Algebraic Geometry (WS 2025)

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(12.1) Projective algebraic sets and their ideals. a) We keep the earlier notation, and let $K \subseteq L$ is a field extension. Then the elements of $A^\sharp$ do *not* induce functions on $\mathbf{P}$. But still, whenever $f \in A^\sharp$ is homogeneous of degree d, for $[x_0, \ldots, x_n] \in L^{n+1}$ we have $f(\lambda \cdot [x_0, \ldots, x_n]) = \lambda^d \cdot f(x_0, \ldots, x_n)$, for all $\lambda \in L^*$, so that $f(\lambda \cdot [x_0, \ldots, x_n]) \neq 0$ if and only if $f([x_0, \ldots, x_n]) \neq 0$. Thus the vanishing set in $\mathbf{P}$ of f is well-defined, by taking representatives in L^{n+1} of the equivalence classes in $\mathbf{P}$.

Hence, let $\mathcal{S} \subseteq A^\sharp$ be a subset consisting of homogenous polynomials. Then

$$\mathbf{V}_L^\sharp(\mathcal{S}) := \{v \in \mathbf{P}; f(v) = 0 \text{ for all } f \in \mathcal{S}\} \subseteq \mathbf{P}$$

is called the **(projective) (*K*-)algebraic subset**, being **defined** by $\mathcal{S}$; then K and L are called its **field of definition** and **field of coordinates**, respectively.

If $I \trianglelefteq A^\sharp$ is homogeneous, let $\mathbf{V}_L^\sharp(I) := \mathbf{V}_L^\sharp(\{f \in I; f \text{ homogeneous}\})$. In particular, we have $\mathbf{V}_L^\sharp(\mathcal{S}) = \mathbf{V}_L^\sharp(\langle \mathcal{S} \rangle)$. Thus by Hilbert's Basis Theorem there are $f_1, \ldots, f_r \in \mathcal{S}$, for some $r \in \mathbb{N}_0$, such that $\mathbf{V}_L^\sharp(\mathcal{S}) = \mathbf{V}_L^\sharp(f_1, \ldots, f_r)$. Hence any projective algebraic set is defined by a homogenous ideal, or alternatively by finitely many homogenous polynomials. Moreover, for $I \subseteq J \trianglelefteq A^\sharp$ homogenous we have $\mathbf{V}_L^\sharp(J) \subseteq \mathbf{V}_L^\sharp(I)$, and $\mathbf{V}_L^\sharp(\sqrt{I}) = \mathbf{V}_L^\sharp(I)$.

For example, we have $\mathbf{V}_L^\sharp(0) = \mathbf{P}$, and $\mathbf{V}_L^\sharp(1) = \emptyset = \mathbf{V}_L^\sharp(\mathcal{X}^\sharp) = \mathbf{V}_L^\sharp(A_+^\sharp)$. More interestingly, an algebraic set defined by a single homogeneous polynomial of degree $d \geq 1$ is called a **(projective) hypersurface** of **degree** d; for $n = 2$ the latter is also called a **(projective) curve**.

A hypersurface of degree 1 is called a **(projective) hyperplane**; in particular, we have $\mathbf{H}_i := \mathbf{V}_L^\sharp(X_i) = \{[x_0\colon \ldots \colon x_n] \in \mathbf{P}; x_i = 0\} = \mathbf{P} \setminus U_i$, for $i \in \{0, \ldots, n\}$, where the elements of $\mathbf{H}_0 = \mathbf{P} \setminus U_0$ are called (for historical reasons) the **points at infinity** of $\mathbf{P}$. Note that this depends on the coordinates chosen, so that any hyperplane can be deemed to be at infinity.

b) Let $V \subseteq \mathbf{P}$ be any subset. Then the **vanishing ideal** of V is defined as

$$\mathbf{I}_K^\sharp(V) := \langle \{f \in A^\sharp \text{ homogeneous}; f(v) = 0 \text{ for all } v \in V\} \rangle \trianglelefteq A^\sharp.$$

Then we have $\mathbf{I}_K^\sharp(V) = \sqrt{\mathbf{I}_K^\sharp(V)}$, and for $V \subseteq W$ we get $\mathbf{I}_K^\sharp(W) \subseteq \mathbf{I}_K^\sharp(V)$; in particular we have $\mathbf{I}_K^\sharp(\emptyset) = A^\sharp$.

c) We consider the interplay between the operators $\mathbf{V}_L^\sharp$ and $\mathbf{I}_K^\sharp$, where entirely similar to the affine case we observe the following:

For any $V \subseteq \mathbf{P}$ and any $I \trianglelefteq A^\sharp$ homogeneous we get $V \subseteq \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))$ and $I \subseteq \mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(I))$, which entails $\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(I))) = \mathbf{V}_L^\sharp(I)$ and $\mathbf{I}_K^\sharp(\mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))) =$

$\mathbf{I}_K^\sharp(V)$. In particular, for algebraic subsets $\mathbf{V}, \mathbf{W} \subseteq \mathbf{P}$ we get $\mathbf{V} \subseteq \mathbf{W}$ if and only if $\mathbf{I}_K^\sharp(\mathbf{W}) \subseteq \mathbf{I}_K^\sharp(\mathbf{V})$, and $\mathbf{V} = \mathbf{W}$ if and only if $\mathbf{I}_K^\sharp(\mathbf{W}) = \mathbf{I}_K^\sharp(\mathbf{V})$; in particular, if $\mathbf{V} \neq \emptyset$ then from $\mathbf{V}_L^\sharp(A_+^\sharp) = \emptyset$ we get $\mathbf{I}_K^\sharp(\mathbf{V}) \subset A_+^\sharp$.

