Algebraic Geometry (WS 2025)

PD Dr. Jürgen Müller, **Lecture 10** (10.11.2025)

(10.1) Corollary. We Keep the earlier notation, and let $\varphi\colon \mathbf{V} \to \mathbf{W}$ be regular. If $\mathbf{V}$ is irreducible, then so are both $\varphi(\mathbf{V}) \subseteq \mathbf{W}$ and $\overline{\varphi(\mathbf{V})} \subseteq \mathbf{W}$.

Proof. Recall that $U := \varphi(\mathbf{V})$ is irreducible if and only if $\mathbf{U} := \overline{U}$ is. Now let $\mathbf{W}', \mathbf{W}'' \subseteq \mathbf{U}$ be closed such that $\mathbf{U} = \mathbf{W}' \cup \mathbf{W}''$. Then we have $\mathbf{V} = \varphi^{-1}(\mathbf{W}') \cup \varphi^{-1}(\mathbf{W}'')$. Since φ is continuous, and $\mathbf{V}$ is irreducible, we may assume that $\varphi^{-1}(\mathbf{W}') = \mathbf{V}$. This implies $U = \varphi(\mathbf{V}) \subseteq \mathbf{W}'$, hence $\mathbf{W}' \subseteq \mathbf{U} = \overline{U} \subseteq \mathbf{W}'$. $\sharp$

(10.2) Example. Let $K = L = \mathbb{C}$, and let $n = 2$, hence $A := \mathbb{C}[X,Y]$. (We leave out the subscripts in the notation for the operators $\mathbf{V}_\mathbb{C}$ and $\mathbf{I}_\mathbb{C}$.)

We consider the plane curve $\mathbf{V}(f) \subseteq \mathbb{C}^2$ given by $f := Y^2 - X^2 \in A$. We have $f = (Y-X)(Y+X) \in A$, so that $\mathbf{V}(f)$ is reducible. We have $\mathbf{V}(f) = \mathbf{V}((Y-X)(Y+X)) = \mathbf{V}(Y-X) \cup \mathbf{V}(Y+X)$. Since both $Y \pm X \in A$ are irreducible, both $\mathbf{V}(Y \pm X)$ are irreducible. Since the above union is irredundant, we conclude that $\mathbf{V}(Y \pm X)$ are the irreducible components of $\mathbf{V}(f)$; note that $\mathbf{V}(Y-X) \cap \mathbf{V}(Y+X) = \{0\} \neq \emptyset$.

Moreover, $\langle f \rangle \trianglelefteq A$ is radical: Let $g \in \sqrt{\langle f \rangle}$, then $f \mid g^k$ for some $k \in \mathbb{N}$; since $Y \pm X \in A$ are non-associate irreducible, and A is factorial, we conclude that both $Y \pm X \mid g$; thus $f \mid g$, that is $g \in \langle f \rangle$. Hence we have $\mathbb{C}[\mathbf{V}(f)] = A/\langle f \rangle = \mathbb{C}[X,Y]/\langle (Y-X)(Y+X) \rangle$; actually $Y \pm X \in \mathbb{C}[\mathbf{V}(f)]$ are zero-divisors.

(10.3) Example. Let $K = L = \mathbb{C}$, and let $n = 2$, hence $A := \mathbb{C}[X,Y]$.

We consider the plane curve $\mathbf{C} := \mathbf{V}(f) \subseteq \mathbb{C}^2$, where $f := Y^2 - X^3 \in A$. Since $X^3 \in \mathbb{C}[X]$ is not a square, f is irreducible, hence so is $\mathbf{C}$. We have $\mathbb{C}[\mathbf{C}] = A/\langle f \rangle = \mathbb{C}[X,Y]/\langle Y^2 - X^3 \rangle \cong \mathbb{C}[X, \sqrt{X^3}] = \mathbb{C}[Y, \sqrt[3]{Y^2}]$, which is a domain (but not a polynomial algebra).

We consider the regular map $\varphi\colon \mathbb{C} \to \mathbf{C}\colon t \mapsto [t^2, t^3]$, which since $f(\varphi(t)) = 0$ is well-defined indeed. Moreover, let $\psi\colon \mathbf{C} \to \mathbb{C}$ be given by $\psi(0,0) = 0$ and $\psi(x,y) = \frac{y}{x}$, for $[x,y] \neq [0,0]$. Then we have $\psi(\varphi(t)) = \frac{t^3}{t^2} = t$ for $t \neq 0$, and $\psi(\varphi(0)) = 0$; and the other way around $\varphi(\psi(x,y)) = [\frac{y^2}{x^2}, \frac{y^3}{x^3}] = [\frac{x^3}{x^2}, \frac{y^3}{y^2}] = [x,y]$ for $[x,y] \neq [0,0]$, and $\varphi(\psi(0,0)) = [0,0]$. Thus φ is bijective.

The associated comorphism is given as $\varphi^*\colon \mathbb{C}[\mathbf{C}] \to \mathbb{C}[T]\colon X \mapsto T^2, Y \mapsto T^3$. Since φ is dominant, we conclude that φ^* is injective. But since $T \notin \varphi^*(\mathbb{C}[\mathbf{C}])$ we conclude that φ^* is not surjective. Hence φ is not an isomorphism, where φ^{-1} actually is a **rational map**. A dominant morphism $\varphi\colon \mathbb{C} \to \mathbf{C}$ is also called a **parametrisation** of $\mathbf{C}$.
