gröbner basis**.) We successively get:

Dividing h by f we get $b' := h - X^2 f = -X^2 Y - X^2 Z + X^2 + Y^3 + Z^3 - 1$, and proceeding further we obtain

$$\begin{array}{rcl} b & := & \frac{1}{3}\big(b' + (XY + XZ - X - Y^2 - 2YZ - Z^2 + 2Y + 2Z - 1)f\big) \\ & = & -Y^2 Z + Y^2 - YZ^2 + 2YZ - Y + Z^2 - Z. \end{array}$$

Similarly, dividing g by f we get $c' := g - Xf = -XY - XZ + X + Y^2 + Z^2 - 1$, and proceeding further we obtain

$$c := \frac{1}{2}\big(c' + (Y + Z - 1)f\big) = Y^2 + YZ - Y + Z^2 - Z.$$

Finally, dividing b by c we get $d := b + (Z - 1)c = Z^3 - Z^2$.

Hence we have $I = \langle f, g, h \rangle = \langle c, b, f \rangle = \langle d, c, f \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$. Now let $[x, y, z] \in \mathbf{V}(I)$. Then from $d = Z(Z - 1)$ we get $z \in \{0, 1\}$. Next, from $c = Y^2 + Y(Z - 1) + Z(Z - 1)$, for $z = 1$ we get $y = 0$, and for $z = 0$ we get $y \in \{0, 1\}$. Finally, from $f = X + Y + Z - 1$, for $[y, z] \in \{[0, 1], [1, 0]\}$ we get $x = 0$, and for $[y, z] = [0, 0]$ we get $x = 1$. Hence we conclude that $\mathbf{V}(I) = \{[0, 0, 1], [0, 1, 0], [1, 0, 0]\}$. (Note that we have three indeterminates, three equations, and a finite variety.) $\sharp$

ii) In the present case, next to the general attack using polynomial division, there is an approach taking advantage of the particular situation:

The symmetric group $\mathcal{S}_3$ acts on $\mathbb{C}[\mathcal{X}]$ by $\mathbb{C}$-algebra automorphisms given by permuting the indeterminates. Since $f, g, h \in \mathbb{C}[\mathcal{X}]$ are $\mathcal{S}_3$-invariant, so is $I = \langle f, g, h \rangle \trianglelefteq \mathbb{C}[\mathcal{X}]$, and thus $\mathcal{S}_3$ also acts on $\mathbf{V}(I)$.

Appealing to the **Newton identities** for symmetric polynomials, entangling power sum polynomials and elementary symmetric polynomials, we infer that

$$e_2 = \frac{1}{2}(f^2 - g + 2f) = XY + XZ + YZ,$$

$$e_3 = \frac{1}{3}(h - f^3 + 3fe_2 - 3g - 3e_2 + 3f) = \frac{1}{6}(2h + f^3 - 3fg + 3f^2 - 3g) = XYZ.$$

Thus we have $I = \langle f, g, h \rangle = \langle f, e_2, e_3 \rangle \trianglelefteq A$. (Noting that $f = e_1 - 1$, we actually have $\mathbb{C}[f, g, h] = \mathbb{C}[f, e_2, e_3] = \mathbb{C}[e_1, e_2, e_3]$.

Now let again $[x, y, z] \in \mathbf{V}(I)$. Then $e_3 = XYZ$ shows, using the $\mathcal{S}_3$-action, that we may assume $x = 0$. Next, $e_2 = XY + XZ + YZ$ shows, using the $\mathcal{S}_3$-action again, that we may assume $y = 0$ as well. Finally $f = X + Y + Z - 1$ yields $z = 1$. Thus in conclusion we get $\mathbf{V}(I) = \{[0, 0, 1], [0, 1, 0], [1, 0, 0]\}$. $\sharp$