We consider (arbitrary) intersections and (finite) unions of algebraic sets:

Firstly, for algebraic sets $\mathbf{V}_i \subseteq \mathbf{P}$, for $i \in \mathcal{I}$, where $\mathcal{I}$ is a (possibly infinite) index set, we have $\mathbf{V}_L^\sharp(\sum_{i\in\mathcal{I}} \mathbf{I}_K^\sharp(\mathbf{V}_i)) = \bigcap_{i\in\mathcal{I}} \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}_i)) = \bigcap_{i\in\mathcal{I}} \mathbf{V}_i$; in particular, an arbitrary intersection of algebraic sets is algebraic again.

Secondly, for algebraic sets $\mathbf{V}, \mathbf{W} \subseteq \mathbf{P}$ we have $\mathbf{V} \cup \mathbf{W} = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}) \cap \mathbf{I}_K^\sharp(\mathbf{W})) = \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(\mathbf{V}) \cdot \mathbf{I}_K^\sharp(\mathbf{W}))$; in particular, a finite union of algebraic sets is algebraic again. (This again is obtained entirely similar to the affine case, by observing that the polynomials occurring can be chosen homogeneous.)

Thus the smallest algebraic set containing $V \subseteq \mathbf{P}$ is given as $\overline{V} := \mathbf{V}_L^\sharp(\mathbf{I}_K^\sharp(V))$.

(12.2) Projective varieties. a) We keep the above notation. By the properties proved above, the set of algebraic subsets of $\mathbf{P}$ form the closed subsets of a topology, called the **(K-)Zariski topology**. Thus an algebraic subset $\mathbf{V} \subseteq \mathbf{P}$ will just be called **(K-)closed**, and it carries the induced topology, also called its Zariski topology.

Going over to vanishing ideals in $A^\sharp$, we conclude that the Zariski topology is Noetherian, so that any closed subset of $\mathbf{P}$ is the finite union of its irreducible components. In view of this, we observe that a closed subset $\mathbf{V} \subseteq \mathbf{P}$ is irreducible if and only if its vanishing ideal $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ is prime:

We have $\mathbf{V} \neq \emptyset$ if and only if $I := \mathbf{I}_K^\sharp(\mathbf{V}) \subset A^\sharp$. To show that I is prime, it suffices to check for $f, g \in A^\sharp$ homogeneous that $fg \in I$ implies $f \in I$ or $g \in I$; thus we may proceed entirely similar to the affine case treated earlier. $\sharp$

b) This entails the following **algebraic-geometric correspondence**: The operator $\mathbf{I}_K^\sharp$ induces an inclusion-reversing (with respect to set-theoretic inclusion) injective correspondence

$$\mathbf{I}_K^\sharp \colon \{\mathbf{V} \subseteq \mathbf{P} \text{ projective } K\text{-closed}\} \to \{A_+^\sharp \neq I \trianglelefteq A^\sharp \text{ radical homogeneous}\},$$

whose inverse on the image of $\mathbf{I}_K^\sharp$ is given by the operator $\mathbf{V}_L^\sharp$. Moreover, a closed subset $\mathbf{V} \subseteq \mathbf{P}$ is irreducible if and only if $\mathbf{I}_K^\sharp(\mathbf{V}) \trianglelefteq A^\sharp$ is prime. In addition, if L is algebraically closed then $\mathbf{I}_K^\sharp$ is surjective, which follows from:

(12.3) Theorem: Hilbert's Nullstellensatz (projective version).
Let L be algebraically closed, let $I \trianglelefteq A^\sharp$ be homogeneous, and let $\mathbf{V} := \mathbf{V}_L^\sharp(I)$. Then precisely one of the following cases occurs:
i) We have $\mathbf{V} = \emptyset$ and $A_+^\sharp \subseteq \sqrt{I} \trianglelefteq A^\sharp = \mathbf{I}_K^\sharp(\mathbf{V})$.
ii) We have $\mathbf{V} \neq \emptyset$ and $\mathbf{I}_K^\sharp(\mathbf{V}) = \sqrt{I} \subset A_+^\sharp \lhd A^\sharp$.

Proof. A subset $\emptyset \neq V \subseteq L^{n+1}$ is called an **(affine) cone** if for any $v \in V$ we have $\langle v \rangle_L \subseteq V$ as well; in particular we have $0_{n+1} \in V$, and $\{0_{n+1}\}$ is a cone.

We may assume that $I \lhd A^\sharp$, and let $\widetilde{\mathbf{V}} := \mathbf{V}_L(I) \subseteq L^{n+1}$ be the associated affine closed subset. Then by Hilbert's Nullstellensatz we have $\widetilde{\mathbf{V}} \neq \emptyset$, such that $\mathbf{I}_K(\widetilde{\mathbf{V}}) = \sqrt{I}$. Then we conclude that $\widetilde{\mathbf{V}}$ is a closed cone, where $\widetilde{\mathbf{V}} \setminus \{0_{n+1}\}$ consists precisely of the equivalence classes with respect to $\sim$ belonging to $\mathbf{V}$.

We have $\widetilde{\mathbf{V}} = \{0_{n+1}\}$ if and only if $\sqrt{I} = \langle \mathcal{X}^\sharp \rangle = A^\sharp_+ \lhd A^\sharp$. In this case we have $\mathbf{V} = \emptyset$ and $\mathbf{I}^\sharp_K(\mathbf{V}) = A^\sharp$; otherwise, $\mathbf{V} \neq \emptyset$ and $\mathbf{I}^\sharp_K(\mathbf{V}) = \mathbf{I}_K(\widetilde{\mathbf{V}}) = \sqrt{I} \subseteq A^\sharp_+$. $\sharp$
