Algebraic Geometry (WS 2025)

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(20.1) Properties of localisations. a) We keep the earlier notation. Then the natural map $\sigma\colon M \to M_{\mathcal{S}}$ is an isomorphism of R-modules if and only if $\rho_M(f)$ is bijective, for all $f \in \mathcal{S}$: Indeed, the latter condition implies that M together with the natural map id_M is a localisation of M at $\mathcal{S}$.

Similarly, $\sigma\colon R \to R_{\mathcal{S}}$ is a ring isomorphism if and only if $\mathcal{S} \subseteq R$ consists of units: Indeed, the latter condition implies that R together with the natural map id_R is a localisation of R at $\mathcal{S}$.

b) An element $f \in R$ is called a **zero-divisor** on M, if $\rho_M(f)$ is not injective, that is there is $0 \neq m \in M$ such that $mf = 0$. Let $\mathcal{Z} = \mathcal{Z}_R(M) \subseteq R$ be the set of zero-divisors on M. In particular, $R \setminus \mathcal{Z}$ is multiplicatively closed; note that $0 \in \mathcal{Z}$ if and only if $R \neq \{0\}$.

For $m \in M$ let the **annihilator** be defined as $\mathrm{ann}_R(m) := \{f \in R; mf = 0\} \trianglelefteq R$; then we have $\mathcal{Z} = \bigcup_{0 \neq m \in M} \mathrm{ann}_R(m) \subseteq R$. For the natural map $\sigma\colon M \to M_{\mathcal{S}}$ we have $\ker(\sigma) = \{m \in M; \mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset\}$: We have $m \in \ker(\sigma)$ if and only if $\frac{m}{1} = 0 \in M_{\mathcal{S}}$, that is $mf = 0$ for some $f \in \mathcal{S}$, which is equivalent to $\mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset$. In particular, σ is injective if and only if $\mathcal{S} \cap \mathcal{Z}_R(M) = \emptyset$. Moreover, we have $M_{\mathcal{S}} = \{0\}$ if and only if $\ker(\sigma) = M$, which is equivalent to $\mathrm{ann}_R(m) \cap \mathcal{S} \neq \emptyset$ for all $m \in M$ (or for an R-module generating set of M).

Similarly, the natural map $\sigma\colon R \to R_{\mathcal{S}}$ is injective if and only if $\mathcal{S} \cap \mathcal{Z}_R(R) = \emptyset$, where $\mathcal{Z}_R(R)$ is the set of zero-divisors of R (including 0 whenever $R \neq \{0\}$). We have $R_{\mathcal{S}} = \{0\}$ if and only if $1 \in \ker(\sigma) \trianglelefteq R$, which since $\mathrm{ann}_R(1) = \{0\}$ is equivalent to $0 \in \mathcal{S}$; in this case we also have $M_{\mathcal{S}} = \{0\}$, for any R-module M.

Example. Let $\mathcal{Z} = \mathcal{Z}_R(R)$. Then $\mathrm{Q}(R) := R_{R \setminus \mathcal{Z}}$ is called the **(full) ring of fractions** of R, and we have a natural embedding $R \to \mathrm{Q}(R)$. In particular, if R is a domain, then $\mathcal{Z} = \{0\}$, and $\mathrm{Q}(R) := R_{R \setminus \{0\}}$ is a field, called the **field of fractions** of R, which again comes with a natural embedding $R \to \mathrm{Q}(R)$.

c) An element $f \in R$ is called **nilpotent** on M, if $\rho_M(f)$ is nilpotent. Let $\mathcal{N}_R(M) \trianglelefteq R$ be the set of nilpotent elements on M. In particular, for $M = R$ we recover the nilradical $\mathcal{N}_R(R) = \mathrm{nil}(R) \trianglelefteq R$; note that $\mathrm{nil}(R) \subseteq \mathcal{N}_R(M)$, and that $0 \in \mathcal{S}$ if and only if $\mathcal{S} \cap \mathrm{nil}(R) \neq \emptyset$.

