

Algebraic Geometry (WS 2025)

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Projective varieties

(11.1) Projective spaces. Let L be a field, and let $n \in \mathbb{N}_0$. Then the **(n -dimensional) projective space** $\mathbf{P} := \mathbf{P}^n(L)$ over L is defined as the set of equivalence classes in $L^{n+1} \setminus \{0_{n+1}\}$ with respect to the equivalence relation given by $v \sim \lambda v$, for all $\lambda \in L^\times$; in particular, $\mathbf{P}^1(L)$ and $\mathbf{P}^2(L)$ are called the **projective line** and the **projective plane**, over L , respectively.

The equivalence class in $\mathbf{P}$ containing the **point** $0_{n+1} \neq [x_0, x_1, \dots, x_n] \in L^{n+1}$ is denoted by $[x_0 : x_1 : \dots : x_n]$, where the entries $x_i \in L$ are called the associated **homogeneous coordinates**. Thus $\mathbf{P}$ can be identified with the set of 1-dimensional L -subspaces of L^{n+1} , via $[x_0 : \dots : x_n] \mapsto \langle [x_0, \dots, x_n] \rangle_L$.

For $i \in \{0, \dots, n\}$ let $U_i := \{[x_0 : \dots : x_n] \in \mathbf{P}; x_i \neq 0\} \subseteq \mathbf{P}$. Then we have $\mathbf{P} = \bigcup_{i=0}^n U_i$, where U_i can be identified with the n -dimensional affine space L^n , via

$$U_i \rightarrow L^n : [x_0 : \dots : x_n] \mapsto \left[\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i} \right],$$

$$L^n \rightarrow U_i : [x_0, \dots, \widehat{x_i}, \dots, x_n] \mapsto [x_0 : \dots : x_{i-1} : 1 : x_{i+1}, \dots, x_n],$$

where we write $[x_0, \dots, \widehat{x_i}, \dots, x_n] := [x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \in L^n$. The above maps are called **dehomogenising** and **homogenising** at position i , respectively.

(11.2) Graded algebras. a) Let K be a field. A K -algebra A is called **(non-negatively) graded**, if there is a decomposition $A = \bigoplus_{d \geq 0} A_d$ as K -vector spaces, such that $A_i A_j \subseteq A_{i+j}$, for $i, j \geq 0$. The K -subspace A_d is called the **homogeneous component of degree d** . Moreover, for $c \in \mathbb{N}_0$ we abbreviate $A_{\leq c} := \bigoplus_{d=0}^c A_d$ and $A_{< c} := \bigoplus_{d=0}^{c-1} A_d$.

Then we have $1 \in A_0$: Writing $1 = \sum_{i \geq 0} e_i$, from $f_d = 1 \cdot f_d = \sum_{i \geq 0} e_i f_d$, where $f_d \in A_d$, we get $f_d = e_0 f_d$; thus we have $f = e_0 f$ for all $f \in A$, implying $e_0 = 1$. Hence if $A \neq \{0\}$, then we have $K \cong K \cdot 1 \leq A_0$; if $A_0 \cong K$, then A is called an **indecomposable** graded K -algebra.

b) An ideal $I \leq A$ is called **homogeneous** if $I = \bigoplus_{d \geq 0} I_d$, where $I_d := I \cap A_d$; in other words, an element of A belongs to I if and only if all its homogeneous components belong to I . We may characterise homogeneous ideals as follows:

i) An ideal I is homogeneous if and only if it is generated by homogeneous elements: If I is homogeneous, then it is generated by its homogeneous elements. Conversely, if I is generated by homogeneous elements, $g \in A_d$ say, then the equation $g \cdot (\sum_{i \geq 0} f_i) = \sum_{i \geq 0} (g f_i)$ implies that the homogeneous components of any element of I belong to I as well, thus I is homogeneous. $\#$

ii) An ideal I is homogeneous if and only if A/I is a graded K -algebra again, with respect to the inherited grading: If I is homogenous, then we have $A/I = (\bigoplus_{d \geq 0} A_d) / (\bigoplus_{d \geq 0} I_d) \cong \bigoplus_{d \geq 0} A_d / I_d$, where $(A_i / I_i) \cdot (A_j / I_j) \subseteq A_{i+j} / I_{i+j}$, for $i, j \geq 0$. Conversely, if A/I is naturally graded again, that is we have $A/I \cong \bigoplus_{d \geq 0} (A_d + I) / I \cong \bigoplus_{d \geq 0} A_d / I_d$, where the second isomorphism follows from the Homomorphism Theorem, then for $f = \sum_{i \geq 0} f_i \in A$ we have $f \in I$ if and only if $f_i \in I_i$ for all $i \geq 0$; thus I is homogeneous. $\sharp$

In particular, if A is indecomposable, then $A_+ := \bigoplus_{d \geq 1} A_d \triangleleft A$ is the unique maximal homogeneous ideal, being called the **irrelevant ideal** (for a reason becoming clear soon); we have $A/A_+ \cong A_0 \cong K$, hence A_+ is maximal.

c) We collect a few straightforward properties of homogeneous ideals:

It is immediate that arbitrary sums, arbitrary intersections, and (finite) products of homogeneous ideals are homogeneous. Moreover, for $I \triangleleft A$ homogeneous and $J \triangleleft A$ we have the following: If J is homogeneous, then so is $(J+I)/I \triangleleft A/I$; if moreover $I \subseteq J$ such that $J/I \triangleleft A/I$ is homogeneous, then J is homogeneous.

Slightly more subtle are the following properties:

i) If $I \triangleleft A$ is homogeneous, then so is $\sqrt{I} \triangleleft A$: Going over to the graded K -algebra A/I , it suffices to show that $\text{nil}(A) \triangleleft A$ is homogeneous. Hence let $0 \neq f = \sum_{i=0}^d f_i \in A$, for some $d \in \mathbb{N}_0$ such that $f_d \neq 0$, be nilpotent, that is there is $k \in \mathbb{N}$ such that $0 = f^k = (f_d)^k \in A/A_{<kd}$. We conclude that $(f_d)^k = 0$, thus f_d is nilpotent as well. Going over to $f - f_d$, the assertion now follows from induction with respect to d . $\sharp$

ii) Finally, given $I \triangleleft A$ homogeneous, in order to check whether I is prime, it suffices to show for $f, g \in A$ homogeneous only, that $fg \in I$ implies $f \in I$ or $g \in I$: Assume that I has the above property, and let $0 \neq f = \sum_{i=0}^d f_i \in A$ and $0 \neq g = \sum_{i=0}^c g_i \in A$, where $f_d \neq 0 \neq g_c$, be arbitrary elements such that $fg \in I$. Then we have $fg = f_d g_c \in A/A_{<(c+d)}$, thus since I is homogeneous we have $f_d g_c \in I$ as well. By assumption, we have $f_d \in I$, say. Going over to $f - f_d$, it follows from induction with respect to $c + d$ that $f \in I$ or $g \in I$. $\sharp$

Example. Let $\mathcal{X} := \{X_1, \dots, X_n\}$ and $\mathcal{X}^\sharp := \{X_0\} \dot{\cup} \mathcal{X}$ be indeterminates, for $n \in \mathbb{N}_0$. Then $A^\sharp := K[\mathcal{X}^\sharp]$ becomes an indecomposable graded K -algebra with respect to the total degree, where the homogeneous component $A_d^\sharp$ is the finite-dimensional K -subspace generated by the monomials of degree d .