Algebraic Geometry (WS 2025)

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(2.1) Example: Curves and surfaces. a) Let $\{X, Y\}$ be indeterminates, and let $\mathbb{R}[X,Y]$ be the associated polynomial $\mathbb{R}$-algebra. Given $0 \neq f \in \mathbb{R}[X,Y]$, the set $\mathbf{V}_\mathbb{R}(f) := \{[x,y] \in \mathbb{R}^2; f(x,y) = 0\}$ is called a **plane curve**. A few examples are depicted in Table 1, in particular exhibiting the geometrical phenomenon of **singularities** (of which there are at most finitely many):

i) ii) $f := Y^2 - X(X-1)(X-\lambda)$ for $\lambda := -1$ and $\lambda := 1$,

iii) $f := Y^2 - X^3$, **iv)** $f := (X^2+Y^2)^3 - 4X^2Y^2$,

v) $f := \left(\begin{array}{c} -3X^5 - 2X^4Y - 3X^3Y^2 + XY^4 + 3Y^5 + 6X^4 + 7X^3Y \\ +3X^2Y^2 - 2XY^3 - 6Y^4 - 3X^3 - 5X^2Y + XY^2 + 3Y^3 \end{array} \right)$.

b) Let $\{X, Y, Z\}$ be indeterminates, and let $\mathbb{R}[X,Y,Z]$ be the associated polynomial $\mathbb{R}$-algebra. Given $0 \neq f \in \mathbb{R}[X,Y,Z]$, the set $\mathbf{V}_\mathbb{R}(f) := \{[x,y,z] \in \mathbb{R}^3; f(x,y,z) = 0\}$ is called a **(hyper-)surface**. For example, the surface given by $f_\lambda := (X^2-1)^3 + \lambda \cdot (Y^2+Z^2)$, where $\lambda > 0$, is also depicted in Table 1.

(2.2) Example: The Sudoku game. Let $N \in \mathbb{N}$. A **Sudoku problem** of size N is an $N^2 \times N^2$ square tableau, covered by N^2 boxes of size $N \times N$, to be filled with numbers in $\mathcal{N} := \{1, \ldots, N^2\}$, such that the entries in each row are pairwise distinct, the entries in each column are pairwise distinct, and the entries in each box are pairwise distinct. To start with, a few entries are prescribed. A Sudoku problem is called **well-posed**, if the prescribed tableau can be completed uniquely. The classical Sudoku game is the case $N = 3$, but $N = 2$ is already suitable to do experiments, while $N = 1$ is trivisl; see Table 2.

Let $\mathcal{X} := \{X_{ij}; i,j \in \mathcal{N}\}$ be indeterminates, and let $A := \mathbb{Q}[\mathcal{X}]$ be the associated polynomial $\mathbb{Q}$-algebra. Then filling in the entry m_{ij} in position $[i,j]$ amounts to specialising $X_{ij} \mapsto m_{ij}$. Hence a completed tableau is translated into the maximal ideal $M := \langle X_{ij} - m_{ij}; i,j \in \mathcal{N} \rangle \lhd A$; note that $\dim_\mathbb{Q}(A/M) = 1$. Moreover, combinatorial conditions required to be fulfilled by the solutions of a Sudoku problem are translated into polynomial conditions as follows:

i) Let X be an auxiliary indeterminate, and let $p := \prod_{k \in \mathcal{N}} (X - k) \in \mathbb{Q}[X]$.

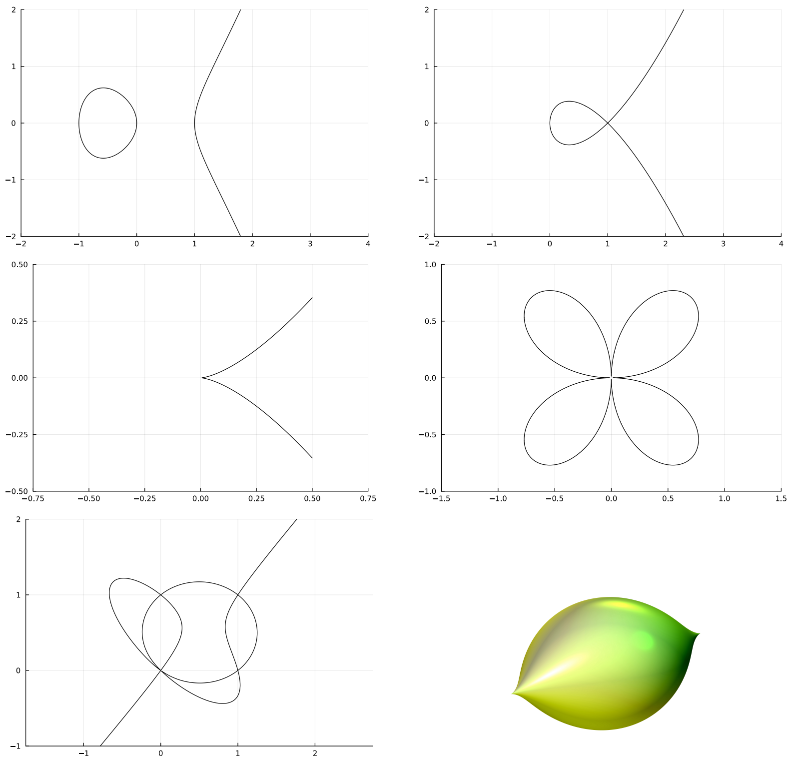
Then the entries being in $\mathcal{N}$ gives rise to $\mathcal{S} := \{p(X_{ij}) \in A; i,j \in \mathcal{N}\}$.