Now, for $f \in R$ let $\mathcal{S} := \{f^k \in R; k \in \mathbb{N}\}$. Then $R_f := R_{\mathcal{S}}$ is called the **localisation of** R **at** f. We have $R_f = \{0\}$ if and only if $f \in \mathrm{nil}(R)$.

i) If R is reduced, then so is R_f: Since $\frac{1}{f} \in R_f$ is a unit, we consider $\frac{g}{1} \in \mathrm{nil}(R_f)$. Thus $\frac{g^k}{1} = 0 \in R_f$, for some $k \in \mathbb{N}$, hence $(gf^r)^k = g^k f^{rk} = 0 \in R$, for some $r \in \mathbb{N}_0$; from $\mathrm{nil}(R) = \{0\}$ we conclude $gf^r = 0 \in R$, thus $\frac{g}{1} = 0 \in R_f$. $\sharp$

ii) For $f, g \in R$ we have a natural isomorphism rings $(R_f)_g \cong R_{fg}$: The image of f and g, with respect to the natural maps, is a unit in both $(R_f)_g$ and R_{fg}. Hence the natural map $R \to R_{fg}$ factors through a natural map $\alpha\colon (R_f)_g \to R_{fg}$, and the natural map $R \to R_f \to (R_f)_g$ factors through a natural map $\beta\colon R_{fg} \to (R_f)_g$. By naturality we have $\alpha\beta = \mathrm{id}_{(R_f)_g}$ and $\beta\alpha = \mathrm{id}_{R_{fg}}$. ♯

iii) Letting T be an indeterminate, we have $R_f \cong R[T]/\langle fT - 1\rangle$: Since $f = \frac{1}{T} \in R[T]/\langle fT - 1\rangle =: S$, the ring homomorphism $R \to R[T]$ extends to a homomorphism of R-algebras given by $\alpha\colon R_f \to S\colon \frac{1}{f} \mapsto T$. Conversely, the homomorphism of R-algebras given by $\widehat{\psi}\colon R[T] \to R_f\colon T \mapsto \frac{1}{f}$ factors through $\langle fT - 1\rangle \trianglelefteq R[T]$, giving rise to $\psi\colon S \to R_f$. Then $\varphi\psi = \mathrm{id}_{R_f}$ and $\psi\varphi = \mathrm{id}_S$. ♯

We recover the **radical membership test**: We have $f \in \mathrm{nil}(R)$, that is f is nilpotent, if and only if $R_f = \{0\}$, which is equivalent to $\langle fT - 1\rangle = R[T]$.

(20.2) Categories again. If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are categories, and $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{E}$ are (covariant) functors, then the **concatenation** $\mathcal{G} \circ \mathcal{F}\colon \mathcal{C} \to \mathcal{E}$ is the (covariant) functor given by

$$\mathcal{G} \circ \mathcal{F}\colon A \mapsto \mathcal{G}(\mathcal{F}(A)) \quad \text{and} \quad (\mathcal{G} \circ \mathcal{F})(\alpha) = \mathcal{G}(\mathcal{F}(\alpha))\colon \mathcal{G}(\mathcal{F}(A)) \to \mathcal{G}(\mathcal{F}(B)),$$

for $A, B \in \mathcal{C}$, and $\alpha\colon A \to B$ a morphism in $\mathcal{C}$; it is immediately checked that this is a functor indeed. The concatenation of contravariant functors, or of covariant and contravariant functors is defined analogously.

Categories $\mathcal{C}$ and $\mathcal{D}$ are called **isomorphic**, if there are functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} = \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{Id}_{\mathcal{D}}$. Note that in this case either both $\mathcal{C}$ and $\mathcal{D}$ are covariant, or both $\mathcal{C}$ and $\mathcal{D}$ are contravariant.

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{C} \to \mathcal{D}$ be (covariant) functors. Then a **natural transformation** $\mathcal{N}\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a map assigning a morphism $\mathcal{N}_A\colon \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$ to all $A \in \mathcal{C}$, such that for any morphism $\alpha\colon A \to B$ in $\mathcal{C}$ we have

$$\big(\mathcal{F}(\alpha) \cdot \mathcal{N}_B\colon \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{G}(B)\big) = \big(\mathcal{N}_A \cdot \mathcal{G}(\alpha)\colon \mathcal{F}(A) \to \mathcal{G}(A) \to \mathcal{G}(B)\big).$$

If the morphisms $\mathcal{N}_A$ are isomorphisms in $\mathcal{D}$, for all $A \in \mathcal{C}$, then $\mathcal{N}$ is called a **natural isomorphism**; in this case we write $\mathcal{F} \cong \mathcal{G}$. Natural transformations and isomorphisms between contravariant functors are defined analogously.

Categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent** if there are (covariant or contravariant) functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G}\circ\mathcal{F} \cong \mathrm{Id}_{\mathcal{C}}$ and $\mathcal{F}\circ\mathcal{G} \cong \mathrm{Id}_{\mathcal{D}}$; in this case we write $\mathcal{C} \cong \mathcal{D}$. In particular, isomorphic categories are (covariantly or contravariantly) equivalent. Typically, 'equivalence', rather than 'isomorphism', is the appropriate notion to say that two categories are 'essentially equal'.