Moreover, prescribing the entries in positions $\mathcal{M} \subseteq \mathcal{N} \times \mathcal{N}$ gives rise to

$$\mathcal{S}_p := \{X_{ij} - m_{ij} \in A; [i,j] \in \mathcal{M}\}.$$

ii) Let Y be another auxiliary indeterminate. Then for $i \in \mathbb{N}_0$ we have $X^i - Y^i = (X-Y) \cdot \sum_{j=0}^{i-1} X^j Y^{i-j-1} \in \mathbb{Q}[X,Y]$. Writing $p := \sum_{i \geq 0} p_i X^i$, where $p_i \in \mathbb{Q}$, we get $p(X) - p(Y) = \sum_{i \geq 0} p_i \cdot (X^i - Y^i)$, showing that $(X - Y) \mid p(X) -$

Table 1: Some plane curves and a surface.



$p(Y) \in \mathbb{Q}[X,Y]$, so that $q(X,Y) := \frac{p(X)-p(Y)}{X-Y} \in \mathbb{Q}[X,Y]$ is a polynomial indeed. Moreover, specialising $Y \mapsto X$ yields $q(X,X) = \sum_{i \geq 1} p_i \cdot iX^{i-1} = (\partial p)(X)$, where ∂ denotes the formal derivative.

Thus for $x, y \in \mathcal{N}$ we get: If $x \neq y$, then $0 = p(x) - p(y) = (x-y)q(x,y)$, implying $q(x,y) = 0$. If $x = y$, then $q(x,x) = (\partial p)(x)$, where since p has the zeroes $\mathcal{N}$, all of which are simple, we conclude that $(\partial p)(x) \neq 0$.

Hence, entries in the same row being pairwise different gives rise to

$$\mathcal{S}_r := \{q(X_{ij}, X_{ik}) \in A]; i,j,k \in \mathcal{N}, j < k\}.$$

Similarly, entries in the same column being pairwise different gives rise to

$$\mathcal{S}_c := \{q(X_{ij}, X_{kj}) \in A; i,j,k \in \mathcal{N}, i < k\}.$$

Table 2: Sudoku problems of size 2 and 3.

1	4	2	3
3	2	1	4
2	3	4	**1**
4	1	3	**2**

1			3
3			1
2			**4**
4			**2**

9	**6**	3	**1**	7	**4**	2	**5**	8
		8	**3**	2	**5**	**6**	4	9
2	5	4	6	8	9	7	3	**1**
8	2	1	**4**	3	7	5	9	6
4	9	**6**	8	5	2	**3**	1	7
		5	**9**	6	**1**	8	2	**4**
5	8	9	7	1	3	4	6	**2**
		7	**2**	4	**6**	**9**	8	5
6	**4**	2	**5**	9	**8**	1	**7**	3

Finally, entries in the same box being pairwise different gives rise to

$$\mathcal{S}_b := \left\{ q(X_{aN+i,bN+j}, X_{aN+k,bN+l}) \in A; \begin{array}{l} a, b \in \{0, \ldots, N-1\}, \\ i, j, k, l \in \{1, \ldots, N\}, i < k, j \neq l \end{array} \right\}.$$

Now, let $I := \langle \mathcal{S}, \mathcal{S}_p, \mathcal{S}_r, \mathcal{S}_c, \mathcal{S}_b \rangle \trianglelefteq A$. Then a completed tableau solves the given Sudoku problem if and only if the associated maximal ideal divides I.

Hence the solutions of the given Sudoku problem are precisely given by the maximal ideals of A dividing I. In particular, the problem is unsolvable if and only if $I = A$, and it is well-posed if and only if $I \lhd A$ is maximal. In order to determine the solutions of the given Sudoku problem explicitly, we have to find a 'good' generating set of I, from which we are able to read off the solutions. (Again, **Gröbner bases** do the job, this time with respect to a **degree-driven** order on monomials.) For example, see Table 2:

i) For $N = 2$ with four suitable entries given (in bold face), there is a unique complete solution (depicted in normal font).

ii) Again for $N = 2$ with four, slightly different entries given, only four more entries are uniquely determined, while there are four complete solutions determined by specialising $x_{2,3} \in \{2, 4\}$ and $x_{4,3} \in \{1, 3\}$ (for example).

iii) For $N = 3$ with 26 suitable entries given, only 49 more entries are uniquely determined, while there are two complete solutions determined by $x_{8,2} \in \{1, 3\}$.